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ON A LINEARITY THEOREM FOR MEASURES

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Abstract. We prove a linearity theorem for modular measures on D-lattices (= lattice ordered effect algebras) and study the consequences for the core of measure games.

Introduction. Let Σ be an algebra of subsets of a given set Ω. Assume $\lambda_1, \cdots, \lambda_n$, μ are (finitely additive) real-valued measures over Σ. By "linearity theorems" we mean theorems which, under suitable conditions, ensure that μ is a linear combination of the measures λ_i.

In [M-M, Theorem 20], the authors, among many other interesting results, proved such a linearity theorem for σ-additive measures on a σ-algebra. The linearity theorem is then applied [M-M, Theorem 21] to characterize those measure games for which the core is made of measures which can be written as $\mu = \sum_i \alpha_i \lambda_i$. We recall that measure games, which play an important role in economic theory (see [A-S], [H-N]), are cooperative games ν of the special form $\nu = g(\lambda_1, \cdots, \lambda_n)$.

Let us observe that [M-M, Theorem 20], in its turn, generalizes the uniqueness theorem of [M] to a multivariate setting. In [A-B] we proved that the uniqueness theorem above cited holds true more generally for measures defined on a very general structure, known under the names of effect algebra ([B-F]) or D-poset ([D-P]). It has been, therefore, natural for us to investigate to what extent the linearity theorem [M-M, Theorem 20] can be generalized allowing the consideration of measures on structures weaker than σ-algebras of sets.

The present paper is devoted to the mentioned investigation and we are able to obtain the linearity theorem for a class of measures on D-posets, namely for modular measures on D-lattices. ([C-K]). This is the content of Theorem 2.1. We also give a finitely additive version (Corollary 2.5) of the linearity theorem.

In Section 3, we consider functions on D-lattices which quite naturally correspond to standard measure games in the sense that they are of the form $\nu = g(\lambda_1, \cdots, \lambda_n)$, with the λ_i being modular measures. The core and the σ-core of these functions are investigated in order to obtain linearity results (Theorem 3.4, Proposition 3.6, Proposition 3.8 and Theorem 3.9) generalizing those in [M-M]. Our results also imply new results for measures on other structures like orthomodular lattices and MV-algebras, which are of the interest in quantum physics ([B-C]) and cooperative game theory ([B-K]). The final Section 4 discusses a different representation of the elements of the core of measure games, namely an integral representation ([M-M, Theorem 14]) of which the previous linearity theorems can be considered special cases. Naturally, we cannot expect in our general setting any integral representation. However, if the measure game $\nu = g(\lambda_1, \cdots, \lambda_n)$ is defined on a σ-complete D-lattice and the λ_i are σ-additive, then every μ in the core of ν can be written as a finite sum whose n terms are series of simple λ_i-continuous modular measures. If, in particular,

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L is a clan of fuzzy sets (also σ-complete), then we can present an integral representation of μ.

1. Preliminaries. To introduce the basic structure to be used as the domain of measures, one can follow two different approaches. One is based on the notion of effect algebra and the other on the notion of D-poset. The two approaches are known to be equivalent, according to theorem 1.3.4 of the book [D-P], which will be our source of information concerning the domain of measures. We shall use the concept of D-poset just for continuity with respect to our previous paper [A-B].

It is worth mentioning that D-posets (or effect algebras) can be used for modelling both unsharp measurement in a quantum mechanic system ([B-F]) and the class of unambiguous events in a decision theoretical framework ([E-Z]).

A poset $(L, \leq)$ with greatest element 1 and smallest element 0 is said to be a *D-poset* if a partial binary operation $\ominus$, to be called *difference,* can be defined on it in such a way that the following conditions hold true $(a, b, c \in L)$:

(1) $b \ominus a$ is defined iff $a \leq b$.
(2) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.
(3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

If L is a D-poset and $a, b \in L$, we set $a^\perp = 1 \ominus a$ and say that a, b are *orthogonal* if $a \leq b^\perp$. When a and b are orthogonal, we write $a \perp b$. Naturally, $(a^\perp)^\perp = a$ and $a \leq b$ implies $a^\perp \geq b^\perp$. For orthogonal elements a, b of L it is possible to define a commutative and associative operation $\oplus$ given by $a \oplus b = (a^\perp \ominus b)^\perp$. By [D-P], if $a, b \in L$ and $a \leq b$, then $b = a \oplus (b \ominus a)$.

Let L be a D-poset. If $a, b \in L$ and $a \leq b$, we set $[a, b] = \{c \in L : a \leq c \leq b\}$. If $a_1, ..., a_n \in L$, we inductively define $a_1 \oplus ... \oplus a_n = (a_1 \oplus ... \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent of permutations of the elements. We say that a finite subset $\{a_1, ..., a_n\}$ of L is *orthogonal* if $a_1 \oplus ... \oplus a_n$ exists. We say that a sequence $\{a_n\}$ is *orthogonal* if, for every n, $\bigoplus_{i \leq n} a_i$ exists. If, moreover, $\sup_n \bigoplus_{i \leq n} a_i$ exists, we set $\bigoplus_{n \in \mathcal{N}} a_n = \sup_n \bigoplus_{i \leq n} a_i$.

If G is a topological Abelian group, a function $\mu : L \to G$ is said to be a *measure* if, for every orthogonal $a, b \in L$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure if and only if, for every $a, b \in L$, with $a \leq b$, $\mu(b \ominus a) = \mu(b) - \mu(a)$. A measure μ is said to be σ*-additive* if, for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $\mu(\bigoplus_n a_n) = \sum_{n=1}^{\infty} \mu(a_n)$. If $G = R^n$, we say that a measure μ is *strongly continuous* if, for every $\varepsilon > 0$ and $a \in L$, there exists an orthogonal family $\{a_1, \cdots, a_r\}$ in L such that $\bigoplus_{i \leq r} a_i = a$ and $\|\mu(b)\| \leq \varepsilon$ whenever $b \in L$ and $b \leq a_i$ for some $i \leq n$. By [A-B, 4.4], μ is strongly continuous if and only if the previous condition is satisfied with $a = 1$.

Let L be a poset. L is said to be σ*-complete* if every countable set in L has a supremum and an infimum. We say that L *has the interpolation property* if, for all sequences $\{a_n\}, \{b_n\}$ in L with $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for each $n \in N$, there exists $a \in L$ such that $a_n \leq a \leq b_n$ for each $n \in N$. Trivially every σ-complete lattice has the interpolation property. We write $a_n \uparrow a$ if $\{a_n\}$ is increasing and $a = \sup_n a_n$. L is said to be σ*-continuous* if $a_n \uparrow a$ in L implies $a_n \wedge b \uparrow a \wedge b$ for every $b \in L$.

If L is a lattice, a function $\mu : L \to G$ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

Definition (1.1). *A D-poset that is also a lattice is said to be a D-lattice.*

Examples

• *Orthomodular lattices are D-lattices.*

An *orthomodular lattice (OML)* is a lattice $(L, \leq)$ with 0, 1, and an order reversing function $' : L \to L$ having the following properties:

(1) $(a')' = a$ and $a \wedge a' = 0$.
(2) If $a \leq b$, then $b = a \vee (b \wedge a')$.

Every OML is a D-lattice if we define $a \ominus b = a \wedge b'$ for $b \leq a$. In this case, $a^{\perp} = a'$ and, if $a \perp b$, then $a \oplus b = a \vee b$.

• *MV-algebras, and therefore in particular clans of fuzzy sets ([B-K]), are D-lattices.*

We recall that an *MV-algebra* is a commutative semigroup $(L, +)$ endowed with a neutral element 0, a special element 1 and an operation $* : L \to L$ such that:

(1) $x + 1 = 1$.
(2) $(x^*)^* = x$.
(3) $x + x^* = 1$.
(4) $0^* = 1$.
(5) $(x^* + y)^* + y = (x + y^*)^* + x$.

If we set $a \vee b = (a \bigodot b^*) + b$ and $a \wedge b = (a + b^*) \bigodot b$, where $a \bigodot b = (a^* + b^*)^*$, then L becomes a distributive lattice if we define $a \leq b$ if and only if $a \wedge b = a$.

Every MV-algebra is a D-lattice if we define $a \ominus b = (b + a^*)^*$ for $b \leq a$. In this case, $a^{\perp} = a^*$ and $a \oplus b = (a^* \ominus b)^*$ if $b \leq a^*$.

If X is a non-empty set, a *clan of fuzzy sets*, according to [B-K], is a family $\mathcal{A}$ of $[0,1]$-valued functions on X such that:

(1) $1 \in \mathcal{A}$.
(2) If $f, g \in \mathcal{A}$, then $\max\{f - g, 0\} \in \mathcal{A}$.

Every clan of fuzzy sets is a D-lattice if we define $f \ominus g = f - g$ for $g \leq f$. In this case, $f^{\perp} = 1 - f$ and, if $f + g \leq 1$, then $f \oplus g = f + g$.

On the other hand, every clan of fuzzy sets is an MV-algebra by defining $f^* = 1 - f$ and $f + g = \min\{f + g, 1\}$.

Remark. In a D-lattice the concepts of modular function and of measure are independent. On the other hand, we have that on an orthomodular lattice every modular function is a measure, and on an MV-algebra every measure is a modular function.

2. The linearity theorem.

Throughout the paper L is a D-lattice. Denote by $\mu : L \to R$ a *bounded modular measure* and by $\lambda : L \to R_+^n$ a *strongly continuous* modular measure ($\lambda = (\lambda_1, \cdots \lambda_n)$).

An element $a \in L$ is said to be λ-*radial* if there exist $b \in L$ and $t \in [0,1]$ such that $\lambda(a) = t\lambda(b) + (1-t)\lambda(b^{\perp})$.

The aim is to prove the following result.

Theorem (2.1). *Suppose that one of the following conditions is satisfied:*

(1) *L has the interpolation property and μ is strongly continuous.*
(2) *L is σ-complete and μ is σ-additive.*

Moreover suppose that there exist a λ-radial element $a_* \in L$ which satisfies the following condition (where $a \in L$) :

(*) $\lambda(a) = \lambda(a_*) \Rightarrow \mu(a) = \mu(a_*)$.

Then there exist real numbers $\alpha_1, \cdots, \alpha_n$ such that, for every $a \in L$,

$$\mu(a) = \sum_{i=1}^{n} \alpha_i \lambda_i(a).$$

If $\lambda(L)$ has full dimension, then the α_i's are unique. Moreover the coefficients α_i can be chosen nonnegative if and only if the following condition is satisfied (where $a \in L$) :

(**) $\lambda_i(a) \geq \lambda_i(a_*)$ for every $i \leq n \Rightarrow \mu(a) \geq \mu(a_*)$.

The proof of (2.1) is based on the following result of [M-M] (see the proof of Theorem 20).

Lemma (2.2). *Let S be any set, and f, g be functions from S to R^n_+ and R, respectively. Assume what follows:*

(1) *The ranges of f and of (f, g) are convex and contain* 0.

(2) *There is a point $a^* \in S$ such that:*

(a) $f(a^*)$ belongs to the relative interior of $f(S)$.

(b) For every $s \in S$, $f(s) = f(a^*)$ implies $g(s) = g(a^*)$.

Then the function g is a linear combination of the components of f; $f(S)$ is full dimensional if and only if the components of f are linearly independent. Moreover, if $f(s) \geq f(a^*) \Rightarrow g(s) \geq g(a^*)$, then the coefficients of the linear combination are nonnegative.

It is also helpful to recall the following results from [B_1, 3.12 and 3.15].

Theorem (2.3). *If L has the interpolation property and $m : L \to R^n$ is a strongly continuous modular measure, then $m(L)$ is convex.*

Proposition (2.4). *If L is σ-complete and $m : L \to R^n$ is a σ-additive modular measure, then m is atomless if and only if it is strongly continuous.*

We recall that a measure $m : L \to G$ is said to be *atomless* if, for every $a \in L$ with $m(a) \neq 0$, there exists $b \leq a$ such that $m(b) \neq 0$ and $m(a \ominus b) \neq 0$.

Proof of Theorem (2.1).

As in [M-M, Proposition 19], we can prove that an element $a \in L$ is λ-radial if and only if $\lambda(a)$ belongs to the relative interior of $\lambda(L)$. Then, by (2.2), we have only to prove that λ and (λ, μ) have convex range. This follows immediately by (2.3) if (1) holds. If (2) holds, by (2.3) we have only to prove that μ is again strongly continuous. By (2.4), it is sufficient to prove that μ is atomless.

(i) First we prove that, for every $b \in L$, $\lambda(b) = 0$ implies $\mu(b) = 0$.

Let $b \in L$ be such that $\lambda(b) = 0$. Since $\lambda_i \geq 0$ for each $i \leq n$, we get $\lambda(b \wedge a_*) = 0$, from which we have $\lambda(a_* \ominus (b \wedge a_*)) = \lambda(a_*)$. By (*), we get $\mu(a_* \ominus (b \wedge a_*)) = \mu(a_*)$, from which we have $\mu(b \wedge a_*) = 0$. On the other hand, since λ is modular, we have also that $0 = \lambda(b \ominus (b \wedge a_*)) = \lambda(b) - \lambda(a_* \wedge b) = \lambda(a_* \vee b) - \lambda(a_*)$. By (*), we get $\mu(a_* \vee b) = \mu(a_*)$. Since μ is modular, we have $\mu(b) = \mu(a_* \wedge b) + \mu(a_* \vee b) - \mu(a_*) = 0$.

(ii) Now we prove that μ is atomless.

Let $\mu(b) \neq 0$. By (i), we have $\lambda(b) \neq 0$. Suppose that, for every $c \leq b$, either $\mu(c) = 0$ or $\mu(b \ominus c) = 0$. Set $J = \{i \in \{1, \cdots, n\} : \lambda_i(b) > 0\}$ and $\lambda^* = (\lambda_j)_{j \in J}$. By (2.8) of

[A-B], λ^* is strongly continuous and therefore by (2.3) $\lambda^*([0,a])$ is convex for every $a \in L$. Then we can find $b_1 \leq b$ such that $\lambda^*(b_1) = \frac{1}{2}\lambda^*(b)$. By assumption, either $\mu(b_1) = 0$ or $\mu(b \ominus b_1) = 0$. Suppose for example $\mu(b_1) = 0$ and set $c_1 = b \ominus b_1$. Then $\mu(c_1) = \mu(b)$ and $\lambda^*(c_1) = \lambda^*(b) = \lambda^*(b_1) = \frac{1}{2}\lambda^*(b)$. In a similar way, if $\mu(b \ominus b_1) = 0$, we set $c_1 = b_1$ and we have again $\mu(c_1) = \mu(b)$ and $\lambda^*(c_1) = \frac{1}{2}\lambda^*(b)$.

Now choose $b_2 \leq c_1$ such that $\lambda^*(b_2) = \frac{1}{2}\lambda^*(c_1)$. We see that either $\mu(b_2) = 0$ or $\mu(c_1 \ominus b_2) = 0$. In fact, suppose $\mu(c_1 \ominus b_2) \neq 0$, i.e. $\mu(c_1) \neq \mu(b_2)$. Then, since $\mu(b) = \mu(c_1)$ and $b = b_2 \oplus (b \ominus b_2)$, we get $\mu(b \ominus b_2) \neq 0$ and therefore, by assumption, $\mu(b_2) = 0$.

Suppose for example that $\mu(b_2) = 0$ and set $c_2 = c_1 \ominus b_2$. Then $\mu(c_2) = \mu(c_1) = \mu(b)$ and $\lambda^*(c_2) = \lambda^*(c_1) - \lambda^*(b_2) = \frac{1}{4}\lambda^*(b)$.

In this way we obtain by induction a decreasing sequence $\{c_n\}$ in L such that, for each positive integer n,

(1) $c_n \leq b$.

(2) $\mu(c_n) = \mu(b)$.

(3) $\lambda^*(c_n) = \frac{1}{2^n}\lambda^*(b)$.

Since L is σ-complete, there exists $c = \inf_n c_n$, and $c \leq b$. Since $0 \leq \lambda_i^*(c_n) = \frac{1}{2^n}\lambda^*(b)$ for each n, we get $\lambda^*(c) = 0$. Moreover, if there exists $i \notin J$ such that $\lambda_i^*(c) \neq 0$, we would have $\lambda_i(b) \neq 0$ since $b \geq c$, a contradiction to the definition of J. Hence $\lambda(c) = 0$. On the other hand, by [A-B, 2.4], we have $\mu(c) = \lim_n \mu(c_n) = \mu(b) \neq 0$, a contradiction to (i). Therefore μ is atomless. $\square$

Now we see that the assumptions (1) and (2) in (2.1) can be removed if we replace the condition (*) with a stronger condition.

Corollary (2.5). *Suppose that there exists a λ-radial element $a_* \in L$ which satisfies the following condition (where $\{a_n\}$ is a sequence of elements of L) :*

$(*)'$ *If* $\lim_n \lambda(a_n) = \lambda(a_*)$ *and* $\{\mu(a_n)\}$ *converges, then* $\lim_n \mu(a_n) = \mu(a_*)$.

Then there exist real numbers $\alpha_1, \cdots, \alpha_n$ such that, for every $a \in L$,

$\mu(a) = \sum_{i=1}^n \alpha_i \lambda_i(a)$. *If $\lambda(L)$ has full dimension, then α_i's are unique. Moreover we can choose $\alpha_i \geq 0$ for each $i \leq n$ if and only if the following condition holds (where $\{a_n\}$ is a sequence of elements of L) :*

$(**)'$ *If* $\lim_n \lambda(a_n) \geq \lambda(a_*)$ *and* $\{\mu(a_n)\}$ *converges, then* $\lim_n \mu(a_n) \geq \mu(a_*)$.

To prove (2.5), it is helpful to recall that, by [A-B, 4.2], every modular measure on L generates a uniformity on L - called *D-uniformity* - which makes the lattice operations, the $\ominus, \oplus$-operations of L and the measure itself uniformly continuous. Moreover (see [A-B-V, Section 2]) every D-uniformity on L generates a *D-congruence,* i.e. a lattice congruence with the following property: if $a \sim b$, $c \sim d$, $c \leq a$ and $d \leq b$, then $a \ominus c \sim d \ominus b$, and the quotient of a D-lattice with respect to a D-congruence is a D-lattice, too. Recall also that, by [A, 3.8], a modular measure $m : L \to R^n$ is bounded if and only if it is *exhaustive*, i.e. for every monotone sequence $\{a_n\}$ in L, $\{m(a_n)\}$ is a Cauchy sequence. By [W_2, 3.2], a modular function is exhaustive if and only if the uniformity it generates is exhaustive, i.e. the latter makes Cauchy every monotone sequence in L.

Proof of (2.5).

Denote by $\mathcal{U}$ the supremum of the uniformities generated by λ and μ. Since λ and μ are bounded, $\mathcal{U}$ is exhaustive.

Denote by $\hat{L}$ the quotient of L with respect to the D-congruence $N(\mathcal{U}) = \bigcap\{U : U \in \mathcal{U}\}$ (see [A-V]) and by $(\tilde{L}, \tilde{\mathcal{U}})$ the uniform completion of $\hat{L}$. Moreover set $\hat{\mu}(\hat{a}) = \mu(a)$ and $\hat{\lambda}(\hat{a}) = \lambda(a)$ for $\hat{a} \in \hat{L}$ and a in the congruence class $\hat{a}$. Denote by $\tilde{\lambda}, \tilde{\mu}$ the uniformly continuous extension of $\hat{\lambda}, \hat{\mu}$ to $\tilde{L}$. By [A-B, 4.3], $\tilde{L}$ is a σ-complete D-lattice and $\tilde{\lambda}$, $\tilde{\mu}$ are σ-additive modular measures. Moreover it is clear that $\tilde{\lambda}$ is strongly continuous, and $\hat{a}_*$ is a $\tilde{\lambda}$-radial set. We prove that $\tilde{\lambda}$ and $\tilde{\nu}$ satisfy condition (*) of (2.1). Let $\tilde{b} \in \tilde{L}$ be such that $\tilde{\lambda}(\tilde{b}) = \tilde{\lambda}(\hat{a}_*)$. Choose a sequence $\{\hat{b}_n\}$ in $\hat{L}$ such that $\lim_n \hat{b}_n = \tilde{b}$ in $(\tilde{L}, \tilde{\mathcal{U}})$. Since $\tilde{\lambda}$ and $\tilde{\mu}$ are $\tilde{\mathcal{U}}$-continuous, we have $\lim_n \tilde{\lambda}(\hat{b}_n) = \tilde{\lambda}(\tilde{b})$ and $\lim_n \mu(b_n) = \lim_n \tilde{\mu}(\hat{b}_n) = \tilde{\mu}(\tilde{b})$. Then we get $\lim_n \lambda(b_n) = \lim_n \hat{\lambda}(\hat{b}_n) = \lim_n \tilde{\lambda}(\hat{b}_n) = \tilde{\lambda}(\tilde{b}) = \tilde{\lambda}(a_*) = \lambda(a_*)$. Because of the assumptions, we get $\lim_n \mu(b_n) = \mu(a_*)$. Therefore $\tilde{\mu}(\tilde{b}) = \lim_n \tilde{\mu}(\hat{b}_n) = \lim_n \mu(b_n) = \mu(a_*) = \tilde{\mu}(\hat{a}_*)$.

Now, using Theorem (2.1), we find real numbers $\alpha_1, \cdots, \alpha_n$ such that $\tilde{\mu} = \sum_{i=1}^n \alpha_i \tilde{\lambda}_i$, from which $\mu = \sum_{i=1}^n \alpha_i \lambda_i$.

In a similar way, if $(**)'$ is satisfied, we obtain that $\tilde{\lambda}$ and $\tilde{\mu}$ satisfy the condition (**) of (2.1) and therefore we can choose $\alpha_i \geq 0$ for each $i \leq n$. □

Theorems (2.1) and (2.5) allow us to obtain information on the structure of the core of a measure game, as in [M-M].

3. The core of measure games.

First of all we extend, in a straightforward way, the classical terminology, and call a *game* any function $\nu : L \to R$ when $\nu(0) = 0$. A game ν is then a *measure game* if there exist a strongly continuous modular measure $\lambda : L \to R_+^n$ and a function $g : \lambda(L) \to R$, with $g(0) = 0$, such that, for every $a \in L$, $\nu(a) = g(\lambda(a))$. If $n = 1$, then ν is said to be a *scalar measure game.*

In the following, $\nu = g(\lambda)$ denotes a measure game.

By adapting the terminology of [M-M], a measure game is said to be *superdifferentiable* in an element $a \in L$ if there exists a bounded modular measure $\mu : L \to R$ such that, for every $b \in L$,

$$(*) \qquad \nu(b) - \nu(a) \leq \mu(b) - \mu(a).$$

The set (possibly empty) of all modular measures which satisfy (*) is called *superdifferential* of ν in a and denoted by $\partial\nu(a)$.

It is easy to see ([M-M, Lemma 3]) that $\partial\nu(a)$ is related to the superdifferential of g by the following result.

Lemma (3.1). *A modular measure of the form $\alpha \cdot \lambda$, with $\alpha \in R^n$, belongs to $\partial\nu(a)$ if and only if $\alpha \in \partial g(\lambda(a))$.*

The *core* and the *σ-core* of ν are the sets

$$C(\nu) = \{\mu : L \to R : \mu \text{ a bounded modular measure, } \mu(1) = \nu(1) \text{ and } \mu \geq \nu\}$$

and

$$C_\sigma(\nu) = \{\mu \in C(\nu) : \mu \text{ is } \sigma\text{-additive}\}.$$

An element $a \in L$ is said to be *ν-linear* if $\nu(a) + \nu(a^\perp) = \nu(1)$. Trivially, if $C(\nu) \neq \emptyset$, we have $\nu(1) \geq \nu(a) + \nu(a^\perp)$ for every $a \in L$.

Lemma (3.2). *If $c \in L$ is ν-linear and $\mu \in C(\nu)$, then $\mu(c) = \nu(c)$.*

Proof. We have $\mu(c) + \mu(c^\perp) = \mu(1) = \nu(1) = \nu(c) + \nu(c^\perp)$ and $\mu \geq \nu$. □

As in [M-M, Theorem 10], the core of ν is related to $\partial\nu$ by the following result. We give the proof here for completeness, although the arguments are basically in [M-M].

Proposition (3.3). *Consider the following conditions:*

(1) *a is ν-linear.*
(2) *$C(\nu) = \partial\nu(a) \cap \partial\nu(a^\perp)$.*
(3) *$\partial\nu(a) \cap \partial\nu(a^\perp) \neq \emptyset$.*

Then, (1) implies (2). Moreover, if $C(\nu) \neq \emptyset$, conditions (1), (2) and (3) are equivalent.

Proof. (i) First we prove that, if $\mu \in \partial\nu(a)$, then $\mu(a) \leq \nu(a)$ and $\nu(1) \leq \nu(a) + \mu(a^\perp)$.

In fact, we have $\nu(1) = \nu(a \oplus a^\perp) \leq \nu(a) + \mu(a \oplus a^\perp) - \mu(a) = \nu(a) + \mu(a^\perp)$ and $0 = \nu(0) \leq \nu(a) + \mu(0) - \mu(a)$, from which we get $\nu(a) \geq \mu(a)$.

(ii) By (i), we obtain that, if $\mu \in \partial\nu(a^\perp)$, then $\mu(a^\perp) \leq \nu(a^\perp)$ and $\nu(1) \leq \nu(a^\perp) + \mu(a)$.

(1) $\Rightarrow$ (2) Because of (3.2), it is clear that $C(\nu) \subseteq \partial\nu(a) \cap \partial\nu(a^\perp)$. Conversely, let $\mu \in \partial\nu(a) \cap \partial\nu(a^\perp)$. By (i) and (1), we get $\mu(a^\perp) \geq \nu(1) - \nu(a) = \nu(a) + \nu(a^\perp) - \nu(a) = \nu(a^\perp)$. By (ii), we get $\mu(a^\perp) = \nu(a^\perp)$. In similar way we obtain that $\mu(a) = \nu(a)$. Then, if $b \in L$, we have $\nu(b) \leq \nu(a) + \mu(b) - \mu(a) = \mu(b)$. Moreover, since a is ν-linear, by (3.2) we have $\nu(1) = \nu(a) + \nu(a^\perp) = \mu(a) + \mu(a^\perp) = \mu(1)$. Then $\mu \in C(\nu)$.

Now suppose $C(\nu) \neq \emptyset$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Since $C(\nu) \neq \emptyset$, we have $\nu(a) + \nu(a^\perp) \leq \nu(1)$. On the other hand, if $\mu \in \partial\nu(a) \cap \partial\nu(a^\perp)$, by (i) and (ii) we have $\nu(1) \leq \nu(a) + \mu(a^\perp) \leq \nu(a) + \nu(a^\perp)$. Then a is ν-linear. □

By (2.1), we obtain the following result.

Theorem (3.4). *Let L be σ-complete. Suppose that $C_\sigma(\nu)$ is a non-empty set and that there exists a ν-linear and λ-radial element $a_* \in L$. Then, if a is any ν-linear element of L, we have $C_\sigma(\nu) = \{\alpha \cdot \lambda : \alpha \in \partial g(\lambda(a)) \cap \partial g(\lambda(a^\perp))\}$. The vectors $\alpha \in R^n$ are uniquely determined if and only if $\lambda(L)$ has full dimension. Moreover, if g is monotone increasing on $\lambda(L)$, then we can choose $\alpha \in R^n_+$.*

Proof. Let $\mu \in C_\sigma(\nu)$. We prove that λ and μ satisfy the assumptions of (2.1). Since a_* is λ-radial, we have only to prove that, for every $b \in L$, $\lambda(b) = \lambda(a^*)$ implies $\mu(b) = \mu(a_*)$.

Let $b \in L$ be such that $\lambda(b) = \lambda(a_*)$. By (3.2), we have $\mu(b) \geq \nu(b) = g(\lambda(b)) = g(\lambda(a_*)) = \nu(a_*) = \mu(a_*)$, and $\mu(b^\perp) \geq \nu(b^\perp) = g(\lambda(b^\perp)) = g(\lambda(a_*^\perp)) = \nu(a_*^\perp) = \mu(a_*^\perp)$. Since $\mu(b) + \mu(b^\perp) = \mu(a_*) + \mu(a_*^\perp)$ and $\mu \geq \nu$, we get $\mu(b) = \mu(a_*)$. Then the first part of the assertion follows by (2.1), recalling (3.1) and (3.3)

Now suppose that g is monotone increasing. We see that λ and μ satisfy the condition (**) of (2.1). Let $b \in L$ be such that $\lambda(b) \geq \lambda(a_*)$. Then we have $\mu(b) \geq \nu(b) = g(\lambda(b)) \geq g(\lambda(a_*)) = \nu(a_*) = \mu(a_*)$. Hence the second part of the assertion follows again by (2.1). □

Remark. Concerning the existence of linear and radial elements of L, the considerations of [M-M, pages 20-21] apply. Indeed, for example, suppose that L has the interpolation property, $C(\nu) \neq \emptyset$ and the game ν is *radially concave,* i.e. there exists a ν-linear element $a \in L$ such that, for every $t \in [0,1]$, $g(t\lambda(a) + (1-t)\lambda(a^\perp)) \geq tg(\lambda(a)) + (1-t)g(\lambda(a^\perp))$. Then there exists a ν-linear and λ-radial element in L.

In fact, if $\lambda(a) = \lambda(a^\perp)$, we have $\lambda(a) = \frac{1}{2}\lambda(1)$. Then $\lambda(a)$ belongs to the relative interior of $\lambda(L)$ and therefore a is λ-radial. If $\lambda(a) \neq \lambda(a^\perp)$ and $t \in]0,1[$, by Theorem (2.3) we can find $b \in L$ such that $\lambda(b) = t\lambda(a) + (1-t)\lambda(a^\perp)$. Then we have $\lambda(b^\perp) = \lambda(1) - \lambda(b) = t\lambda(a^\perp) + (1-t)\lambda(a)$. Therefore $\nu(b) + \nu(b^\perp) \geq g(\lambda(b)) + g(\lambda(b^\perp)) \geq t\nu(a) + (1-t)\nu(a^\perp) + t\nu(a^\perp) + (1-t)\nu(a) = \nu(a) + \nu(a^\perp) = 1$. On the other hand, since $C(\nu) \neq \emptyset$, we have $\nu(b) + \nu(b^\perp) \leq \nu(1)$. Hence b is ν-linear and λ-radial.

Now, to give a description of $C(\nu)$, we need some results, which allow us to extend [M-M, Lemmas 34 and 35] to our setting.

Lemma (3.5). *Suppose $C(\nu) \neq \emptyset$ and there exists a ν-linear element $a \in L$ such that g is lower semicontinuous in $\lambda(a)$ and in $\lambda(a^\perp)$. Then g is continuous in $\lambda(a)$ and in $\lambda(a^\perp)$.*

Proof. Let $\{b_n\}$ be a sequence in L such that $\lim_n \lambda(b_n) = \lambda(a))$. We prove that

$\lim_n g(\lambda(b_n)) = g(\lambda(a))$. Recall that, since $C(\nu) \neq \emptyset$, we have $\nu(1) \geq \nu(b_n) + \nu(b_n^\perp)$. Then we obtain $\limsup_n g(\lambda(b_n)) + \liminf_n g(\lambda(b_n^\perp)) \leq \lim sup_n[g(\lambda(b_n)) + g(\lambda(b_n^\perp))] = \lim sup_n[\nu(b_n) + \nu(b_n^\perp)] \leq \nu(1) = g(\lambda(1))$. Since g is lower semicontinuous in $\lambda(a)$ and in $\lambda(a^\perp)$, we get $\lim sup_n g(\lambda(b_n)) \leq g(\lambda(1)) - \liminf_n g(\lambda(b_n^\perp)) \leq g(\lambda(1)) - g(\lambda(a^\perp)) = \nu(1) - \nu(a^\perp) = \nu(a) = g(\lambda(a)) \leq \liminf_n g(\lambda(b_n))$.

In a similar way we prove that g is continuous in $\lambda(a^\perp)$. □

In the next result we use that, if $c \geq a, b$, then $(c \ominus a) \vee (c \ominus b) = c \ominus (a \wedge b)$ ([D-P, 1.8.2]). Recall that L is said to be σ-continuous if $a_n \uparrow a$ in L implies $a_n \wedge b \uparrow a \wedge b$ for every $b \in L$.

Proposition (3.6). *Suppose that L is σ-continuous and λ is σ-additive. Let $a \in L$ be ν-linear such that g is lower semicontinuous in $\lambda(a)$ and in $\lambda(a^\perp)$. Then $C(\nu) = C_\sigma(\nu)$.*

Proof. Let $\mu \in C(\nu)$. We prove that μ is σ-additive. By [A-B, 2.4], it is sufficient to prove that $a_n \uparrow 1$ implies $\lim_n \mu(a_n) = \mu(1)$.

Let $a_n \uparrow 1$. Since L is σ-continuous, we have $a_n \wedge a \uparrow a$, $a_n \vee a \uparrow 1$ and $a_n^\perp \vee a^\perp \downarrow a^\perp$. Since λ is σ-additive, by [A-B, 2.4] we get

$$\lim_n \lambda(a_n \wedge a) = \lambda(a), \lim_n (a_n \vee a) = \lambda(1), \lim_n \lambda(a_n^\perp \vee a^\perp) = \lambda(a^\perp).$$

Now we prove that:

(1) $\lim_n \mu(a_n \wedge a) = \mu(a)$.
(2) $\lim_n \mu(a_n \vee a) = \mu(1)$.

(1) Since by (3.5) g is continuous in $\lambda(a)$ and in $\lambda(a^\perp)$, we have $\liminf_n \mu(a \wedge a_n) \geq \liminf_n \nu(a \wedge a_n) = \liminf_n g(\lambda(a \wedge a_n)) = g(\lambda(a)) = \nu(a) = \nu(1) - \nu(a^\perp) = g(\lambda(1)) - \lim_n g(\lambda(a_n^\perp \vee a^\perp)) = \nu(1) - \lim_n \nu(a_n^\perp \vee a^\perp) \geq \mu(1) - \liminf_n \mu(a_n^\perp \vee a^\perp) = \limsup_n[\mu(1) - \mu(a_n^\perp \vee a^\perp)] = \limsup_n[\mu(1) - \mu(1 \ominus (a_n \wedge a))] = \limsup_n \mu(a_n \wedge a)$. Then $\lim_n \mu(a_n \wedge a) = \nu(a) = \mu(a)$ by (3.2).

(2) We have $\liminf_n \mu(a_n \vee a) \geq \liminf_n \nu(a_n \vee a) = \liminf_n g(\lambda(a_n \vee a)) = g(\lambda(1)) = \nu(1) = \mu(1)$.

Since $a_n = (a_n \wedge a) \oplus (a_n \ominus (a_n \wedge a))$ and μ is modular, we have $\mu(a_n) = \mu(a_n \wedge a) + \mu(a_n \ominus (a_n \wedge a)) = \mu(a_n \wedge a) + \mu(a_n \vee a) - \mu(a)$. Then, by (1) and (2), we get $\lim_n \mu(a_n) = \mu(1)$. □

In the next results, we use the following notation: if $\mu : L \to R$ is a modular measure, we set, for $a \in L$,

$$\mu^+(a) = \sup\{\mu(b) : b \leq a\}, \mu^-(a) = \sup\{-\mu(b) : b \leq a\}.$$

Moreover we denote by $|\mu|$ the total variation of μ, i.e. for $a \in L$,

$$|\mu|(a) = \sup\{\sum_{i=1}^r |\mu(a_i) - \mu(a_{i-1})| : r \in N, 0 = a_0 \leq a_1 \leq \cdots \leq a_r = a\}.$$

By $[B_1, 3.11]$, $|\mu|$ is a modular measure, $|\mu|$ is bounded if and only if μ is bounded and, for $a \in L$,

$$|\mu|(a) = \sup\{\sum_{i=1}^{r} |\mu(a_i)| : r \in N, \{a_1, \cdots, a_r\} \text{ orthogonal family in } L, \bigoplus_{i=1}^{r} a_i = a\}.$$

Moreover recall that, by [D-P, 1.1.6], if $a \perp b$, $c \leq a$ and $d \leq b$, then $a \leq a \oplus b$, $(a \oplus b) \ominus a = b$, $c \oplus d \leq a \oplus b$ and we have that $e = a \oplus b$ if and only if $b = e \ominus a$.

Proposition (3.7). *If μ is a bounded modular measure, then μ^+ and μ^- are modular measures, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$.*

Proof. (i) First we prove that, for every $a \in L$,

$$(*) \qquad \mu^+(a) = \sup\{\sum_{i=1}^{r} [\mu(a_i) - \mu(a_{i-1})]^+ : r \in N, 0 = a_0 \leq a_1 \leq \cdots \leq a_r = a\}$$

and

$$(**) \qquad \mu^-(a) = \sup\{\sum_{i=1}^{r} [\mu(a_i) - \mu(a_{i-1})]^- : r \in N, 0 = a_0 \leq a_1 \leq \cdots \leq a_r = a\}$$

where naturally we mean, for $x \in R$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

Denote by $\overline{\mu}(a)$ the right hand side of (*). The inequality $\mu^+(a) \leq \overline{\mu}(a)$ is trivial. Let $0 = a_0 \leq a_1 \leq \cdots \leq a_r = a$ be a chain in L. For $i \in \{1, \cdots, r\}$, set $b_i = a_i \ominus a_{i-1}$. By [A-B, 2.1], $\{b_1 \cdots, b_r\}$ is an orthogonal family and $b_1 \oplus \cdots \oplus b_r = a$. Moreover we have $\mu(b_i) = \mu(a_i) - \mu(a_{i-1})$ for each $i \in \{1, \cdots, r\}$. Set $I = \{i \in \{1, \cdots, r\} : \mu(b_i) \geq 0\}$. If $I = \emptyset$, we have $\sum_{i=1}^{r} [\mu(a_i) - \mu(a_{i-1}]^+ = 0 \leq \mu^+(a)$. If $I \neq \emptyset$, we have $\sum_{i=1}^{r} [\mu(a_i) - \mu(a_{i-1}]^+ = \sum_{i \in I} \mu(b_i) = \mu(\bigoplus_{i \in I} b_i) \leq \mu^+(a)$, since $\bigoplus_{i \in I} b_i \leq \bigoplus_{i=1}^{r} \mu(b_i) = a$. Taking the supremum, we get $\overline{\mu}(a) \leq \mu^+(a)$.

In a similar way we can prove (**).

(ii) By (i) and $[B_2$, X.6.10], we get that μ^+ and μ^- are modular functions, with $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. We have only to prove that μ^+, μ^- are measures. Since μ is a measure and $\mu^- = \mu^+ - \mu$, it is sufficient to prove that μ^+ is a measure.

Let $a, b \in L$ be such that $a \perp b$ and $0 = b_0 \leq b_1 \leq \cdots \leq b_r = a \oplus b$. We have $0 = b_0 \wedge a \leq b_1 \wedge a \leq \cdots \leq b_n \wedge a = (a \oplus b) \wedge a = a$. Moreover, if we set $c_i = (b_i \vee a) \ominus a$ for $i \leq r$, we have $c_0 = (b_0 \vee a) \ominus a = 0 \leq c_1 \leq \cdots \leq c_n = (b_n \vee a) \ominus a = (a \oplus b) \ominus a = b$. Since μ is modular, we have $\mu(b_i) - \mu(b_{i-1}) = \mu(b_i \vee a) - \mu(b_{i-1} \vee a) + \mu(b_i \wedge a) - \mu(b_{i-1} \wedge a) = \mu(c_i) - \mu(c_{i-1}) + \mu(b_i \wedge a) - \mu(b_{i-1} \wedge a)$. Then $[\mu(b_i) - \mu(b_{i-1})]^+ \leq \mu^+(b) + \mu^+(a)$. Taking the supremum, we get $\mu^+(a \oplus b) \leq \mu^+(a) + \mu^+(b)$.

Now let $\varepsilon > 0$ and choose $c \leq a$, $d \leq b$ such that $\mu^+(a) < \mu(c) + \varepsilon$ and $\mu^+(b) < \mu(d) + \varepsilon$. Since $a \perp b$, we have also $c \perp d$. Then $\mu^+(a) + \mu^+(b) < \mu(c \oplus d) + 2\varepsilon \leq \mu(a \oplus b) + 2\varepsilon$, since $c \oplus d \leq a \oplus b$. Since ε is arbitrary, we get $\mu^+(a) + \mu^+(b) \leq \mu^+(a \oplus b)$. □

Recall that g is lower Lipschitzian in x_0 if there exist $\varepsilon, c > 0$ such that $\|x - x_0\| \leq \varepsilon$ implies $[g(x) - g(x_0)]^- \leq c\|x - x_0\|$.

Proposition (3.8). *Suppose that $C(\nu) \neq \emptyset$ and g is lower Lipschitzian in* 0 *and in* $\lambda(1)$. *Then every $\mu \in C(\nu)$ is strongly continuous. Moreover, if λ is σ-additive, then $C(\nu) = C_\sigma(\nu)$.*

Proof. We prove that there exists $c > 0$ such that, for every $\mu \in C(\nu)$ and $b \in L$, $|\mu|(b) \leq \frac{c}{n} \sum_{i=1}^{n} \lambda_i(b)$.

Let $\mu \in C(\nu)$. Since $\mu \geq \nu$ and $\mu(1) = \nu(1)$, for $a \in L$ we have $\nu(a) \leq \mu(a) = \mu(a) + \nu(1) - \mu(1)$, that is

$$g(\lambda(a)) - g(\lambda(1)) \leq -\mu(a^{\perp}) \ \forall \ a \in L. \tag{*}$$

Since g is lower Lipschitzian in $\lambda(1)$, we can find $k, \varepsilon > 0$ such that $\|\lambda(a) - \lambda(1)\| \leq \varepsilon$ implies

$$[g(\lambda(a)) - g(\lambda(1))]^{-} \leq k\|\lambda(a) - \lambda(1)\|. \tag{**}$$

By (*) and (**), we obtain that, for every $a \in L$, $\|\lambda(a^{\perp})\| \leq \varepsilon$ implies $\mu(a^{\perp}) \leq k\|\lambda(a^{\perp})\|$. Then, if $\|\lambda(a)\| \leq \varepsilon$, for every $b \leq a$ we have $\mu(b) \leq k\|\lambda(b)\| \leq k\|\lambda(a)\|$, from which $\mu^{+}(a) \leq k\|\lambda(a)\|$. In a similar way, using that g is lower Lipschitzian in 0, we can find $k', \varepsilon' > 0$ such that, for every $a \in L$, $\|\lambda(a)\| \leq \varepsilon'$ implies $\mu^{-}(a) \leq k'\|\lambda(a)\|$. By (3.7), we obtain $\overline{\varepsilon}, c, c' \geq 0$ such that, for every $a \in L$, $\|\lambda(a)\| \leq \overline{\varepsilon}$ implies $|\mu|(a) \leq c'\|\lambda(a)\| \leq c\lambda^{*}(a)$, where $\lambda^{*}(a) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i(a)$. Since every λ_i is strongly continuous, it is easy to see that λ^{*} is strongly continuous, too. Then, if $a \in L$, we can find an orthogonal family $\{a_1, \cdots, a_r\}$ in L such that $\bigoplus_{i \leq r} a_i = a$ and $\lambda^{*}(a_i) \leq \overline{\varepsilon}$ for each $i \leq r$. Then, for every $a \in L$, we have $|\mu|(a) = \sum_{i=1}^{r} |\mu|(a_i) \leq c\sum_{i=1}^{r} \lambda^{*}(a_i) = c\lambda^{*}(a)$. Hence μ is strongly continuous. Moreover, if λ is σ-additive, recalling [A-B, 2.4], we obtain that μ is σ-additive, too. □

Now, as a consequence of (2.1) and (2.5), we can prove the following result (compare with [M-M, Theorem 21]).

Theorem (3.9). *Suppose that $C(\nu)$ is a non-empty set and there exists a ν-linear and λ-radial element $a_* \in L$. Moreover suppose that one of the following conditions is satisfied:*

(1) *g is lower semicontinuous in $\lambda(a_*)$ and in $\lambda(a_*^{\perp})$.*
(2) *L is σ-complete and σ-continuous, λ is σ-additive and there exists a ν-linear element $a \in L$ such that g is lower semicontinuous in $\lambda(a)$ and in $\lambda(a^{\perp})$.*
(3) *L has the interpolation property and g is lower Lipschitzian in 0 and in $\lambda(1)$.*

Then, if a is any ν-linear element of L, we have $C(\nu) = \{\alpha \cdot \lambda : \alpha \in \partial g(\lambda(a) \cap \partial g(\lambda(a^{\perp}))\}$. The vectors $\alpha \in R^n$ are uniquely determined if and only if $\lambda(L)$ has full dimension. Moreover, if g is monotone on $\lambda(L)$, then we can choose $\alpha \in R^n_+$.

Proof. As in (3.4), we obtain that every $\mu \in C(\nu)$ satisfies the condition (*) of (2.1) and, if g is monotone, also the condition (**) of (2.1) is satisfied. Moreover, if (2) is satisfied, by (3.6) we have $C(\nu) = C_{\sigma}(\nu)$. If (3) is satisfied, then by (3.8) every $\mu \in C(\nu)$ is strongly continuous. Then, in both the cases (2) and (3), the assertion follows from (2.1), recalling (3.1) and (3.3).

Now suppose that (1) is satisfied and observe that, by (3.5), g is continuous in $\lambda(a_*)$ and in $\lambda(a_*^{\perp})$. Let $\mu \in C(\nu)$. We prove that μ and λ satisfy the condition $(*)'$ of (2.5). Let $\{b_n\}$ be a sequence in L such that $\lim_n \lambda(b_n) = \lambda(a_*)$ and $\{\mu(b_n)\}$ converges. By (3.2), we have

$$\lim_n \mu(b_n) \geq \lim_n \nu(b_n) = \lim_n g(\lambda(b_n)) = g(\lambda(a_*)) = \nu(a_*) = \mu(a_*),$$

$$\lim_n \mu(b_n^{\perp}) \geq \lim_n \nu(b_n^{\perp}) = \lim_n g(\lambda(b_n^{\perp})) = g(\lambda(a_*^{\perp})) = \nu(a_*^{\perp}) = \mu(a_*^{\perp}),$$

from which $\lim_n \mu(b_n) = \mu(a_*)$.

In a similar way, we see that, if g is monotone, λ and ν satisfy also the condition $(**)'$ of (2.5). Then the assertion follows from (2.5). □

Remark. Suppose that L is σ-complete, ν is *exact* (i.e. $C(\nu) \neq \emptyset$ and, for every $a \in L$, $\nu(a) = \min\{\mu(a) : \mu \in C_{\sigma}(\nu)\}$), and $C(\nu) \subseteq span\{\lambda_1, \cdots, \lambda_n\}$. Then there exists a ν-linear and λ-radial element in L.

In fact, since by (2.2) $\lambda(L)$ is convex, we can find $a \in L$ such that $\lambda(a) = \frac{1}{2}\lambda(1)$. We see that a is ν-linear. Observe that, if $\mu \in C_\sigma(\nu)$, there exist $\alpha_1, \cdots, \alpha_n$ such that $\mu(a) = \sum_{i=1}^n \alpha_i \lambda_i(a) = \frac{1}{2}\sum_{i=1}^n \alpha_i\lambda_i(1) = \frac{1}{2}\mu(1) = \frac{1}{2}\nu(1)$. Since ν is exact, we get $\nu(a) = \frac{1}{2}\nu(1)$. In a similar way we see that $\nu(a^\perp) = \frac{1}{2}\nu(1)$. Then a is ν-linear.

We conclude with the scalar case.

Corollary (3.10). *Suppose $n = 1$. Suppose that there exists a ν-linear and λ-radial element a_* in L, and one of the conditions of (3.9) is satisfied. Then, if $C(\nu) \neq \emptyset$, we have $C(\nu) = \{\frac{g(\lambda(1))}{\lambda(1)}\lambda\}$. Moreover, if L has the interpolation property, then $C(\nu) \neq \emptyset$ if and only if, for every $x \in [0, \lambda(1)]$, $g(x) \leq \frac{g(\lambda(1))}{\lambda(1)}x$.*

Proof. Let $\mu \in C(\nu)$. By (3.9), we can find a real number α such that, for every $a \in L$, $\mu(a) = \alpha\lambda(a)$. Then $\alpha = \frac{\mu(1)}{\lambda(1)} = \frac{g(\lambda(1))}{\lambda(1)}$, from which (2) holds.

Now suppose that L has the interpolation property and $C(\nu) \neq \emptyset$. By (2.2), if $x \in [0, \lambda(1)]$, we can find $a \in L$ such that $\lambda(a) = x$. Then $g(x) = g(\lambda(a)) = \nu(a) \leq \frac{g(\lambda(1))}{\lambda(1)}\lambda(a) = \frac{g(\lambda(1))}{\lambda(1)}x$.

The converse is trivial since, if we set $\mu(a) = g(\lambda(1))\lambda(1)\lambda(a)$ for $a \in L$, then $\mu \in C(\nu)$. □

4. A representation of σ-core of measure games.

In this section, on the basis of a Radon-Nikodym type theorem for σ-additive modular measures, we give a description of the σ-core of measure games on σ-complete D-lattices.

Recall that an element $c \in L$ is said to be *central* if, for every $a \in L$, $a = (a \wedge c) \vee (a \wedge c^\perp)$. By [D-P, 1.9.14], the set $C(L)$ of all central elements of L is a Boolean algebra.

Let $\mu, m : L \to R$ be modular measures. We denote by $\mathcal{U}(\mu)$ and $\mathcal{U}(m)$ the D-uniformities generated by μ and m (see Section 3). We write $\mathcal{U}(\mu) \leq \mathcal{U}(m)$ whenever $\mathcal{U}(\mu)$ is coarser than $\mathcal{U}(m)$, and $\mathcal{U}(\mu) \wedge \mathcal{U}(m) = 0$ whenever the infimum of $\mathcal{U}(\mu)$ and $\mathcal{U}(m)$ is the trivial uniformity.

We say that $\mu << m$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mu|(a) < \varepsilon$ whenever $a \in L$ and $|m|(a) < \delta$.

We say that $\mu \perp m$ if, for every $\varepsilon > 0$, there exists $a \in L$ such that $|\mu|(a) < \varepsilon$ and $|m|(a^\perp) < \varepsilon$.

By [A-V, 3.3 and 3.4], $\mu << m$ if and only if $\mathcal{U}(\mu) \leq \mathcal{U}(m)$ and $\mu \perp m$ if and only if $\mathcal{U}(\mu) \wedge \mathcal{U}(m) = 0$.

Proposition (4.1). *Let $\mu, m : L \to R$ be σ-additive modular measures and suppose L σ-complete. Then $\mu << m$ if and only if $\mu(a) = 0$ whenever $a \in L$ and $m(b) = 0$ for each $b \leq a$.*

Proof. $\Rightarrow$ is trivial.

$\Leftarrow$ Set $\mathcal{U} = \mathcal{U}(\mu) \vee \mathcal{U}(m)$. Denote by $\hat{L}$ the quotient of L with respect to $\mathcal{U}$, and by $\hat{\mathcal{U}}$ the quotient uniformity generated by $\mathcal{U}$. By [A-B, 4.3], $\hat{L}$ is a D-lattice. By [W_1, 6.3, 7.1.9 and 8.1.4], $(\hat{L},\hat{\mathcal{U}})$ is complete and $\hat{\mathcal{U}}$ is o.c. Set $\hat{\mu}(\hat{a}) = \mu(a)$ and $\hat{m}(\hat{a}) = m(a)$ for $a \in \hat{a} \in \hat{L}$. By [A-B, 4.3], $\hat{\mu}$ and $\hat{m}$ are well-defined modular measures on $\hat{L}$. Denote by $\overline{\mu}$ and $\overline{m}$, respectively, the restrictions of $\hat{\mu}$ and $\hat{m}$ to $C(\hat{L})$. By [W_3, 3.4], $\overline{\mu}$ and $\overline{m}$ are o.c. measures on the complete Boolean algebra $C(\hat{L})$. By [A-V, 4.1], $\hat{\mu}(\hat{b}) = 0$ for every $\hat{b} \in C(\hat{L})$ with $\hat{b} \leq \hat{a}$, implies $\hat{\mu}(\hat{b}) = 0$ for every $\hat{b} \in \hat{L}$ with $\hat{b} \leq \hat{a}$. Then from the assumption we derive that $\overline{\mu} << \overline{m}$. By [A-B-V, 4.1], we obtain $\mu << m$. □

Proposition (4.2). *(Riesz decomposition property). Let $\mu, m, \tau : L \to R$ be bounded modular measures such that $\mu << m + \tau$. Then there exist bounded modular measures $\mu_1, \mu_2 : L \to R$ such that $\mu = \mu_1 + \mu_2$, $\mu_1 << m$ and $\mu_2 << \tau$.*

Proof. By [A-V, 3.6], we can find bounded modular measures $\mu_1, \mu_2 : L \to R$ such that $\mu = \mu_1 + \mu_2$, $\mu_1 << m$, $\mu_2 \perp m$ and $\mathcal{U}(\mu) = \mathcal{U}(\mu_1) \vee \mathcal{U}(\mu_2)$. Then we have

$$(*) \qquad \mathcal{U}(\mu_2) \wedge \mathcal{U}(m) = 0,$$

and

$$(**) \qquad \mathcal{U}(\mu_2) \leq \mathcal{U}(\mu_1) \vee \mathcal{U}(\mu_2) = \mathcal{U}(\mu) \leq \mathcal{U}(m + \tau) \leq \mathcal{U}(m) \vee \mathcal{U}(\tau).$$

Since the exhaustive D-uniformities on L form a Boolean algebra (see [A-V, 2.9]) and the uniformities generated by bounded modular measures are exhaustive by [W_2, 3.8] and [A-B-V, 2.6], by (*) and (**) we derive $\mathcal{U}(\mu_2) \leq \mathcal{U}(\tau)$, i.e. $\mu_2 << \tau$. □

In what follows, if $c \in L$, we denote by m_c the function defined by $m_c(a) = m(c \wedge a)$ for all $a \in L$. In general, m_c is not a modular measure. Nevertheless, if c is central, then m_c is a modular measure by [A-B-V, 2.2].

A function $f : L \to R$ is said to be *a simple m-continuous function* if it belongs to the linear space generated by $\{m_c : c \in L\}$. We denote by $S_m(L)$ the linear space of all m-continuous simple functions on L which are modular measures.

We need the following result, which corresponds to the Radon-Nikodym type theorem in the case of finite additivity ([B-B]).

Lemma (4.3)([A-B-V, 3.2]). *Let $m : L \to [0, +\infty[$ and $\mu : C(L) \to R$ be bounded modular measures such that $\mu << m_{|C(L)}$. Then there exists a sequence $\{m_k\} \subseteq S_m(C(L))$ such that the function μ' defined as $\mu'(a) = \sum_{k=1}^{\infty} m_k(a)$ for $a \in L$ is a modular measure which extends μ and the convergence is uniform with respect to $a \in L$.*

Theorem (4.4). *(Radon-Nikodym theorem). Suppose L is σ-complete and let $m : L \to [0, +\infty[$ and $\mu : L \to R$ be σ-additive modular measures such that $\mu << m$. Then μ belongs to the closure of $S_m(L)$ with respect to the topology of the uniform convergence. Moreover, if L is a clan of fuzzy sets on a set Ω, then there exists a function $h : \Omega \to R$ which is integrable with respect to the restriction $\overline{m}$ of m to the σ-algebra $\mathcal{A}$ of crisp sets in L such that, for every $f \in L$, $\mu(f) = \int fh \, d\overline{m}$.*

Proof. (i) First suppose L complete and m o.c.

Since $\mu << m$, μ is o.c., too. Since $\mu_{|C(L)} << m_{|C(L)}$, by (4.3) we can find a sequence $\{m_k\} \subseteq S_m(C(L))$ such that the function $\mu' : L \to R$ defined as $\mu'(a) = \sum_{k=1}^{\infty} m_k(a)$ for $a \in L$ is a modular measure which extends $\mu_{|C(L)}$, and the convergence is uniform with respect to $a \in L$. Moreover, by [A-B-V, 3.1], every m_k is o.c. and therefore μ' is o.c., too. Then μ and μ' are o.c. modular measures on L which coincide on $C(L)$. By [A-B-V, 2.7], we obtain $\mu = \mu'$ on L.

(ii) Now we consider the general case.

Denote by $\hat{L}$ the quotient of L with respect to $\mathcal{U}(m)$, and set $\hat{\mu}(\hat{a}) = \mu(a)$ and $\hat{m}(\hat{a}) = m(a)$ for $a \in \hat{a} \in \hat{L}$. As in (4.1), we can see that $\hat{L}$ is a complete D-lattice, and $\hat{\mu}, \hat{m}$ are well-defined o.c. modular measures. By (i), we can find a sequence $\{\hat{m}_k\} \subseteq S_{\hat{m}}(C(\hat{L}))$ such that $\hat{\mu}(\hat{a}) = \sum_{k=1}^{\infty} \hat{m}_k(\hat{a})$ for $\hat{a} \in \hat{L}$. Set $m_k(a) = \hat{m}_k(\hat{a})$ for $a \in L$. Then m_k are modular measures, $m_k \in S_m(L)$, and $\mu(a) = \sum_{k=1}^{\infty} m_k(a)$ for $a \in L$.

(iii) Now suppose that L is a clan of fuzzy sets on a set Ω.

Since L is σ-complete, by [B-L-W, 3.1 and 3.2], for each $f \in L$ we have $\mu(f) = \int f \, d\overline{\mu}$, where $\overline{\mu}$ is the restriction of μ to $\mathcal{A}$, and f is the uniform limit of a sequence of $\mathcal{A}$-simple functions. Since $\overline{\mu} << \overline{m} = m_{|\mathcal{A}}$, by the Radon-Nikodym theorem we can find an $\overline{m}$-integrable function $h : \Omega \to R$ such that $\overline{\mu}(A) = \int_A h \, d\overline{m}$ for each $A \in \mathcal{A}$. Hence, for every $f \in L$, we have $\mu(f) = \int fh \, d\overline{m}$. □

In the next result $\nu = g(\lambda)$ is a measure game as in Section 3.

Corollary (4.5). *Suppose L σ-complete and λ σ-additive. Then, for every $\mu \in C_\sigma(\nu)$, there exist n sequences $\{\lambda_k^i\}$ of simple λ_i-continuous modular measures on L such that $\mu = \sum_{i=1}^n \sum_{k=1}^\infty \lambda_k^i$. Moreover, if L is a clan of fuzzy sets on a set Ω, then for every $\mu \in C_\sigma(\nu)$ there exist n functions $h_i : \Omega \to R$ such that, for each $i \le n$, h_i is integrable with respect to the restriction $\overline{\lambda}_i$ of λ_i to the σ-algebra $\mathcal{A}$ of crisp sets in L, and $\nu(f) = \sum_{i=1}^n \int fh_i \, d\overline{\lambda}_i$ for every $f \in L$.*

Proof. Let $\mu \in C_\sigma(\nu)$, and set $\lambda^* = \frac{1}{n} \sum_{i=1}^n \lambda_i$. We prove that $\mu << \lambda^*$, applying (4.1). Indeed, if $\lambda^*(b) = 0$, we have $\lambda(b) = 0$ and then $\lambda(b^\perp) = \lambda(1)$. Since $\mu \ge \nu$ and $\mu(1) = \nu(1)$, we have $\mu(b) \ge g(\lambda(b)) = g(0) = 0$ and $\mu(b^\perp) \ge g(\lambda(b^\perp)) = g(\lambda(1)) = \nu(1) = \mu(1)$, whence $\mu(b) = \mu(1) - \mu(b^\perp) \le 0$. Hence $\mu(b) = 0$. By (4.2), we can find n modular measures $\mu_i : L \to R$ such that $\mu_i << \lambda_i$ for each $i \le n$ and $\mu = \sum_{i=1}^n \mu_i$. Applying (4.4) to each μ_i, we obtain the assertion. □

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