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STOPPING A STOCHASTIC INTEGRAL PROCESS AS CLOSE AS POSSIBLE TO THE ULTIMATE VALUE OF A FUNCTIONAL

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Abstract. We extend to stochastic integral processes with deterministic integrands the results previously proved by Graversen, Peskir and Shiryaev, on stopping Brownian motion as close as possible to the ultimate value of a functional.

1. Introduction

In this short note, we extend to stochastic integral processes with deterministic integrands the results previously proved by Graversen, Peskir and Shiryaev for Brownian Motion (BM), concerning stopping the process as close as possible to the ultimate value of a functional. Let us consider the process

$$X(t) = X_0 + \int_0^t \sigma(s) dB_s \tag{1.1}$$

where B_t is a standard Brownian Motion; we assume that $\sigma(t)$ is a non-negative deterministic function. Notice that the quadratic variation of $X(t)$, $\langle X \rangle_t = \int_0^t \sigma^2(s)ds$, turns out to be a deterministic increasing process, thus $X(t)$ is a martingale which is also a Gaussian process. Then we consider the optimal stopping problem:

$$w^* = \inf_{\tau \in [0,1]} E[X(\tau) - S_1]^2 \tag{1.2}$$

where the infimum is taken over all stopping times $\tau \in [0,1]$ of $X(t)$, and S_t is a given functional of the process $X(t)$ for $t \in [0,1]$. We consider two cases: in the first one S_t is the maximum process of $X(t)$ in $[0,t]$, i.e. $S_t = \max_{s \in [0,t]} X(s)$, in the second case S_t is the time average of $X(t)$ in the interval $[0,t]$, i.e. $S_t = \frac{1}{t} \int_0^t X(s)ds$.
Since S_1 is a random quantity whose value depends on the entire path of the process $X(t)$ over the interval $[0,1]$, its ultimate value is at any time $t \in [0,1)$, unknown. The optimal stopping problem above consists in finding the particular (random) time τ^* at which the process $X(t)$ must be terminated so that $w^* = E[X(\tau^*) - S_1]^2$, in this way the value of the process at τ^* is as "close" as possible to the ultimate value S_1, where "closeness" is meant in the sense of mean-square distance. Our results generalize the analogous ones found in (Graversen, Peskir and Shiryaev, 2000) for BM; for instance, they have applications in Stochastic Finance, when an optimal decision (i.e. a stopping time) has to be based on a prediction of the future behaviour e.g. of a stoch price $X(t)$ of the form (1.1).

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2. Statement of the problem and main results

Throughout this section $X(t)$ is the solution of the stochastic differential equation:

$$dX(t) = \sigma(t)dB_t, \ X(0) = X_0 \tag{2.1}$$

where B_t is a standard Brownian Motion, $\sigma(t) \geq 0$ is a (deterministic) bounded continuous function; we denote by $\langle X \rangle_t = \int_0^t \sigma^2(s)ds$, the quadratic variation of X.

2.1. **The case of the maximum functional.** Our aim is to solve the optimal stopping problem:

$$w_* = \inf_{\tau \in [0,1]} E[X(\tau) - S_1]^2 \tag{2.2}$$

where the infimum is taken over all stopping times $\tau \in [0, 1]$ of $X(t)$, and
$S_1 = \max_{s \in [0,1]} X(t)$. First, we recall the following result holding for BM.

Theorem 2.1. *(Graversen, Peskir and Shiryaev, 2000) Let us consider the optimal stopping problem (2.2), with $X(t) \equiv B_t$ and $X_0 = 0$ (i.e. $\sigma(\cdot) = 1$ and $S_t \equiv S_t^0 \doteq \max_{s \in [0,t]} B_s$). Then the value $w_* \equiv w_*^0$ is given by*

$$w_*^0 = 2\Phi(z_*^0) - 1 = 0.73.... \tag{2.3}$$

where $z_*^0 = 1.12....$ is the unique root of

$$4\Phi(z_*^0) - 2z_*^0\phi(z_*^0) - 3 = 0 \tag{2.4}$$

and

$$\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}, \ \Phi(x) = \int_{-\infty}^x \phi(u)du$$

Moreover $w_*^0 = E[B_{\tau_*^0} - S_1^0]^2$, where

$$\tau_*^0 = \inf\{0 \leq t \leq 1 : S_t^0 - B_t \geq z_*^0 \sqrt{1-t}\} \tag{2.5}$$

The following holds:

Theorem 2.2. *Let us consider the optimal stopping problem (2.2), where $X(t)$ satisfies (2.1), and denote by $\rho(t) = \langle X \rangle_t$ the quadratic variation of the process X. We suppose that*

$$\rho(+\infty) = +\infty \tag{2.6}$$

Then:

$$w_* = \rho(1) w_*^0 \tag{2.7}$$

where w_*^0 is given by (2.3). Moreover, the infimum in (2.2) is attained at τ^*, which is given by:

$$\tau_* = \inf\{0 < t < \rho(1) : S_t^0 - B_t \geq z_*^0 \sqrt{\rho(1) - t}\} \tag{2.8}$$

where z_*^0 is given by (2.4).

Proof.
We assume $X_0 = 0$; the general case is reduced to this by considering the process $Z(t) \doteq X(t) - X_0$. Thanking to (2.6), by a random-time change (see e.g. Krylov, 1994) we can write: $X(t) = \tilde{B}_{\langle X \rangle_t} = \tilde{B}_{\rho(t)}$, where $\tilde{B}$ is a suitable BM. Then:

$$S_1 = \max_{s\in[0,1]} X(s) = \max_{s\in[0,1]} \tilde{B}_{\rho(s)} = \max_{u\in[0,\rho(1)]} \tilde{B}_u$$

So:

$$w_* = \inf_{\tau\in[0,1]} E\left[X(\tau) - S_1\right]^2 = \inf_{u\in[0,\rho(1)]} E\left(\tilde{B}_u - \max_{u\in[0,\rho(1)]} \tilde{B}_u\right)^2 \tag{2.9}$$

Now, by using the scaling property of BM, $\tilde{B}_{\rho(1)\cdot s}/\sqrt{\rho(1)} \doteq W_s$ is also BM, so setting $s = u/\rho(1)$, the last expression of the infimum can be written:

$$\rho(1) \cdot \inf_{s\in[0,1]} E\left[\left(W_s - \max_{r\in[0.1]} W_r\right)^2\right]$$

Therefore, from Theorem 1.1 (2.7) follows. Moreover, by the scaling property of BM again (see Remark 4 of (Graversen, Peskir and Shiryaev, 2000)), we obtain (2.8).

2.2. **The case of the integral functional.** We consider here the optimal stopping problem (1.2), with $S_t = \frac{1}{t}\int_0^t X(s)ds$, i.e. S_1 is the time average of the process $X(t)$ in the interval $[0,1]$. First, we state a representation result for the functional $I_1 \doteq S_1$.

Lemma 2.3. *The following formula holds:*

$$I_1 = \int_0^1 X(s)ds = X(0) + \int_0^1 \sigma(s)(1-s)dB_s \tag{2.10}$$

Proof.
By Itô's formula, we get:

$$\int_0^t X(s)ds = tX(t) - \int_0^t s\sigma(s)dB_s \tag{2.11}$$

By using that:

$$X(1) = X(0) + \int_0^1 \sigma(s)dB_s \tag{2.12}$$

and taking $t = 1$ in (2.11), we easily obtain (2.10).

Remark 2.4. *In the special case when $X(t) \equiv B_t$, formula (2.10) gives $I_1 = \int_0^1 (1-t)dB_t$, as already stated in (Graversen, Peskir and Shiryaev, 2000).*

We associate to I_1 the martingale:

$$M_t \doteq X(0) + \int_0^t \sigma(s)(1-s)dB_s \tag{2.13}$$

Note that $M_1 = I_1$; moreover, it is easy to see that:

$$M_t = (1-t)X(t) + \int_0^t X(s)ds \tag{2.14}$$

Theorem 2.5. *For any stopping time $\tau \in [0,1]$:*

$$E[X(\tau)-I_1]^2 = E\left(\int_0^\tau (2s-1)\sigma^2(s)ds\right) + \int_0^1 \sigma^2(s)(1-s)^2 ds \tag{2.15}$$

Thus the infimum over all stopping times $\tau \in [0,1]$ can be esplicitly calculated by finding the minimum of the function

$\tau \to \psi(\tau) \doteq \int_0^\tau (2s-1)\sigma^2(s)ds + \int_0^1 \sigma^2(s)(1-s)^2 ds$ and τ_* is the value at which ψ attains its minimum.

Proof.
We assume $X_0 = 0$; the general case is reduced to this by considering the process $Z(t) \doteq X(t) - X_0$. We have:

$$E[X(\tau)-I_1]^2 = E[X(\tau)^2] - 2E[X(\tau)I_1] + E[I_1^2] \tag{2.16}$$

By Itô's formula, the first addendum in the right-hand member of (2.16) is:

$$E[X(\tau)^2] = E\left(\int_0^\tau \sigma^2(s)ds\right) \tag{2.17}$$

Moreover, by (2.10) the third addendum in the right-hand member of (2.16) is:

$$\int_0^1 \sigma^2(s)(1-s)^2 ds \tag{2.18}$$

An application of the optional sampling theorem for martingale to bounded stopping times, gives that, for any stopping time $\tau \le 1$:

$$E[X(\tau)I_1] \equiv E[X(\tau)M_1] = E[X(\tau)M_\tau] \tag{2.19}$$

By using that $d(X(t)M_t) = X(t)dM_t + M_t dX(t) + \sigma^2(t)(1-t)dt$ we obtain:

$$E[X(\tau)M_\tau] = E\left(\int_0^\tau X(t)\sigma(t)(1-t)dB_t\right) \tag{2.20}$$

$$+E\left(\int_0^\tau M_t\sigma(t)dB_t\right) + E\left(\int_0^\tau \sigma^2(t)(1-t)dt\right)$$

which is equal to

$$E\left(\int_0^\tau \sigma^2(t)(1-t)dt\right) \tag{2.21}$$

since the others terms vanish by the optional sampling theorem again. Thus, (2.15) follows by (2.17), (2.18) and (2.21).

Remark 2.6. *Note that, unlike Theorem 2.2, we do not require that $\langle X\rangle_\infty = \infty$. We observe that, although I_1 is unknown because it is a random quantity whose value depends on the entire path of the process $X(t)$ over the time interval $[0,1]$, the optional sampling theorem permits to write $E[X(\tau)I_1]$ as $E[X(\tau)M_\tau]$ which involves values of the process until the time τ.*
In the case of BM ($\sigma(\cdot)=1$) (2.15) furnishes

$$\inf_{\tau\in[0,1]} E[X(\tau)-I_1]^2 = \inf_{\tau\in[0,1]} E(\tau^2 - \tau + \frac{1}{3}) = \frac{1}{12}$$

and the infimum is attained at $\tau^* = \frac{1}{2}$, as already found in (Graversen, Peskir and Shiryaev, 2000).

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