ON GENERALIZED STRONG A-SUMMABILITY

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Abstract. In an unpublished paper A. Gülcü introduced a generalized notion of strong summation termed "strong A-summability in the wide sense". We investigate this notion and clarify some points left unclear by the initial paper of Gülcü. We settle integrability of the A-distribution function under condition of finitely strong A-summability in the wide sense and analyze necessity of conditions to conclude finitely strong A-summability in the wide sense from A-distributional summability. In particular, we prove a sharp direct theorem as well as its corresponding converse theorem to describe connection of these notions. We also clarify connection of finitely strong A-summability in the wide sense and usual A-summability and compute the sum of a sequence from information about its generalized A-strong summation or its A- distribution. The paper ends with comments on the original work of A. Gülcü.

§0. Introduction

Strong summability (of a sequence (x_k) or corresponding partial sums $s_k := x_1 + \cdots + x_k$, to sum s) as a sharpening of Cesáro-1 summability, is a classical notion, meaning $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |s_k - s| = 0$. The notion appears in the work of Hardy and Littlewood already with a more general setting of H_q-strong summability with $q > 0$, see [3] or [7, Vol. II, pp. 180-181]. Its importance grew further after the well-known result of Marcinkiewicz [4] on a.e. pointwise strong summability of Fourier series of integrable functions, see also [7, Vol. II, pp. 184-188].

For a given sequence (s_k) let $F_n(y) := \frac{1}{n}\#\{k \le n \ : \ s_k < y\}$. When $\lim\limits_{n\to\infty} F_n(y) = F(y)$ all over $(-\infty, \infty)$ least an exceptional set of at most countable many points, it is said that (s_k) is summable in distribution to $F(y)$ as limit distribution.

Strong summation was also considered on lacunary subsequences $(n_j) \subset \mathbb{N}$ [1], with the averaging over $k = 1, \ldots, n$ replaced by averaging over $k = n_j + 1, \ldots, n_{j+1}$ with $h_j := n_{j+1} - n_j \to \infty$.

In 1983 Yoneda [6] gave a further generalization of the notion of strong or H_1-strong summability, called "strong summability in the wide sense to sum S". In the extension the role of the classical sum s is taken over by a certain function $S(y)$, considered the "generalized strong sum" of the sequence.

Definition 1 (Yoneda). *For a sequence (x_k) and corresponding (s_k) put $S_n(y) := \frac{1}{n}\sum_{k=1}^{n} |s_k - y|$. We say that (x_k) (or, equivalently, (s_k)) is strongly summable in the wide sense to $S(y)$, if for all $-\infty < y < \infty$ we have $\lim_{n\to\infty} S_n(y) = S(y)$.*

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Example 1. The sequence $x_k := (-1)^k 2$ has $s_k = -1+(-1)^k$, which oscillates, hence is *not* strongly summable, but is strongly summable in the wide sense to $S(y) = \max\{1, |y-1|\}$.

The above extended definition can be considered for subsequences $(n_j) \subset \mathbb{N}$, too. In fact, Nuray and Savas deals with sequences strongly summable in the wide sense to $S(y)$ along some subsequence (n_j), see [5].

Let $A = (a_{nk})_{k=1,\infty}^{n=1,\infty}$ be a Toeplitz matrix. Letting $\sum_k$ stand for $\sum_{k=1}^{\infty}$, it is said that (s_k) is A-summable to s if $\lim_{n\to\infty} \sum_k a_{nk} s_k = s$, and that it is strongly A-summable to s if $\lim_{n\to\infty} \sum_k a_{nk} |s_k - s| = 0$.

Note that treating the sequence of the partial sums (s_k) in place of the original sequence (x_k) requires reshaping original Toeplitz matrices $T = (t_{kn})_{k=1,\infty}^{n=1,\infty}$, written for (x_k), by defining new entries $a_{nk} := t_{n,k} - t_{n,(k+1)}$ of a corresponding matrix A. Following [7, Chapter III, pp. 74-75], we deal exclusively with Toeplitz matrices A corresponding partial sums (s_k) as above throughout the paper.

For regular Toeplitz matrices $\lim_{n\to\infty} \sum_{j=k}^{\infty} a_{nj} = 1$ (for all fixed $j, n \in \mathbb{N}$) and also uniform boundedness of $\sum_k |a_{nk}|$ are required. Regularity of A means those properties throughout this paper.

Finally, we always assume that the Toeplitz matrix A is positive, meaning $a_{nk} \geq 0$, $(k, n \in \mathbb{N})$, that is, all s_ks are considered with nonnegative weights only.

Denoting the indicator function of positive reals by χ, i.e. $\chi(x) = 1$ for $x > 0$ and $\chi(x) = 0$ for $x \leq 0$, one has the corresponding notion of convergence in distribution. Let a positive, regular Toeplitz matrix A and a sequence (s_k) be given. Put

$$R_n(y) := \sum_k a_{nk} \chi(y - s_k). \tag{1}$$

When $\lim_{n\to\infty} R_n(y) = R(y)$ all over $(-\infty, \infty)$ least an exceptional set of at most countable many points, (s_k) is said to be A-summable in distribution with $R(y)$ as its A-limit distribution.

If (s_k) is summable to s, then it is A-summable in distribution with $R(y) = \chi(y - s)$ for any positive and regular Toeplitz summation matrix A.

In 1998 the following notion was introduced in [2], as a common generalization of strong summation in the wide sense, defined in [6], and strong summation in the wide sense along some subsequence, considered in [5].

Definition 2 (Gülcü). *For any given positive Toeplitz matrix A and sequence (s_k) let*

$$T_n(y) := \sum_k a_{nk} |s_k - y| \ . \tag{2}$$

When $\lim_{n\to\infty} T_n(y) =: T(y)$ exists in $[0, \infty]$ for all y from $(-\infty, \infty)$, we say that (s_k) is strongly A-summable in the wide sense, or, for short, w.s.s. A-summable, to $T(y)$. If $T(y) < \infty$ $(\forall y \in \mathbb{R})$, we say that (s_k) is finitely strongly A-summable in the wide sense (or f.w.s.s. A-summable) to $T(y)$.

It is easy to see that if a sequence (x_k) is summable to s, $(s_k \to s$ as $k \to \infty)$, then for any positive, regular Toeplitz matrix A it is strongly A-summable in the wide sense to $T(y) = |y - s|$.

In this paper we investigate strong A-summability in the wide sense. We recover the content of [2] and present a couple of new results, sharpening and extending results of Gülcü. In particular, we correct some errors and show why a normalized definition of A-summability in distribution is more natural to use in the context. The paper ends with comments on comparison with [2].

§1. Preliminaries

Lemma 1 (Gülcü). *Assume that (s_k) is w.s.s. A-summable to $T(y)$ for the positive, regular Toeplitz matrix A. Then we have*

i) $|T(y) - T(z)| \le |y - z|$ *for all* $-\infty < y, z < \infty$,

ii) T *is a nonnegative and convex function,*

iii) $T(y) \ge \limsup_{n\to\infty} \left| \sum_k a_{nk} s_k - y \right|$,

iv) $T(y) \ge \max \left\{ \limsup_{n\to\infty} \sum_k a_{nk} s_k - y,\ y - \liminf_{n\to\infty} \sum_k a_{nk} s_k \right\}$,

v) If $T(y_0) = 0$ *for some* y_0*, then both* (s_k) *and* $(|s_k|)$ *are A-summable to* $s = y_0$ *and* $|y_0|$*, respectively.*

Proof. (i) follows from the triangle inequality and regularity ($\limsup_{n\to\infty} \sum_{j=k}^{\infty} a_{nj} = 1$).

(ii) is obvious for $|y - s_k|$, hence for positive a_{nk} even for (2). Taking limits keeps these properties, hence the assertion.

(iii) follows from $T_n(y) = \sum_k a_{nk}|s_k - y| \ge |\sum_k a_{nk}(s_k - y)| \ge |\sum_k a_{nk}s_k - y| - |\sum_k a_{nk} - 1|\,|y|$ and regularity of A.

(iv) is immediate from (iii) as $|a - b| = \max\{a - b, b - a\}$.

(v) Since $||s_k| - |y_0|| \le |s_k - y_0|$, it is immediate that $T(y_0) = 0$ implies $T^*(|y_0|) = 0$, where T^* is the corresponding A-strong sum function in the wide sense for the sequence $(|s_k|)$. Hence it suffices to deal with the case of (s_k). Furthermore, (iii) with y_0 immediately yields $\lim_{n\to\infty} \sum_k a_{nk}s_k = y_0$. □

Proposition 1. *Let (s_k) be any sequence and A be any positive, regular Toeplitz matrix. Then the following are equivalent.*

i) (s_k) *is strongly A-summable to sum s.*

ii) (s_k) *is finitely w.s.s. A-summable to* $T(y) := |y - s|$.

iii) (s_k) *is finitely w.s.s. A-summable to* $T(y)$ *and* $T(s) = 0$.

Proof. (i) ⇒ (ii). We have

$$\left| T_n(y) - \sum_k a_{nk}|y - s| \right| \le \sum_k a_{nk}\, ||y - s| - |y - s_k|| \le \sum_k a_{nk}|s - s_k|\ ,$$

which tends to zero by assumption. On the other hand regularity gives $\left(\sum_k a_{nk}\right)|y - s| \to |y - s|$, hence the assertion.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i). Clearly we have $T_n(s) := \sum_k a_{nk}|s - s_k| \to T(s) = 0$, whence (i). □

Lemma 2. *Let r be a nondecreasing function with* $r(-\infty) = 0$ *and* $r(\infty) = b < \infty$*. Then we have*

$$t(y) := \int_{-\infty}^{\infty} |u|dr(u + y) = \int_{y}^{\infty} (b - r(u))\,du + \int_{-\infty}^{y} r(u)du\ . \tag{3}$$

Moreover, if $t(y)$ is finite, then we have $\lim_{v\to-\infty} vr(v) = 0$ *and* $\lim_{w\to\infty} w(b - r(w)) = 0$.

Proof. When both forms of $t(y)$ in (3) are $+\infty$, then there is nothing to prove. Hence we can assume that at least one of these expressions is finite. In both forms of $t(y)$, both the measures and the integrands involved are nonnegative, hence they exist in the extended sense - both as Lebesgue integrals and as improper Riemann integrals - being either $+\infty$, or $< \infty$.

Take now any $v < \min\{0, y\} \leq \max\{0, y\} < w$. Partial integration yields

$$\int_y^w (b - r(u))\, du + \int_v^y r(u)du = (b - r(w))(w - y) + r(v)(y - v) + \int_v^w |u - y| dr(u) \ .$$

As $r(\infty) = b$ and $r(-\infty) = 0$, to conclude the proof it suffices to show $\lim_{v\to-\infty} vr(v) = 0$ and $\lim_{w\to\infty} w(b - r(w)) = 0$, whenever *any of the two forms* of $t(y)$ in (3) happens to be $< \infty$. If the right hand side of (3) is finite, then for large enough $z = z(\epsilon)$ and $w > z$, $-v > z$ we necessarily have $(w-z)(b-r(w)) \leq \int_z^\infty (b-r) < \epsilon$ and $(-v-z)r(v) \leq \int_{-\infty}^{-z} r < \epsilon$. Therefore in view of $r(\infty) = b$ and $r(-\infty) = 0$ it follows that $\limsup_{v\to-\infty} vr(v) \leq \epsilon$ and $\limsup_{w\to\infty} w(b - r(w)) \leq \epsilon$, hence the assertion. Similarly, if the first expression is finite, then $(w - y)(b - r(w)) = (w - y) \int_{w-y}^\infty dr(u + y) \leq \int_{w-y}^\infty |u|dr(u + y) < \epsilon$ for large enough w, giving $\limsup_{w\to\infty} w(b - r(w)) \leq \epsilon$, (and similarly for $v \to -\infty$), whence the assertion obtains. □

Lemma 3 (Gülcü). *For any sequence* (s_k) *and any* $n \in \mathbb{N}$ *we have*

$$T_n(y) = \int_{-\infty}^{\infty} |u|dR_n(u + y) = \int_y^{\infty} \left(\sum_k a_{nk} - R_n(u) \right) du + \int_{-\infty}^{y} R_n(u)du \ . \tag{4}$$

Furthermore, if $T_n(0)$ *is finite, then we have*

$$\sum_k a_{nk}s_k = \int_{-\infty}^{\infty} udR_n(u) = \int_0^{\infty} \left(\sum_k a_{nk} - R_n(u) \right) du - \int_{-\infty}^{0} R_n(u)du \ . \tag{5}$$

Proof. Note that $d\chi = \delta_0$ (the Dirac delta measure at 0), hence positivity of A entails that all measures $a_{nk}d\chi(\cdot - s_k) = a_{nk}\delta_{s_k}$ are nonnegative. It is easy to see that $|x - y| = \int_{-\infty}^\infty |u - y|d\chi(u - x)$ for any $-\infty < x, y < \infty$. Applying it in (2), positivity justifies interchanging summation and integration, leading to

$$T_n(y) = \sum_k a_{nk} \int_{-\infty}^{\infty} |u - y|d\chi(u - s_k) = \int_{-\infty}^{\infty} |u - y|dR_n(u) \ .$$

This gives the first part of (4), while a reference to Lemma 2 supplies the second part, too.

Regarding (5) note that *finite* existence of $T_n(0)$ is just absolute convergence of $\sum_k a_{nk}s_k$. Hence also the left hand side of (5) is absolute convergent and we obtain

$$\sum_k a_{nk}s_k = \sum_k a_{nk} \int_{-\infty}^{\infty} ud\chi(u - s_k) = \int_{-\infty}^{\infty} udR_n(u) \ , \tag{6}$$

interchanging the integral and the sum being permitted by absolute convergence of the integral in view of (4) (which has already been proved). Since $T_n(0) < \infty$, the last part of Lemma 2 yields $\lim\limits_{w\to\infty} w\left(\sum_k a_{nk} - R_n(w)\right) = 0$ and $\lim\limits_{v\to-\infty} vR_n(v) = 0$. Splitting the interval of integration in (6) at 0 and referring to these limit relations, partial integration yields the last part of (4). □

Lemma 4. *Let A be a positive and regular Toeplitz matrix and assume that the sequence (s_k) is A-summable in distribution with the distribution function R. Then if for some n we have $(R_n - R) \in L^1(0,\infty)$, then we also have $\sum_k a_{nk} = 1$.*

Proof. For the monotone functions R and R_n we have $\lim_\infty R_n = \sum_k a_{nk}$ and $\lim_\infty R = 1$. If these limits at ∞ are not equal, then with $\epsilon < |\sum_k a_{nk} - 1|/3$, and for u large enough, $|R(u) - 1| < \epsilon$ and also $|R_n(u) - \sum_k a_{nk}| < \epsilon$ ensures $|R(u) - R_n(u)| > \epsilon$. That, however, implies $\int_0^\infty |R - R_n| = \infty$. □

§2. Connections between w.s.s. A-summability and in A-distribution

Theorem 1 (Gülcü). *If a sequence (s_k) is finitely w.s.s. A-summable to $T(y)$, then it is also A-summable in distribution with*

$$R(y) = \frac{1}{2}\left\{1 + \frac{d}{dy}T(y)\right\} \quad \text{a.e. .} \tag{7}$$

Proof. From Lemma 3, second part of (4) we easily infer for any $-\infty < x < y < \infty$

$$T_n(x) - T_n(y) = (y - x)\sum_k a_{nk} - 2\int_x^y R_n \ . \tag{8}$$

Observe that χ, hence for positive A also R_n, are nonnegative and non-decreasing. By regularity, $\sum_k a_{nk}$ is bounded, hence (R_n) form a uniformly bounded sequence of non-decreasing functions. Note that also the total variation of each R_n equals to $\sum_k a_{nk}$ (it's sup norm) by monotonicity. Thus Helley's second theorem applies providing at least a subsequence (n_k) and a function R so that $\lim_{k\to\infty} R_{n_k}(u) = R(u)$ everywhere except for a countable set. It is then obvious that R is a non-decreasing function of y.

By Lebesgue's dominated convergence theorem we have

$$\lim_{k\to\infty}\int_x^y R_{n_k}(u)du = \int_x^y R(u)du \tag{9}$$

for all $-\infty < x < y < \infty$. Therefore we are led to

$$T(x) - T(y) = (y - x) - 2\int_x^y R(u)du \ . \tag{10}$$

Now differentiation yields (7) for all points of $\mathbb{R} \setminus E$ where E is the (countable) set of points of jumps of the monotone function R.

Note that (10) or (7) immediately implies that R is uniquely determined a.e., hence not only a subsequence, but also the full sequence (R_n) converges a.e. to R. This concludes the proof. □

Theorem 2. *If a sequence (s_k) is A-summable in distribution, and, moreover, $T_n(z) \to T(z) \in \mathbb{R}$ for a certain point $z \in \mathbb{R}$, then (s_k) is also finitely w.s.s. A-summable to*

$$T(y) = (z - y) + T(z) + 2\int_z^y R(u)du \ . \tag{11}$$

Proof. Lemma 3, as in the proof of Theorem 1, yields (8). As $n \to \infty$, on the right hand side the sum tends to 1 by regularity of A, and the integral converges to $\int_x^y R$ by Lebesgue's dominated convergence theorem. So in view of $T_n(z) \to T(z)$, also $T_n(y)$ must converge as all other terms of (8) do so. Hence we are led to (11). □

Theorem 3. *Let the sequence* (s_k) *be A-summable in distribution with* $R(u)$ *(i.e.* $R_n(u) \to R(u)$ *a.e. as* $n \to \infty$*). Moreover, assume the following further conditions:*

i) $R_n - R \to 0$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$,

ii) $1 - R \in L^1(0,\infty)$ *and* $R \in L^1(-\infty, 0)$.

Then (s_k) *is also finitely w.s.s. A-summable in the wide sense to*

$$T(y) := \int_y^\infty (1 - R(u))\,du + \int_{-\infty}^y R(u)du \ . \tag{12}$$

Proof. By Condition (i), $\lim_{n\to\infty} \int_0^\infty (1-R_n) = \int_0^\infty (1-R)$ and $\lim_{n\to\infty} \int_{-\infty}^0 R_n = \int_{-\infty}^0 R$, whether these latter integrals are finite or infinite. Note that we do not - and cannot, in general - state finiteness of them unless referring to Condition (ii), which provides exactly this finiteness.

Condition (i) ensures $(R_n - R) \in L^1(0,\infty)$, hence by Lemma 4 $\sum_k a_{nk} = 1$. Thus applying (4) of Lemma 3 for an arbitrary $y \in \mathbb{R}$ we find

$$T_n(y) = \int_{-\infty}^\infty |u| dR_n(u+y) = \int_y^\infty (1 - R_n(u))\,du + \int_{-\infty}^y R_n(u)du \ .$$

Again, in general these can be finite or infinite as well, but at present Conditions (i) and (ii) imply finiteness for n large. Now letting $n \to \infty$ we obtain by Condition (i) the convergence of $T_n(y)$ to the (now finite) integrals on the right hand side of (12). □

There is a variant of Theorem 3, which hold for the general case, whether (12) is finite or infinite. However, we emphasize that *finite* w.s.s. A-summability is not stated now.

Theorem 4 (Gülcü). *Let the sequence* (s_k) *be A-summable in distribution with* $R(u)$*, and assume* $R_n - R \to 0$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$*. Then* (s_k) *is also w.s.s. A-summable to* $T(y)$ *in* (12).

Let us note that Condition (ii) of Theorem 3 can be deduced easily (instead of assuming it), if we make either the restriction that (s_k) be bounded, or in case A is a *triangular* Toeplitz matrix (meaning $a_{nk} = 0$ for $k > n$). In the latter case $R_n(u)$ is constant $\sum_k a_{nk}$, i.e. with the notation (13) below, $\widetilde{R}_n(u) = 1$ for $u > \max_{k\le n} s_k$, which together with Condition (i) implies $(1 - R) \in L^1(0,\infty)$; $R \in L^1(-\infty, 0)$ follows similarly. We thus also have

Corollary 1. *Asume that A is a regular, positive triangular matrix. Let the sequence* (s_k) *be A-summable in distribution with* $R(u)$*, and assume* $R_n - R \to 0$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$*. Then* (s_k) *is also f.w.s.s. A-summable to* $T(y)$ *in* (12).

In particular, for considerations of Cesaro-1 summation by Yoneda the triangular condition, hence the above Corollary, is in effect.

§3. Tightness of T and L^1 convergence of normalized A-distribution

In connection to Theorems 3 and 4, the natural question of necessity of all conditions arises. Next we exhibit by example that Condition (ii) can not be dropped from Theorem 3.

Then in Theorem 5 we will prove that Condition (ii) is satisfied whenever a sequence (s_k) is finitely w.s.s. A-summable. These two results show independence and necessity of Condition (ii). We also analyze Condition (i), showing how to get around of this seemingly artificially strong restriction by a suitably modified approach to A-summability in distribution. In Theorem 5 it will also be seen that *normalized* A-summability distribution functions do necessarily satisfy Condition (i), too.

In what follows we denote $\mathcal{P} = (p_n)$ the (increasing) sequence of primes, and $\mathcal{Q}$ the set of all prime powers. Clearly the subsequences $\mathcal{Q}^{(n)} := (p_n^j)_{j=1}^{\infty}$ form a disjoint partition of $\mathcal{Q}$.

Example 2. Let A be defined by $a_{nk} := 0$ for $k \notin \mathcal{Q}^{(n)}$ and $a_{nk} := 1/(j(j+1))$ if $k = p_n^j \in \mathcal{Q}^{(n)}$. Let the sequence (s_k) be defined by $s_k := 0$ for $k \notin \mathcal{Q}$, and $s_k = j$ if $k = p^j$ with $p = p_n \in \mathcal{P}$. We have

i) A is a normalized, positive, regular Toeplitz matrix,

ii) $R_n(u) = R(u) = 0 \quad (\forall u \leq 0)$ and $R_n(u) = R(u) = 1 - 1/\lceil u \rceil \quad (\forall u > 0)$, hence also $R_n \to R$ a.e. and Condition (i) of Theorem 3, but not Condition (ii) of the same Theorem, are met,

iii) (s_k) is w.s.s. A-summable to $T(y) \equiv \infty$.

Proof. (i). Clearly A is positive, and for any fixed n we have $\sum_k a_{nk} = 1$. Moreover, for m fixed and for any $n > m$ (hence $p_n > m$), $a_{nk} = 0$ for all $k \leq m$, the first nonzero weight occurring at $k = p_n$ in the n^{th} row sequence of A. Thus for any $m \in \mathbb{N}$ we get $\sum_{k=m}^{\infty} a_{nk} = 1$ whenever $n > m$, proving regularity of A.

(ii). It is obvious that all $R_n(u) = 0$ for $u \leq 0$, as all s_k are at least 0. On the other hand for positive u we find $R_n(u) = \sum_k a_{nk}\chi(u - s_k) = \sum_j a_{n,p_n^j}\chi(u - j) = \sum_{j<u} \frac{1}{j(j+1)} = 1 - 1/\lceil u \rceil$.

(iii) By Lemma 3, (4) we have $T_n(y) \geq \int_y^{\infty}(1 - R_n) = \int_{y^+}^{\infty} 1/\lceil u \rceil du = \infty$. □

Remark 1. With a suitable modification we can easily make e.g. $T_n(0)$ oscillating between any prescribed (nonnegative) finite or infinite values. See Example 4 for a similar argument.

We have seen that $(R - R_n) \to 0$ in $L^1(\mathbb{R})$ is useful to prove f.w.s.s. A-summability. However, Lemma 4 clarified that this restricts generality considerably, as then we must have $\sum_k a_{nk} = 1$ for all but a finite number of values of n. There is no reason to restrict the definition of f.w.s.s. A-summability to these normalized matrices only. Hence we look for a modified approach to handle even those cases when A is only regular, but not necessarily normalized. It is then natural to consider the following definition.

$$\widetilde{R}_n := \frac{R_n}{\sum_k a_{nk}} = \frac{1}{\sum_k a_{nk}} \sum_k a_{nk}\chi(\cdot - s_k). \tag{13}$$

Clearly this is equivalent to considering $\widetilde{A}$ in place of A with rows normalized by $\widetilde{a}_{nm} := a_{nm}/\sum_k a_{nk}$ to have sum of weights exactly 1. It is easy to see that for any continuity point y of $\widetilde{R}$, hence also almost everywhere, $\widetilde{R}_n(y) \to R(y)$ exactly when $R_n(y) \to \widetilde{R}(y)$ for $n \to \infty$. That is, the sequence (s_k) is A-summable in distribution with R if and only if it is $\widetilde{A}$-summable in distribution with R.

Also, in view of Lemma 3, (4), we can write

$$\widetilde{T}_n(y) := \frac{T_n(y)}{\sum\limits_k a_{nk}} = \int\limits_{-\infty}^{\infty} |u| d\widetilde{R}_n(u + y) = \int\limits_{y}^{\infty} (1 - \widetilde{R}_n(u))du + \int\limits_{-\infty}^{y} \widetilde{R}_n(u)du\ . \tag{14}$$

Thus regularity of A and strong A-summability in the wide sense to $T(y)$ implies $\widetilde{T}_n(y) \to T(y) \quad (n \to \infty)$, too. In other words, (s_k) is (finitely) w.s.s. $\widetilde{A}$-summable to $T(y)$ if and only if it is (finitely) w.s.s. A-summable to $T(y)$.

Finally, in order to formulate our next result we need to introduce a further notion. Take the function T (when it exists, i.e. when (s_k) is f.w.s.s. A-summable to it), and define its "tightness function" as

$$\Phi(x) := \frac{T(x) + T(-x) - 2x}{2} \ . \tag{15}$$

Lemma 5. *If (s_k) is f.w.s.s. A-summable then* (15) *is a nonnegative, non-increasing function. In particular, the limit*

$$\tau := \tau(A, (s_k)) := \Phi(\infty) := \lim_{x \to \infty} \Phi(x) \tag{16}$$

exists, and $0 \le \tau \le T(0)$.

Proof. In view of Lemma 1 (i) $\Phi : [0, \infty) \to \mathbb{R}$ is a Lipshitz-1 (hence absolutely continuous) function with $\Phi(0) = T(0)$ and, in regard of Theorem 1, (7), for a.a. x we have $\Phi'(x) = (R(x) - 1) - R(-x) \le 0$. Hence Φ is non-increasing, and $\Phi(\infty) \le \Phi(x) \le \Phi(0) = T(0)$.

On the other hand for any fixed value of x, f.w.s.s. A-summability entails $\Phi(x) = \lim_{n\to\infty} \widetilde{\Phi}_n(x)$ with the obvious meaning $\widetilde{\Phi}_n(x) := (\widetilde{T}_n(x) + \widetilde{T}_n(-x) - 2x)/2$. It follows from (14) that

$$\widetilde{\Phi}_n(x) = \int_x^{\infty} (1 - \widetilde{R}_n(u))du + \int_{-\infty}^{-x} \widetilde{R}_n(u)du \ , \tag{17}$$

which represents a nonnegative, non-increasing function of x. By *finite* w.s.s. A-summability, $\widetilde{T}_n(y)$ is finite (at least for $n \ge n_0$) and we have $\widetilde{R}_n \in L^1(-\infty, 0)$ and $(1 - \widetilde{R}_n) \in L^1(0, \infty)$ by Lemma 3, (4) and (14). So let us note on passing that these tail integrals tend to zero when $x \to \infty$, giving $\widetilde{\Phi}_n(\infty) = 0$. Also, from $\widetilde{\Phi}_n(x) \ge 0$ we conclude $\Phi(x) \ge 0$, as needed. Now the statements regarding τ follow. □

It may seem peculiar that, although $\widetilde{\Phi}_n(\infty) = 0$, we can not conclude $\tau = 0$. We'll see later in Example 3 that this is essential. But first let us see what can be said in this generality.

Theorem 5. *Assume that with a positive and regular Toeplitz matrix A, the sequence (s_k) is finitely w.s.s. A-summable to $T(y)$. Then the distribution function R defined by* (7) *satisfies $(1 - R) \in L^1(0, \infty)$ and $R \in L^1(-\infty, 0)$. Moreover, we have*

$$T(y) = \tau + \int_y^{\infty} (1 - R(u))\, du + \int_{-\infty}^{y} R(u)du = \tau + \int_{-\infty}^{\infty} |u| dR(u + y) \ , \tag{18}$$

with τ defined in (16).

Proof. By Theorem 1, (s_k) is A-summable in distribution with (7), and in view of Theorem 2, (11) holds for the functions T and R. On combining (11) for $z = L$ and also $z = -L$, where $L > 0$ is a (large) parameter, a calculation gives

$$T(y) = \Phi(L) + \int_y^L (1 - R) + \int_{-L}^y R \ .$$

Since the functions under the integral signs are nonnegative, the limits of these integrals exist - at least in the extended sense in $[0,\infty]$ - when $L \to \infty$. Taking into account Lemma 5, $L \to \infty$ yields the first equality of (18). As $T(y)$ is finite, it follows that $(1-R) \in L^1(0,\infty)$ and $R \in L^1(-\infty, 0)$.

Since A is positive and regular, R is non-decreasing with $R(-\infty) = 0$ and $R(\infty) = 1$. In view of Lemma 2 we can thus infer equality of the two expressions given in (18) for $T(y)$. $\square$

Theorem 6. *Assume that with a positive and regular Toeplitz matrix A, the sequence (s_k) is finitely w.s.s. A-summable to T. Let the function R be defined by* (7). *Denote $\widetilde{R}_n$ and $\widetilde{T}_n$ according to* (13) *and* (14), *respectively. Then the following are equivalent.*

i) When $n \to \infty$, we have $\widetilde{T}_n \to T$ uniformly on $\mathbb{R}$;

ii) T is tight, i.e. with τ defined in (16), *we have $\tau = 0$;*

iii) For the f.w.s.s. A-sum function T we have the integral formula

$$T(y) = \int_{-\infty}^{\infty} |u| dR(u+y) = \int_y^{\infty} (1 - R(u))\, du + \int_{-\infty}^{y} R(u) du \ ; \tag{19}$$

iv) When $n \to \infty$, $(\widetilde{R}_n - R) \to 0$ in $L^1(\mathbb{R})$ norm.

Proof. By Theorem 1, (s_k) is A-summable in distribution with (7), i.e. also $\widetilde{R}_n \to R$ a.e. $(n \to \infty)$, and in view of Theorem 5, $(1-R) \in L^1(0,\infty)$ and $R \in L^1(-\infty,0)$.

(ii)⇔(iii) by Theorem 5, hence it suffices to show implications (i)⇒(ii)⇒(iv)⇒(i).

(i)⇒(ii). With the notations in the proof of Lemma 5, we find also $\widetilde{\Phi}_n \to \Phi$ *uniformly* with $n \to \infty$, while we have already seen that $\widetilde{\Phi}_n(\infty) = 0$. Whence $\tau = \Phi(\infty) = 0$.

(ii)⇒(iv). Let us define for any fixed value $y \in \mathbb{R}$ the functions

$$f_n(u) := \chi(u-y) + \operatorname{sign}(y-u)\widetilde{R}_n(u) \quad \text{and} \quad f(u) := \chi(u-y) + \operatorname{sign}(y-u)R(u) \ .$$

Then we have $0 \le f_n, f \le 1$, $\widetilde{T}_n(y) = \int_{\mathbb{R}} f_n \to T(y)$, and by A-summability in distribution also $f_n \to f$ a.e. on $\mathbb{R}$. Thus an application of Fatou's theorem yields $\int_{\mathbb{R}} f \le T(y) < \infty$, hence $f \in L^1(\mathbb{R})$. (Taking $y = 0$ this immediately gives $R \in L^1(-\infty,0)$ and $(1-R) \in L^1(0,\infty)$.)

So let us prove $\lim_{n\to\infty} \|\widetilde{R}_n - R\|_{L^1(\mathbb{R})} = 0$. Take any $\epsilon > 0$. As $R \in L^1(-\infty,0)$ and $(1-R) \in L^1(0,\infty)$, for a large $L(\epsilon)$, chosen suitably, and for any $L \ge L(\epsilon)$ we have $\int_L^{\infty}(1-R) + \int_{-\infty}^{-L} R < \epsilon$. From this and the triangle inequality we get

$$\begin{aligned}
\int_{-\infty}^{\infty} |R - \widetilde{R}_n| &\le \int_{-\infty}^{-L} (\widetilde{R}_n + R) + \int_L^{\infty} ((1-\widetilde{R}_n) + (1-R)) + \int_{-L}^{L} |R - \widetilde{R}_n| \\
&\le \int_{-\infty}^{-L} \widetilde{R}_n + \int_L^{\infty} (1-\widetilde{R}_n) + \epsilon + \int_{-L}^{L} |R - \widetilde{R}_n| \\
&= \widetilde{\Phi}_n(L) + \epsilon + \int_{-L}^{L} |R - \widetilde{R}_n| \ ,
\end{aligned} \tag{20}$$

using (17) for $x = L$ in the last line.

Let now $n \to \infty$. By Lebegue's dominated convergence theorem the integral on the finite interval $[-L, L]$ converges to 0, and by condition of f.w.s.s. A-summability $\widetilde{T}_n(\pm L) \to T(\pm L)$, i.e. also $\widetilde{\Phi}_n(L) \to \Phi(L)$, hence

$$\limsup_{n\to\infty} \int_{-\infty}^{\infty} |R - \widetilde{R}_n| \le \Phi(L) + \epsilon \ . \tag{21}$$

This holding true for all $\epsilon > 0$ and all $L \ge L(\epsilon)$, first taking $L \to \infty$ (and using the condition $\tau = 0$) and then letting $\epsilon \to 0$ we infer $\limsup_{n\to\infty} \int_{-\infty}^{\infty} |R - \widetilde{R}_n| = 0$, whence (iv).

(iv)⇒(i). Consider the functions $f_n(u)$ and $f(u)$ as above, and put $\epsilon_n := \int_{\mathbb{R}} |f - f_n| = \int_{\mathbb{R}} |\widetilde{R}_n - R|$. Since $\widetilde{T}_n(y) = \int_{\mathbb{R}} f_n$, from condition (iv) we find $|\widetilde{T}_n(y) - \int_{\mathbb{R}} f| \le \epsilon_n \to 0$ as $n \to \infty$, uniformly for all $y \in \mathbb{R}$. However, $\widetilde{T}_n(y) \to T(y)$ in the pointwise sense, hence also the uniform limit is $T(y)$, and (i) obtains. □

§4. A-summation of f.w.s.s. A-summable sequences

Theorem 7. *Let A be a positive and regular Toeplitz matrix, and (s_k) be a sequence finitely w.s.s. A-summable to $T(y)$. Write*

$$S := \int_{-\infty}^{\infty} u dR(u) = \int_{0}^{\infty} (1 - R(u))\, du - \int_{-\infty}^{0} R(u) du \ , \tag{22}$$

where $R(u)$ is defined by (7) *and the integrals converge absolutely by Theorem 5. Then we have*

$$S - \tau \le \liminf_{n\to\infty} \sum_k a_{nk} s_k \le \limsup_{n\to\infty} \sum_k a_{nk} s_k \le S + \tau \ , \tag{23}$$

with τ defined in (16).

Proof. Since (s_k) is f.w.s.s. A-summable to $T(y)$, Theorem 5 entails that the integrals in (22) are absolutely convergent. Since $T_n(0) \to T(0)$, by regularity of A also $\widetilde{T}_n(0) \to T(0)$ when $n \to \infty$. Also, the sum $\sum_k a_{nk} s_k$ is absolutely convergent since $T_n(0)$ is finite (at least for $n \ge n_0$). Moreover, analogously to the normalization in (14) we can write

$$\widetilde{S}_n := \frac{S_n}{\sum_k a_{nk}} := \frac{\sum_k a_{nk} s_k}{\sum_k a_{nk}} = \int_{-\infty}^{\infty} u d\widetilde{R}_n(u) = \int_{0}^{\infty} (1 - \widetilde{R}_n(u)) du - \int_{-\infty}^{0} \widetilde{R}_n(u) du \ , \tag{24}$$

taking into account (5) of Lemma 3. Clearly we have

$$|S_n - S| \le |S_n - \widetilde{S}_n| + |\widetilde{S}_n - S| \le \left| 1 - \frac{1}{\sum_k a_{nk}} \right| T_n(0) + |\widetilde{S}_n - S| \ ,$$

where the first expression is a product of a 0 sequence and a convergent one. Thus

$$\limsup_{n\to\infty} |S_n - S| \le \limsup_{n\to\infty} |\widetilde{S}_n - S| \le \limsup_{n\to\infty} \int_{-\infty}^{\infty} |R - \widetilde{R}_n| \ . \tag{25}$$

So the proof hinges upon estimating the deviation of R and $\widetilde{R}_n$ in $L^1(\mathbb{R})$ norm as $n \to \infty$. This was already accomplished in the proof of Theorem 6, in the part (ii)⇒(iv). Taking up the argument at (21), and taking the limits first in $L \to \infty$ and then in $\epsilon \to 0$ the same way, we now obtain by (16)

$$\limsup_{n\to\infty} \int_{-\infty}^{\infty} |R - \widetilde{R}_n| \le \tau \ . \tag{26}$$

Combining (25) and (26) yields (23). □

Corollary 2 (Gülcü). *Let A be a positive and regular Toeplitz matrix, and (s_k) be a sequence finitely w.s.s. A-summable to $T(y)$. If we also have one - and hence all - of the equivalent assertions (i)-(iv), listed in Theorem 6, then (s_k) is A-summable to S defined in* (22).

From Proposition 1 and Corollary 2 we see the hierarchy "strong A-summability $\Rightarrow$ f.w.s.s. A-summability with $\tau = 0 \Rightarrow A$-summability", in all cases the formula (22) providing the A-sum of the series.

We have not seen, however, if $\tau > 0$ is possible, and whether the estimates of (23) are best possible. Next we aim at showing these by examples.

Let $\beta > 0$ be any parameter, and let the numerical sequences (b_n) and (α_n) be defined as

$$b_j := \alpha_{[(j+1)/2]} + (-1)^j \beta \quad \text{with} \quad 0 \le \alpha_j < \beta \quad \text{for all } j \in \mathbb{N} \ . \tag{27}$$

Consider the sequence (s_k) defined by

$$s_k := \begin{cases} b_{k+1-m} 3m & \text{if} \quad k = 3m-2 \quad \text{or} \quad k = 3m-1 \ , \quad m \in \mathbb{N}, \\ 0 & \text{if} \quad k = 3m \ , \qquad\qquad m \in \mathbb{N} \ , \end{cases} \tag{28}$$

that is, $(s_k) = 3b_1,\ 3b_2,\ 0, 6b_3,\ 6b_4,\ 0,\ , 9b_5,\ 9b_6,\ 0,\ \dots$ with $3mb_{2m-1},\ 3mb_{2m},\ 0$ on the $(3m-2)^{\mathrm{nd}}$, $(3m-1)^{\mathrm{st}}$ and $(3m)^{\mathrm{th}}$ places, respectively.

Define the normalized, positive Toeplitz matrix $A = (a_{nk})$ as follows.

$$a_{nk} := \begin{cases} \frac{1}{n} & \text{if} \quad k = 3m \quad \text{with} \quad m < n, \\ \frac{1}{2n} & \text{if} \quad k = 3n-2 \quad \text{or} \quad k = 3n-1, \\ 0 & \quad \text{otherwise} \ . \end{cases} \tag{29}$$

Lemma 6. *With the definitions* (27), (28) *and* (29) *above, the following assertions hold true.*

i) $\widetilde{S}_n = S_n = \sum_k a_{nk} s_k = \alpha_n$, *for all* $n \in \mathbb{N}$.

ii)

$$\widetilde{R}_n(u) = R_n(u) = \begin{cases} 0 & \textit{for} \quad -\infty < u \le (\alpha_n - \beta)n, \\ \frac{1}{2n} & \textit{for} \quad (\alpha_n - \beta)n < u \le 0, \\ 1 - \frac{1}{2n} & \textit{for} \quad 0 < u \le (\alpha_n + \beta)n, \\ 1 & \textit{for} \quad (\alpha_n + \beta)n \le u < \infty \ . \end{cases} \tag{30}$$

iii) For all $u \in \mathbb{R}$ the limit distribution function is $R(u) = \chi(u)$ and the integral $S := \int_{\mathbb{R}} |u| dR(u)$ is just 0.

iv)

$$\widetilde{T}_n(y) = T_n(y) = \begin{cases} |y| + \alpha_n & \text{if} \quad -\infty < y \le (\alpha_n - \beta)n, \\ (1-\frac{1}{n})|y| + \beta & \text{if} \quad (\alpha_n - \beta)n \le y \le (\alpha_n + \beta)n, \\ |y| - \alpha_n & \text{if} \quad (\alpha_n + \beta)n \le y < \infty \ . \end{cases} \tag{31}$$

v) The f.w.s.s. A-sum function is $T(y) = |y| + \beta$ for all $y \in \mathbb{R}$.

Proof. Since A is normal (meaning $\sum a_{nk} = 1$ for all $n \in \mathbb{N}$), all normalized quantities $\widetilde{R}_n$, $\widetilde{S}_n$, $\widetilde{T}_n$ coincide with their counterparts R_n, S_n and T_n, respectively.

(i). Fix n. Observe that then by construction there are only two $s_k \neq 0$ with non-vanishing weights a_{nk}, namely $s_{3n-2} = b_{2n-1} = n(\alpha_n - \beta)$ and $s_{3n-1} = b_{2n} = n(\alpha_n + \beta)$, both with weights $a_{n,3n-2} = a_{n,3n-1} = \frac{1}{2n}$. This gives (i) immediately.

(ii). In (1) we have to replace s_k from the sum S_n by the corresponding $\chi(u - s_k)$, that is, for most of the cases we insert just $\chi(u)$. There are only two different terms, namely $\chi(u - n(\alpha_n - \beta))$ and $\chi(u - n(\alpha_n + \beta))$, both with weights $\frac{1}{2n}$. In other words, R_n is left continuous, piecewise constant, and has jumps $\frac{1}{2n}$, $(1 - \frac{1}{n})$ and $\frac{1}{2n}$ at $n(\alpha_n - \beta)$, 0 and $n(\alpha_n + \beta)$, respectively, while $R_n(-\infty) = 0$. Whence (ii) obtains.

(iii). Since $\frac{1}{2n} \to 0$, $R_n(u) \to \chi(u)$ uniformly, hence a.e., too.

(iv). There are at least three ways to compute T_n, perhaps the simplest being to take into account $T_n' = 2R_n - 1$ a.e., coming from (8) similarly to (7). Using this and computing, say, $T_n(0) = (|n(\alpha_n - \beta)| + |n(\alpha_n + \beta)|)/(2n) = \beta$, (iv) follows from (ii). Of course, a direct calculation by (2) or using (4) is equally possible.

(v). Subtracting $T_n(y)$ from $|y| + \beta$ we get

$$g_n(y) := (|y| + \beta) - T_n(y) = \begin{cases} \beta - \alpha_n & \text{if} \quad -\infty < y \le (\alpha_n - \beta)n, \\ \frac{1}{n}|y| & \text{if} \quad (\alpha_n - \beta)n \le y \le (\alpha_n + \beta)n, \\ \alpha_n + \beta & \text{if} \quad (\alpha_n + \beta)n \le y < \infty \ . \end{cases} \tag{32}$$

Observe that $|\alpha_n| < \beta$ and we have $0 \le g_n(y) \le |y|/n$, hence the assertion. $\square$

Example 3. Let the parameters $0 < \beta$ and σ with $|\sigma| \le \beta$ be arbitrary. Let us choose any sequence (α_n) with $|\alpha_n| < \beta$ so that $\lim_{n\to\infty} \alpha_n = \sigma$; moreover, take the corresponding sequence (b_k) of (27). Then the matrix A and the sequence s_k, defined in (29) and (28), respectively, has the following properties.

i) The sequence (s_k) is f.w.s.s. A-summable to $T(y) = |y| + \beta$,

ii) The tightness of the f.w.s.s. A-sum function T of the sequence (s_k) is $\tau = \beta$;

iii) The A-limiting distribution of s_k is $R(u) = \chi(u)$;

iv) The integral $S := \int_{\mathbb{R}} |u| dR(u)$ is 0;

v) The sequence (s_k) is A-summable to $s = \sigma$.

Proof. Everything follows easily from the preceeding Lemma 6. $\square$

Remark 2. It can be observed on this example that neither of the statements (i)-(iv) of Theorem 6 are satisfied; moreover, $\int_{\mathbb{R}} |R - \widetilde{R}_n| = \tau$ for all $n \in \mathbb{N}$. Still, if we take $\sigma = 0$, the A-sum may hit the value of the integral S; hence equality of them is not equivalent to the assertions listed. Also, the tightness function (15) is constant $\tau = T(0) = \beta$ and thus the estimate $\tau \le T(0)$ of Lemma 5 can be sharp. Compare Corollary 3.

Modifying our example a little we can even obtain non-A-summable sequences with tightness τ and proving simultaneous sharpness of the two-sided estimates (23) in Theorem 7.

Example 4. Let the parameters $\beta > 0$ and σ, ρ with $-\beta \le \rho \le \sigma \le \beta$ be arbitrary. Let us choose any sequence (α_n) with $|\alpha_n| < \beta$ so that $\liminf_{n\to\infty} \alpha_n = \rho$ and $\limsup_{n\to\infty} \alpha_n = \sigma$; moreover, take the corresponding sequence (b_k) of (27). Then the matrix A and the sequence (s_k), defined in (29) and (28), respectively, has the properties (i)-(iv) of Example 3. Furthermore, the sequence (s_k) satisfies $\liminf_{n\to\infty} \sum_k a_{nk} s_k = \rho$ and $\limsup_{n\to\infty} \sum_k a_{nk} s_k = \sigma$, too.

§5. Asymptotes of $T(y)$ and A-summability

A real function f is said to have left asymptote $l(x) = ax+b$, if $\lim\limits_{-\infty}\{f-l\} = 0$. Similarly, $p(x) = cx + d$ is a right asymptote of f if $\lim\limits_{\infty}\{f - p\} = 0$. According to Proposition 1, if (s_k) is strongly A-summable then T has both asymptotes, and their equation determines the value of the sum s. This is not a special case only: the functions $T(y)$, when existing finitely, always have asymptotes.

Proposition 2. *Let A be a positive and regular Toeplitz matrix, and (s_k) be a sequence finitely w.s.s. A-summable to $T(y)$. Let τ denote the tightness parameter defined in* (15) *and* (16)*. Then T has asymptotes $l(y) = -y+a$ and $p(y) = y+b$ of slope ± 1 with $a = S+\tau$ and $b = -S+\tau$, where S is defined in* (22) *and the integrals are absolutely convergent. In particular, $\tau = (a+b)/2$ and if, furthermore, $a+b = 0$, then $\tau = 0$ and (s_k) is A-summable to $a = -b = S$.*

Proof. In view of Theorem 5 $R \in L^1(-\infty, 0)$ and $(1 - R) \in L^1(0,\infty)$. In particular, both integrals in the formula defining S in (22) as well as in (18) are finite. On combining Theorems 5 and 2, we obtain for any $y < 0$

$$T(y) + y - S - \tau = 2\int_{-\infty}^{y} R\ .$$

Since we now have $R \in L^1(-\infty, 0)$, this tail integral tends to 0 with $y \to -\infty$. That leads to the assertion with the left asymptote $l(y) = -y + S + \tau$. The right asymptote $p(y) = y - S + \tau$ obtains similarly. The special case $\tau = 0$ is then obvious. □

Remark 3. Neither $\tau = 0$, nor $a = S$ or $b = -S$ is necessary for A-summability. On the other hand, the case of tightness (i.e., of $\tau = 0$) is thus "asymptotically the same" as strong summability, see Proposition 1.

A consequence, also obtainable by considering some shifted version of (s_k), is the following improvement on the estimate $\tau \le T(0)$ from Lemma 5.

Corollary 3. *Let A be a positive and regular Toeplitz matrix, and (s_k) be a sequence finitely w.s.s. A-summable to $T(y)$. Then the "tightness parameter"* (16) *of T satisfies $0 \le \tau \le \min_{\mathbb{R}} T$.*

Proof. By Lemma 1 (ii) T is convex, therefore the lines of the asymptotes l and p are below the graph of T. Hence also $\max\{l, p\} \le T$ and the same is true for the minima of $q(x) := \max\{l(x), p(x)\}$ and $T(x)$. But $\min_{\mathbb{R}} \max\{l, p\}$ is attained at $(a - b)/2$, where the lines intersect; and their (common) value is $(a + b)/2$ there. Since $(a + b)/2 = \tau$, we find $\tau \le \min_{\mathbb{R}} T$. □

Remark 4. If $\min_{\mathbb{R}} T = 0$, then we have strong A-summability, cf. Proposition 1.

§6. A-summation from conditions weaker than f.w.s.s. A-summability

In fact, to conclude A-summability we need not assume full f.w.s.s. A-summability.

Theorem 8. *Let A be a positive and regular Toeplitz matrix, and (s_k) be an arbitrary sequence. Assume that*

i) With some function R, $R_n \to R$ a.e. in (x,∞);

ii) For a certain $z \in [x,\infty)$, $T_n(z)$ converges finitely to $Z := T(z) := \lim_{n\to\infty} T_n(z)$.

Then we have

i) $T(y)$ exists finitely for all $y \geq x$. Moreover, $T(y) = y - z + Z - 2\int_z^y (1-R)$ for all $y \geq x$,

ii) T has right asymptote $p(y) = y + b$ with $b := Z - z - 2\int_z^\infty (1-R)$.

Furthermore, if also $(\widetilde{R}_n - R) \to 0$ in $L^1(x,\infty)$, then the sequence (s_k) is A-summable to $s = -b = z - Z + 2\int_z^\infty (1-R)$.

Proof. By Fatou's theorem, $\int_x^\infty (1-R) \leq \liminf_{n\to\infty} \int_x^\infty (\sum_k a_{nk} - R_n)$ which is at most $\sum_k a_{nk}(z-x) + T_n(z) \to (z-x) + T(z)$, in view of the integral representation (4) of T_n in Lemma 3. Thus $(1-R) \in L^1(x,\infty)$.

From here we can argue as in the proof of Theorem 2: Lemma 3 yields (8) (now with z in place of x) and for the interval I between y and z $R_n \to R$ together with $T_n(z) \to T(z) = Z$ provides similarly to (11) $T(y) = z - y + Z + 2\int_z^y R = y - z + Z - 2\int_z^y (1-R)$, i.e. part (i).

Since $(1-R) \in L^1(x,\infty)$ and $z, y \in [x,\infty)$, the integral $\int_z^y (1-R)$ converges to $\int_z^\infty (1-R)$ as $y \to \infty$. That yields the asymptote $p(y)$ as stated in part (ii).

Assume now $(\widetilde{R}_n - R) \to 0$ in $L^1(x,\infty)$. Since $T_n(0) = \sum_k a_{nk}|s_k| \leq \sum_k a_{nk}|z| + T_n(z)$, $T_n(0)$ is finite (and, in fact, uniformly bounded), hence also (5) of Lemma 3 applies here. On combining (4) and (5) we find

$$T_n(y) + \sum_k a_{nk}s_k = y\sum_k a_{nk} + 2\int_y^\infty \left(\sum_k a_{nk} - R_n(u)\right) du \ ,$$

that is

$$\frac{1}{\sum_k a_{nk}} \sum_k a_{nk}s_k = y - \widetilde{T}_n(y) + 2\int_y^\infty (1-\widetilde{R}_n) \ .$$

Since the right hand side converge with $n \to \infty$, so does the left hand side. By regularity and $(1-\widetilde{R}_n) \to (1-R)$ in $L^1(x,\infty)$, we are led to

$$\lim_{n\to\infty} \sum_k a_{nk}s_k = y - T(y) + 2\int_y^\infty (1-R) \ .$$

Substituting the formula of part (i) for $T(y)$ gives the statement. □

Remark 5. Analogous statements hold for the negative halfline.

Remark 6. Note that in place of Condition (i) and $(\widetilde{R}_n - R) \to 0$ in $L^1(x, \infty)$ it suffices to assume that $(1 - \widetilde{R}_n)$ forms a Cauchy sequence in $L^1(x, \infty)$. Indeed, then by completeness of $L^1(x, \infty)$ there exists an L^1-limit $(1 - R)$, which implies $R_n \to R$ a.e., too.

Corollary 4 (Gülcü). *Let A be a positive and regular Toeplitz matrix. If (s_k) is bounded from above by M and $T_n(z)$ converges to the finite value $Z := T(z) := \lim_{n\to\infty} T_n(z)$ for a certain $z \geq M$, then*

i) $T(y)$ exists finitely for all $y \geq M$. Moreover, $T(y) = y - z + Z$ for all $y \geq M$.

ii) T has right asymptote $p(y) = y - z + Z$.

iii) (s_k) is A-summable to s with $s = S := z - Z$.

Proof. By the boundedness condition, $\widetilde{R}_n(y) \equiv 1$ for all $n \in \mathbb{N}$ and all $y > M$. Hence Theorem 8 applies. □

In Theorem 8 it can still happen that for $y < x$ $T_n(y)$ diverges, however between finite bounds. Also, if we allow that z be less than x in Condition (ii), then it may happen that $T_n(y)$ diverges for all $y > x$, and, even for all $y \neq z$. Also, if we assume only Conditions (i) and (ii), but not L^1-convergence, or if we allow that z be less than x in Condition (ii), then the sequence (s_k) may be A-summable and may not be A-summable as well. These will be seen from the next examples.

Example 5. Let $A := (a_{nk})$ with $a_{nk} := \delta_{n,k}$, where $\delta_{n,k}$ stands for the Kronecker symbol, i.e. $\delta_{n,k} = 1$ or 0 according to $n = k$ or not. Clearly, A is a normalized (hence regular), positive Toeplitz matrix. Let $s_k := (-1)^k$. Then (s_k) is a bounded sequence which is not w.s.s. A-summable as $T_n(y) = |y - (-1)^n|$ oscillates for $y \neq 0$; however, for $z := 0$ $T(0) = 1$ exists, and also $R_n(u) \equiv R(u) := \chi(u)$ for all points u with $|u| > 1$ as $R_n(u) = \chi(u - (-1)^n)$. Furthermore, (s_k) is not A-summable as $S_n := \sum_k a_{nk} s_k = (-1)^n$ oscillates between ± 1. Hence $z \geq x$ in Condition (ii) of Theorem 8 (or $z \geq M$ from Corollary 4) can not be dropped.

Assuming existence of $T(z)$ for some point z and also further conditions like $(1 - \widetilde{R}_n) \to (1 - R)$ in $L^1(z, \infty)$ or even boundedness of (s_k), one would like to obtain more, e.g. f.w.s.s. A-summability. However, that is not possible, as is seen from the next example.

Example 6. Let $A := (a_{nk})$ with $a_{nk} := \delta_{n,k}/2 + \delta_{n+2,k}/2$. Clearly, A is a normalized, positive Toeplitz matrix. Let $s_k := \Re\{i^k\}$ (where $i := \sqrt{-1}$, as usual). Then (s_k) is a bounded sequence which is not w.s.s. A-summable. However, for $|y| \geq 1$ $T(y)$ exists finitely and is $|y|$, and for $|u| > 1$ we have $R_n(u) \equiv R(u) := \chi(u)$.

Proof. For any $y \in \mathbb{R}$ we have $\widetilde{R}_n(u) = R_n(u) = \chi(u)$ for n odd and $\widetilde{R}_n(u) = R_n(u) = (\chi(u-1) + \chi(u+1))/2$ for n even. Thus $\widetilde{R}_n(u) = R_n(u) = \chi(u)$ for all $|u| > 1$, hence the limit exists and is $R(u) = \chi(u)$, while for $|u| < 1$ $R_n(u)$ oscillates between $1/2$ and $\chi(u)$. Computing $T_n(y)$ either directly from Definition 2 or from Lemma 3 we find $T_n(y) = |y|$ for n odd and $T_n(y) = \max\{1, |y|\}$ for n even. Thus, for $|y| < 1$ $T(y)$ does not exist and (s_k) is not w.s.s. A-summable, while $T(y) = |y|$ for all $|y| > 1$. □

Remark 7. Observe that in the above example the sequence (s_k) is A-summable to 0 as $S_n := \sum_k a_{nk} s_k = 0$ for all n. This is a general phenomenon, necessarily occurring in view of Theorem 8 or Corollary 4 - in spite of lack of f.w.s.s. A-summability around 0 - if certain one-sided conditions are assumed only.

§7. Historical remarks

The manuscript [2] was written sometimes in 1998 - certainly before January 1999 - and was seen by the present author in early 2000. The generalized definition, some of the results and the attempts to push them further clearly suggested the basic idea; however, it contained errors and a couple of the proofs were incomplete. Unfortunately, the project did not come to harvest, as A. Gülcü has never published a revised paper. To the contrary, it seems that he has abandoned not only this work, but also mathematical research as a whole. After many futile attempts to find and to contact him, finally the present author decided to finish this work by himself.

In the sequel results proved basically correctly by Gülcü are attributed to him throughout. Lack of any published version of [2] justifies detailing even these assertions[1]. However, sometimes the proofs were reshaped even for these claims, and the build-up of the material is reordered for good reasons. Thus a few comments about content of [2] are in order here.

Definition 2 is due to Gülcü but restricted to the finite case only. For us it seemed to be practical to allow $+\infty$ here and distinguish finite w.s.s. A-summability within the notion: correction of several errors from [2] and general formulation of lemmas and relations justify our choice. Lemma 1 and $(iii) \Leftrightarrow (ii)$ from Proposition 1 are from [2], but some arguments - e.g. proof of (iv) - are not correct there. To prove Lemma 3 [2] follows [6] and [5], considering a refining sequence of partitions of $\mathbb{R}$, but the proof is incomplete. Theorem 1 is taken from [2], adding the observation that not only a subsequence, but also the whole sequence (R_n) must necessarily be convergent a.e., thus cutting a lengthy argument of [2] short. Also Theorem 2 is from [2], but not the proof being erroneous there.

The main results of our paper stem from the question posed by an invalid statement of [2] with essentially stating that Theorem 3 holds with Condition (ii) as a part of the result, and not as a condition. Here it is clarified that validity of Theorem 3 requires Condition (ii), while without it we can state only Theorem 4. However, if the summation method involves a *triangular* matrix A, then - as presented in Corollary 1 - one can even drop Condition (ii).

Corollary 2 can be attributed to Gülcü as in [2, Theorem 4] he presents a longer, direct proof - based on Lemma 1 - for the version with assuming Condition (iii) from the equivalent assumptions listed in Theorem 6. However, equivalence with the other conditions, as well as the phenomenon perceived here as the "tightness" τ, are described here for the first time.

In [2, Theorem 5] also the easy Corollary 4 is deduced directly from the definitions; the more general material, as well as the examples constructed, are new.

We have left (iii) and (iv) of Lemma 1 in the formulation only for historical reasons. Clearly these constitute a half of A-summability and asymptote evaluations of the A-sum, obtained here also under weaker conditions - e.g. not assuming full f.w.s.s. A-summability of (s_k).

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