# BOREL PRODUCTS OF RIESZ SPACE-VALUED MEASURES ON TOPOLOGICAL SPACES

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Abstract. The purpose of the paper is to show the existence and uniqueness of the Borel product of two Dedekind complete Riesz space-valued $\sigma$-measures on completely regular spaces. It is also shown that the operation making such a Borel product is jointly continuous with respect to weak order convergence of measures.

## 1. Introduction

Let $X$ be a Hausdorff space and $V$ a Dedekind complete Riesz space. Denote by $\mathcal{B}(X)$ the $\sigma$-field of all Borel subsets of $X$. A $V$-valued $\sigma$-measure on $X$ is a finitely additive set function $\mu : \mathcal{B}(X) \to V$ such that $\mu(\cup_{n=1}^{\infty} A_n) = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \mu(A_k)$ whenever $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$. If $V$ possesses a Hausdorff vector topology $\tau$ for which each upper bounded, increasing sequence in $V$ converges in the $\tau$-topology to its least upper bound, $V$-valued $\sigma$-measures are ordinary topological vector space-valued measures that are fairly well understood; see Diestel and Uhl [7], Dinculeanu [8], and Kluvánek and Knowles [13]. But $V$ need not possess any such topology; see Floyd [9].

In usual measure theory on topological spaces, the notion of a Borel product measure is of central importance. In Wright [21, Theorem 1.7], a general definition is given of what is meant by a Borel product of two $\sigma$-measures on locally compact spaces in the case that those $\sigma$-measures take values in monotone complete, partially ordered vector spaces. Further, some conditions which imply the existence and uniqueness of such product measures are established.

The main purpose of the paper is to show the existence and uniqueness of the Borel product of two Dedekind complete Riesz space-valued $\sigma$-measures on completely regular spaces. In the process of the extension, the crucial step is to find a necessary and sufficient condition that yields an analogue of the Riesz representation theorem in our setting, that is, a condition that a given positive linear mapping $T$ from $C(X)$, the space of all bounded, continuous, real-valued functions on $X$, into a Dedekind complete Riesz space $V$ can be uniquely represented by a $V$-valued $\sigma$-measure $\mu$ on $X$ such that $T(f) = \int_X f d\mu$ for all $f \in C(X)$. A successful analogue of the Riesz representation theorem was first proved by Wright [18, Theorem 4.1] in the case that $X$ is compact. See also [19, Theorem 1] for the case that $X$ is locally compact.

In Section 2 we recall some basic facts about Riesz spaces and give some preliminary results about Riesz space-valued $\sigma$-measures and regularities of such measures on a topological space. The results explained in the preceding paragraph are obtained in Sections 3 and 4.

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When both $X$ and $Y$ are compact, some results of the paper reduce to the special case of the results in Wright [18, 19, 20, 21]. However, our work will be needed to develop the theory of weak order convergence of Riesz space-valued $\sigma$-measures, in which we usually assume that the involved measures are defined on metric spaces or more generally on completely regular spaces; see Boccuto and Sambucini [3] and [10, 11, 12].

As an application in this light, we show in Section 5 that the operation making the Borel product of two Riesz space-valued $\sigma$-measures is jointly continuous with respect to weak order convergence of measures.

## 2. Notation and preliminaries

All the topological spaces in this paper are supposed to be Hausdorff. Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all natural numbers, respectively. In this section we recall some basic facts about Riesz spaces and give some preliminary results about regularities of Riesz space-valued $\sigma$-measures.

2.1. *Riesz spaces.* A Riesz space is said to be *Dedekind complete* if every non-empty subset that is bounded above has a least upper bound. Every Dedekind complete Riesz space is Archimedean (Zaanen [22, Theorem 12.3]).

Let $V$ be a Riesz space and put $V^+ := \{u \in V : u \geq 0\}$. Given a net $\{u_\alpha\}_{\alpha\in\Gamma}$ in $V$ and $u \in V$ we write $u_\alpha \downarrow u$ to mean that it is decreasing and $\inf_{\alpha\in\Gamma} u_\alpha = u$. The meaning of $u_\alpha \uparrow u$ is analogous. A net $\{u_\alpha\}_{\alpha\in\Gamma}$ is said to *converge in order* to $u$ and is denoted by $u_\alpha \xrightarrow{o} u$ or $\lim_{\alpha\in\Gamma} u_\alpha = u$ if there is a net $\{p_\alpha\}_{\alpha\in\Gamma}$ in $V$ with $p_\alpha \downarrow 0$ such that $|u - u_\alpha| \leq p_\alpha$ for all $\alpha \in \Gamma$. In [22, Lemma 10.1 and Theorem 10.2] some properties of order convergence are formulated and proved for sequences in a Riesz space, but the analogous properties are also valid for nets with appropriate modifications. See [12, Proposition 1] for precise formulae.

In this paper we also need the notions of the *limes superior* and the *limes inferior* of a net in a Riesz space. Let $\{u_\alpha\}_{\alpha\in\Gamma}$ be an order bounded net in a Dedekind complete Riesz space $V$. Then $x_\beta := \sup_{\alpha\geq\beta} u_\alpha$ exists in $V$ for each $\beta \in \Gamma$, and a net $\{x_\beta\}_{\beta\in\Gamma}$ is decreasing and bounded below. By the Dedekind completeness of $V$, there is an element $x \in V$ such that $x_\beta \downarrow x$ and we write $x := \limsup u_\alpha$. Similarly, we write $y := \liminf u_\alpha$, where $y_\beta := \inf_{\alpha\geq\beta} u_\alpha$ for each $\beta \in \Gamma$ and $y_\beta \uparrow y$. The properties of the limes superior and limes inferior of nets in $V$ are very much analogous to the properties of those of nets in $\mathbb{R}$. See [12, Proposition 2] for precise formulae.

Let $e \in V$ with $e > 0$. Denote by $V_e$ the principal ideal generated by the element $e$, that is, $V_e := \{u \in V : |u| \leq re \text{ for some } r \in \mathbb{R} \text{ with } r > 0\}$. Then, $V_e$ is an AM-space with order unit $e$ under the order unit norm $\|u\|_e := \inf\{r > 0 : |u| \leq re\}$, so that by the Kakutani-Krein theorem (Schaefer [16, Theorem II.7.4]), it is isometrically and lattice isomorphic to $C(S)$, the space of all (bounded) continuous real-valued functions on a compact Hausdorff space $S$. Since $V$ is Dedekind complete, so also is $V_e$. Hence $S$ is Stonean, that is, the closure of every open subset of $S$ is also open [16, Corollary to Proposition II.7.7]. See also Aliprantis and Burkinshaw [1] and Luxemburg and Zaanen [15] for further information on Riesz spaces.

2.2. *$\sigma$-measures.* Let $X$ be a topological space. Denote by $\mathcal{B}(X)$ the $\sigma$-field of all Borel subsets of $X$, that is, the $\sigma$-field generated by the open subsets of $X$. Denote by $C(X)$ the Banach lattice of all bounded, continuous, real-valued functions on $X$ with lattice norm $\|f\| := \sup_{x\in X} |f(x)|$.

Let $V$ be a Dedekind complete Riesz space. A finitely additive, positive set function $\mu : \mathcal{B}(X) \to V$ is called a *$\sigma$-measure* on $X$ if whenever $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$ then $\mu(\cup_{n=1}^{\infty} A_n) = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \mu(A_k)$. We emphasize that only measures taking positive values are considered. As in the scalar case, every $\sigma$-measure has the monotone sequential continuity from below and from above, in other words whenever $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{B}(X)$ then $\mu(\cup_{n=1}^{\infty} A_n) = \sup_{n\in\mathbb{N}} \mu(A_n)$ and whenever it is decreasing then $\mu(\cap_{n=1}^{\infty} A_n) = \inf_{n\in\mathbb{N}} \mu(A_n)$, respectively.

In Wright [18, 20] an integral of a measurable real-valued function with respect to a $V$-valued $\sigma$-measure is constructed, and successful analogues of the monotone convergence theorem and the Lebesgue convergence theorem are obtained. We shall use them freely in this paper.

2.3. *Some regularities of $\sigma$-measures.* As in usual measure theory on topological spaces we shall need to introduce some regularities for Riesz space-valued $\sigma$-measures to develop the theory. Let $X$ be a topological space and $V$ a Dedekind complete Riesz space.

**Definition 1.** Let $\mu$ be a $V$-valued $\sigma$-measure on $X$.

(i) $\mu$ is said to be *quasi-regular* if whenever $G$ is an open subset of $X$ then
$$\mu(G) = \sup\{\mu(F) : F \subset G \text{ and } F \text{ is closed}\}.$$

(ii) $\mu$ is said to be *quasi-Radon* if whenever $G$ is an open subset of $X$ then
$$\mu(G) = \sup\{\mu(K) : K \subset G \text{ and } K \text{ is compact}\},$$
and it is said to be *tight* if the above condition holds for $G = X$.

(iii) $\mu$ is said to be *$\tau$-smooth* if whenever $\{G_\alpha\}_{\alpha\in\Gamma}$ is an increasing net of open subsets of $X$ with $G = \cup_{\alpha\in\Gamma} G_\alpha$ then $\mu(G) = \sup_{\alpha\in\Gamma} \mu(G_\alpha)$.

*Remark.* The notions of regular and Radon $\sigma$-measures will be defined by the same way as in the scalar case. We omit their definitions because they are not used in the present paper.

**Lemma 1.** *Let $\mu$ be a $V$-valued $\sigma$-measure on $X$.*

(i) *$\mu$ is quasi-regular if and only if for each open subset $G$ of $X$ there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $V$ with $p_\alpha \downarrow 0$ and $\{F_\alpha\}_{\alpha\in\Gamma}$ of closed subsets of $X$ such that $F_\alpha \subset G$ and $\mu(G - F_\alpha) \leq p_\alpha$ for all $\alpha \in \Gamma$.*

(ii) *$\mu$ is quasi-Radon if and only if for each open subset $G$ of $X$ there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $V$ with $p_\alpha \downarrow 0$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$ such that $K_\alpha \subset G$ and $\mu(G - K_\alpha) \leq p_\alpha$ for all $\alpha \in \Gamma$.*

(iii) *$\mu$ is tight if and only if there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $V$ with $p_\alpha \downarrow 0$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$ such that $\mu(X - K_\alpha) \leq p_\alpha$ for all $\alpha \in \Gamma$.*

*Further, the above nets $\{F_\alpha\}_{\alpha\in\Gamma}$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ can be chosen to be increasing.*

*Proof.* (i) The proof of "if" part: Let $G$ be an open subset of $X$ and put $u := \sup\{\mu(F) : F \subset G \text{ and } F \text{ is closed}\}$. It follows from assumption that $\mu(G) \leq \mu(F_\alpha) + p_\alpha \leq u + p_\alpha$ for all $\alpha \in \Gamma$. Since $p_\alpha \downarrow 0$, we have $\mu(G) \leq u$. Whereas, $u \leq \mu(G)$ and thus $\mu(G) = u$.

The proof of "only if" part: Let $G$ be an open subset of $X$ and put $\mathcal{F} := \{F : F \subset G \text{ and } F \text{ is closed}\}$. Then $\mathcal{F}$ becomes a directed set under the partial ordering defined by the usual set inclusion. For each $F \in \mathcal{F}$, put $p_F := \mu(G - F)$. Then $\{p_F\}_{F\in\mathcal{F}}$ is a decreasing net and $\inf\{p_F : F \in \mathcal{F}\} = 0$. Thus, the nets $\{p_F\}_{F\in\mathcal{F}}$ and $\{F\}_{F\in\mathcal{F}}$ are the seeking ones.

The proofs of assertions (ii) and (iii) are analogous. $\square$

**Lemma 2.** *Let $\mu$ be a $V$-valued $\sigma$-measure on $X$. Then the following two conditions are equivalent.*

(i) *$\mu$ is tight and quasi-regular.*

(ii) *$\mu$ is quasi-Radon.*

*Proof.* The implication that (ii) implies (i) is obvious and we prove the implication that (i) implies (ii).

Let $G$ be an open subset of $X$. By Lemma 1 the tightness and quasi-regularity of $\mu$ imply that there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ and $\{q_\beta\}_{\beta\in\Lambda}$ in $V$ with $p_\alpha \downarrow 0$ and $q_\beta \downarrow 0$, $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$, and $\{F_\beta\}_{\beta\in\Lambda}$ of closed subsets of $G$ such that

$$\mu(X - K_\alpha) \le p_\alpha \quad \text{and} \quad \mu(G - F_\beta) \le q_\beta \tag{1}$$

for all $\alpha \in \Gamma$ and $\beta \in \Lambda$.

Since the index sets $\Gamma$ and $\Lambda$ are directed sets, so is $\Gamma \times \Lambda$ under the canonical coordinatewise ordering.

For each $(\alpha, \beta) \in \Gamma \times \Lambda$, put $D_{\alpha,\beta} := K_\alpha \cap F_\beta$ and $r_{\alpha,\beta} := p_\alpha + q_\beta$. Then each $D_{\alpha,\beta}$ is a compact subset of $G$ and it follows from (1) that

$$\begin{aligned}\mu(X) - \mu(D_{\alpha,\beta}) &\le \mu(X - K_\alpha) + \mu(X - F_\beta)\\ &\le p_\alpha + \mu(X - G) + q_\beta\\ &= \mu(X) - \mu(G) + p_\alpha + q_\beta.\end{aligned}$$

Thus $\mu(G - D_{\alpha,\beta}) \le r_{\alpha,\beta}$ for all $(\alpha, \beta) \in \Gamma \times \Lambda$. Since $r_{\alpha,\beta} \downarrow 0$ by [15, Theorem 15.8], the quasi-Radonness of $\mu$ follows from Lemma 1. □

**Lemma 3.** *Every quasi-Radon $V$-valued $\sigma$-measure $\mu$ on $X$ is $\tau$-smooth.*

*Proof.* Let $\{G_\alpha\}_{\alpha\in\Gamma}$ be an increasing net of open subsets of $X$ with $G = \cup_{\alpha\in\Gamma} G_\alpha$. Then $\mu(G)$ is obviously an upper bound of $\{\mu(G_\alpha)\}_{\alpha\in\Gamma}$.

Let $u \in V$ satisfying $\mu(G_\alpha) \le u$ for all $\alpha \in \Gamma$. If $K$ is a compact subset of $G$, then there is $\alpha_0 \in \Gamma$ such that $K \subset G_{\alpha_0}$, so that $\mu(K) \le \mu(G_{\alpha_0}) \le u$. Since $\mu$ is quasi-Radon, we have $\mu(G) = \sup\{\mu(K) : K \subset G \text{ and } K \text{ is compact}\} \le u$. Hence $\mu(G) = \sup_{\alpha\in\Gamma} \mu(G_\alpha)$ and this implies the $\tau$-smoothness of $\mu$. □

The following result can be proved as in the case of scalar measures; see [11, Proposition 4] for the proof.

**Proposition 1.** *Let $\mu$ be a $\tau$-smooth $V$-valued $\sigma$-measure on $X$. Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly bounded, increasing net of lower semicontinuous real-valued functions on $X$. Then $\int_X f d\mu = \lim_{\alpha\in\Gamma} \int_X f_\alpha d\mu = \sup_{\alpha\in\Gamma} \int_X f_\alpha d\mu$ whenever $f = \sup_{\alpha\in\Gamma} f_\alpha$ is the pointwise supremum of $f_\alpha$.*

The following lemma will be used to prove the uniqueness of a representing measure in our Riesz representation theorem; see Theorem 1 below.

**Lemma 4.** *Assume that $X$ is completely regular. Let $\mu$ and $\nu$ be $\tau$-smooth $V$-valued $\sigma$-measures on $X$. If $\int_X f d\mu = \int_X f d\nu$ for each $f \in C(X)$ then $\mu = \nu$ on $\mathcal{B}(X)$.*

*Proof.* Let $G$ be an open subset of $X$. We first show that $\mu(G) = \nu(G)$. Since $\chi_G$ is lower semicontinuous, there is an increasing net $\{f_\alpha\}_{\alpha\in\Gamma}$ of continuous real-valued functions on $X$ such that $0 \le f_\alpha \le \chi_G$ for all $\alpha \in \Gamma$ and $\chi_G(x) = \sup_{\alpha\in\Gamma} f_\alpha(x)$ for all $x \in X$ [5, Chapter IX, Section 1, Proposition 5]. Since $\mu$ and $\nu$ are $\tau$-smooth, it follows from assumption and Proposition 1 that

$$\mu(G) = \int_X \chi_G d\mu = \sup_{\alpha\in\Gamma} \int_X f_\alpha d\mu = \sup_{\alpha\in\Gamma} \int_X f_\alpha d\nu = \int_X \chi_G d\nu = \nu(G).$$

Next we show that $\mu = \nu$ on $\mathcal{B}(X)$. Put $\mathcal{A} := \{A \in \mathcal{B}(X) : \mu(A) = \nu(A)\}$. By the preceding paragraph, $\mathcal{A}$ contains all open subsets of $X$. It is readily seen that $\mathcal{A}$ is a Dynkin system. Thus it follows from the Dynkin system theorem [2, Theorem 4.1.2] that $\mathcal{A}$ contains $\mathcal{B}(X)$, and this implies that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}(X)$. □

Denote by $\Theta$ the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$. A Dedekind complete Riesz space $V$ is said to *weakly $\sigma$-distributive* if $\inf_{\theta\in\Theta}\sup_{i\in\mathbb{N}} q_{i,\theta(i)} = 0$ whenever $\{q_{i,j}\}$ is an order bounded, double sequence in $V$ satisfying $q_{i,j} \geq q_{i,j+1}$ for each $i,j \in \mathbb{N}$ and $\inf_{j\in\mathbb{N}} q_{i,j} = 0$ for each $i \in \mathbb{N}$. By Lemma 2 and [12, Theorem 3], every weakly $\sigma$-distributive, Dedekind complete Riesz space-valued $\sigma$-measure on any complete separable metric space is quasi-Radon.

## 3. An analogue of the Riesz representation theorem

Let $X$ be a topological space and $V$ a Dedekind complete Riesz space. Denote by $B(X)$ the Banach lattice of all bounded, Borel measurable real-valued functions on $X$ with lattice norm $\|f\| := \sup_{x\in X}|f(x)|$.

In this section we give a necessary and sufficient condition (tightness condition) which assures the validity of an analogue of the Riesz representation theorem for a positive linear mapping from $C(X)$ into $V$. We first extend [18, Proposition 4.1] to the case that $X$ is not necessarily compact.

**Proposition 2.** *Let $X$ be a completely regular space and $Y$ a compact space. Let $T : C(X) \to C(Y)$ be a positive linear mapping. Assume that there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $C(Y)$ with $p_\alpha \downarrow 0$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$ such that $T(f) \leq p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_\alpha) = \{0\}$. Put $N := \{y \in Y : \inf_{\alpha\in\Gamma} p_\alpha(y) > 0\}$. Then there is a mapping $\tilde{T} : B(X) \to B(Y)$ such that*

(i) *$\tilde{T}$ is positive and linear,*

(ii) *for each $f \in C(X)$, $\tilde{T}(f)(y) = T(f)(y)$ for all $y \notin N$,*

(iii) *if $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $B(X)$ which converges pointwise to $f$, then $f \in B(X)$ and*
$$\tilde{T}(f)(y) = \lim_{n\to\infty} \tilde{T}(f_n)(y) \text{ for all } y \in Y,$$

(iv) *if $f$ is a lower semicontinuous real-valued function on $X$, then*
$$\tilde{T}(f)(y) = \sup\{T(g)(y) : 0 \leq g \leq f, g \in C(X)\} \text{ for all } y \notin N,$$
*and hence $\tilde{T}(f)$ is lower semicontinuous on $Y - N$.*

*Proof.* For each $y \notin N$, define a positive linear functional $\varphi_y$ on $C(X)$ by $\varphi_y(f) := T(f)(y)$ for all $f \in C(X)$.

Fix $\varepsilon > 0$ and $y \notin N$. Since $\inf_{\alpha\in\Gamma} p_\alpha(y) = 0$, there is $\alpha_0 \in \Gamma$ with $0 \leq p_{\alpha_0}(y) < \varepsilon$. It follows from assumption that $\varphi_y(f) = T(f)(y) < \varepsilon$ whenever $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_{\alpha_0}) = \{0\}$. Then, it is easily verified that the functional $\varphi_y$ satisfies assumptions of Proposition III.2.1 [6], so that there is a finite, positive Radon measure $m_y$ on $X$ such that $T(f)(y) = \int_X f dm_y$ for all $f \in C(X)$.

For each $f \in B(X)$, put
$$\tilde{T}(f)(y) := \begin{cases} \int_X f dm_y, & y \notin N, \\ 0, & y \in N. \end{cases}$$
We shall prove that $\tilde{T}$ satisfies properties (i)–(iv) and it maps $B(X)$ into $B(Y)$.

The proofs of properties (i) and (ii) are obvious, and property (iii) follows from a standard argument in measure theory. Property (iv) follows from [4, Chapter IX, Section 5, Proposition 1].

It remains to prove that $\tilde{T}$ maps $B(X)$ into $B(Y)$. Let $f \in B(X)$ and put $h_f(y) := \tilde{T}(f)(y) = \int_X f dm_y$ for all $y \notin N$. Since $T$ is norm bounded, it is easy to prove that $\tilde{T}$ maps $B(X)$ into the space of all bounded, real-valued functions on $Y$. Since the pointwise infimum of continuous functions $p_\alpha$ is lower semicontinuous, $N = \{y \in Y : \inf_{\alpha\in\Gamma} p_\alpha(y) > 0\}$

is a Borel subset of $Y$. Thus, to prove that $\tilde{T}(f) \in B(Y)$ we have only to show that the function $h_f$ is Borel measurable on $Y - N$.

Put $\mathcal{A} := \{A \in \mathcal{B}(X) : h_{\chi_A} \in B(Y - N)\}$. By property (iv), $\mathcal{A}$ contains all open subsets of $X$, and by property (iii) it is readily seen that $\mathcal{A}$ is a Dynkin system. Thus, it follows from the Dynkin system theorem that $\mathcal{A}$ contains $\mathcal{B}(X)$, so that $h_f \in B(Y - N)$ whenever $f$ is a Borel measurable simple function on $X$.

Since each bounded, Borel measurable function is the pointwise limit of uniformly bounded sequence of Borel measurable simple functions [2, Theorem 1.5.5], it follows from property (iii) that $h_f \in B(Y - N)$ for all $f \in B(X)$, and the proof is complete. $\square$

Let $S$ be a compact Stonean space. Denote by $\mathcal{M}$ the $\sigma$-ideal of all meagre Borel subsets of $S$. Let $\kappa$ be a canonical $C(S)$-valued $\sigma$-measure on $S$ such that

($\kappa$1) $\mathcal{M}$ is the kernel of $\kappa$,

($\kappa$2) $\kappa(A) = \chi_A$ for each clopen subset $A$ of $S$.

The existence of $\kappa$ follows from [18, page 118] and $\kappa$ is called the *Birkhoff-Ulam $C(S)$-valued $\sigma$-measure* on $S$.

The following lemma has been already given in [18] implicitly.

**Lemma 5.** *Let $\kappa$ be the Birkhoff-Ulam $C(S)$-valued $\sigma$-measure on $S$. Then $\int_S f d\kappa = f$ for all $f \in C(S)$.*

*Proof.* Fix $f \in C(S)$. Let $\Phi$ be the set of all finite linear combinations with real coefficients of the characteristic functions of clopen subsets of $S$. Since $S$ is totally disconnected, $\Phi$ separates points of $S$, and it clearly contains a non-zero constant function. By the real Stone-Weierstrass theorem, $\Phi$ is dense in $C(S)$, so that there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C(S)$ such that $f_n$ uniformly converges to $f$ on $S$. Thus, it follows from [18, Proposition 3.5] that $\int_S f_n d\kappa \xrightarrow{o} \int_S f d\kappa$.

By property ($\kappa$2) we have $\int_S f d\kappa = f$ for all $f \in \Phi$, so that $f_n \xrightarrow{o} \int_S f d\kappa$. Hence $\int_S f d\kappa = f$ by [22, Theorem 15.4], and the proof is complete. $\square$

By Proposition 2 we naturally reach the following definition.

**Definition 2.** Let $X$ be a topological space and $V$ a Riesz space. We say that a positive linear mapping $T : C(X) \to V$ satisfies the *tightness condition* if there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $V$ with $p_\alpha \downarrow 0$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$ such that $T(f) \leq p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_\alpha) = \{0\}$.

We now give an analogue of the Riesz representation theorem for a Dedekind complete Riesz space-valued positive linear mapping.

**Theorem 1.** *Let $X$ be a completely regular space and $V$ a Dedekind complete Riesz space. Let $T : C(X) \to V$ be a positive linear mapping. Then the following two conditions are equivalent.*

(i) *$T$ satisfies the tightness condition.*

(ii) *There is a quasi-Radon $V$-valued $\sigma$-measure $\mu$ on $X$ such that*

$$T(f) = \int_X f d\mu \quad \text{for all } f \in C(X). \tag{2}$$

*Further, the $\mu$ is uniquely determined by* (2) *and the quasi-Radonness of $\mu$.*

*Proof.* The uniqueness of $\mu$ follows from Lemmas 3 and 4, and we shall prove the existence of $\mu$.

Put $e := T(1)$ and observe that $T$ maps $C(X)$ into the principal ideal $V_e$ generated by the element $e$. We may assume that each $p_\alpha$ in Definition 2 is an element of $V_e$. Indeed, if we put $p'_\alpha := p_\alpha \wedge e$ for each $\alpha \in \Gamma$, then the net $\{p'_\alpha\}_{\alpha\in\Gamma}$ is contained in $V_e$ and satisfies condition of Definition 2.

As we stated in Section 2, the space $V_e$ is lattice isomorphic to $C(S)$ for some compact Stonean space $S$. Every lattice isomorphism $\theta$ from a Riesz space $U$ onto another Riesz space $W$ is positive, and its inverse mapping $\theta^{-1}$ is also a lattice isomorphism from $W$ onto $U$. Further, $\theta$ preserves arbitrary supremum and infimum [1, Theorem 7.9]. Consequently, to prove the existence of $\mu$ we may assume that $V = C(S)$ with $S$ a compact Stonean space.

Let $\kappa$ be the Birkhoff-Ulam $C(S)$-valued $\sigma$-measure on $S$. Let $\tilde{T} : B(X) \to B(S)$ be a positive linear mapping constructed in Proposition 2.

For each $A \in \mathcal{B}(X)$, put $\mu(A) := \int_A \tilde{T}(\chi_A)d\kappa$. Then by the same argument used in the proof of [18, Theorem 4.1], together with Lemma 5, it can be proved that $\mu$ is a $C(S)$-valued $\sigma$-measure on $X$ such that

(a) $T(f) = \int_X f d\mu$ for all $f \in C(X)$, and
(b) $\mu(G) = \sup\{T(f) : 0 \le f \le \chi_G, f \in C(X)\}$ for every open subset $G$ of $X$.

Thus, it remains to prove that $\mu$ is quasi-Radon. To do this, by Lemma 2 we have only to prove the tightness and quasi-regularity of $\mu$.

Since $T$ satisfies the tightness condition, there are nets $\{p_\alpha\}_{\alpha\in\Gamma} \subset C(S)$ with $p_\alpha \downarrow 0$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$ such that $T(f) \le p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \le f \le 1$ and $f(K_\alpha) = \{0\}$. Thus, it follows from property (b) that

$$\mu(X - K_\alpha) = \sup\{T(f) : 0 \le f \le \chi_{X-K_\alpha}, f \in C(X)\} \le p_\alpha$$

for all $\alpha \in \Gamma$, so that $\mu$ is tight.

Next we prove the quasi-regularity of $\mu$. Let $G$ be an open subset of $X$. Fix $f \in C(X)$ with $0 \le f \le \chi_G$ and put $F_n := \{x \in X : f(x) \ge 1/n\}$ for all $n \in \mathbb{N}$. Then $\{F_n\}_{n\in\mathbb{N}}$ is an increasing sequence of closed subsets of $X$ and $\cup_{n=1}^\infty F_n = \{x \in X : f(x) > 0\} \subset G$. Thus, $T(f) = \int_X f d\mu \le \mu(\cup_{n=1}^\infty F_n) = \sup_{n\in\mathbb{N}} \mu(F_n)$, so that

$$T(f) \le \sup\{\mu(F) : F \subset G \text{ and } F \text{ is closed}\} \le \mu(G).$$

Hence, the quasi-regularity of $\mu$ follows from property (b). □

The tightness condition in the above theorem is automatically satisfied if $X$ is compact, so that Theorem 1 reduces to [18, Theorem 4.1] and a special case of [20, Theorem 4.5]. See also [19, Theorem 1]. However, our work will be needed to develop the theory of weak order convergence of Riesz space-valued $\sigma$-measures, in which we usually assume that the involved measures are defined on metric spaces or more generally on completely regular spaces.

## 4. A Borel product of $\sigma$-measures

Let $X$ and $Y$ be topological spaces. Recall that $B(X)$ is the Banach lattice of all bounded, Borel measurable, real-valued functions on $X$ with lattice norm $\|f\| := \sup_{x\in X} |f(x)|$. Similar definitions are made for $B(Y)$ and $B(X \times Y)$.

Throughout this section, let $U$, $V$, and $W$ be Dedekind complete Riesz spaces, and $\langle\cdot,\cdot\rangle$ be a bilinear mapping from $U \times W$ into $V$ that is bipositive, in other words $\langle u, w\rangle \in V^+$ whenever $u \in U^+$ and $w \in W^+$. We assume that $\langle\cdot,\cdot\rangle$ is *completely proper*, that is, the following two conditions are satisfied.

(i) Whenever $\{u_\alpha\}_{\alpha\in\Gamma}$ is an upper bounded, increasing net in $U$ and $w \in W^+$ then $\sup_{\alpha\in\Gamma} \langle u_\alpha, w\rangle = \langle \sup_{\alpha\in\Gamma} u_\alpha, w\rangle$.
(ii) Whenever $\{w_\alpha\}_{\alpha\in\Gamma}$ is an upper bounded, increasing net in $W$ and $u \in U^+$ then $\sup_{\alpha\in\Gamma} \langle u, w_\alpha\rangle = \langle u, \sup_{\alpha\in\Gamma} w_\alpha\rangle$.

The following example gives some of completely proper, bilinear and bipositive mappings defined on Dedekind complete Riesz spaces; see also [21].

**Example.** (1) Denote by $F(I)$ the Dedekind complete Riesz space of all real-valued functions on an arbitrary non-empty set $I$. Then the mapping $\langle\cdot,\cdot\rangle : F(I)\times F(I)\to F(I)$ defined by $\langle f,g\rangle := fg$ for all $f,g\in F(I)$ is completely proper.

(2) In the following, let $(\Omega,\mathcal{A},m)$ be a $\sigma$-finite measure space. Denote by $L^0(\Omega)$ the super Dedekind complete Riesz space of all $m$-measurable real-valued functions on $(\Omega,\mathcal{A},m)$. Let $L^p(\Omega)$ $(1\le p\le\infty)$ be the usual Lebesgue spaces. Since $L^p(\Omega)$ are all ideals in $L^0(\Omega)$, they are also super Dedekind complete. Further, $L^p(\Omega)$ is a Banach lattice having order continuous norm for every $1\le p<\infty$, but $L^\infty(\Omega)$ is an example of Banach lattices with norm failing to be order continuous. The following mappings are all completely proper:

- The mapping $\langle\cdot,\cdot\rangle : L^0(\Omega)\times L^0(\Omega)\to L^0(\Omega)$ defined by $\langle f,g\rangle := fg$ for all $f,g\in L^0(\Omega)$.
- Let $1\le p\le\infty$ and $1/p+1/q=1$. The mapping $\langle\cdot,\cdot\rangle : L^p(\Omega)\times L^q(\Omega)\to L^1(\Omega)$ defined by $\langle f,g\rangle := fg$ for all $f\in L^p(\Omega)$ and $g\in L^q(\Omega)$ and the mapping $\langle\cdot,\cdot\rangle : L^p(\Omega)\times L^q(\Omega)\to\mathbb{R}$ defined by $\langle f,g\rangle := \int_\Omega fg\,dm$ for all $f\in L^p(\Omega)$ and $g\in L^q(\Omega)$.
- The mapping $\langle\cdot,\cdot\rangle : L^\infty(\Omega)\times L^\infty(\Omega)\to L^\infty(\Omega)$ defined by $\langle f,g\rangle := fg$ for all $f,g\in L^\infty(\Omega)$.
- Assume that $(\Omega,\mathcal{A},m)$ is a finite measure space. The mapping $\langle\cdot,\cdot\rangle : L^\infty(\Omega)\times L^\infty(\Omega)\to\mathbb{R}$ defined by $\langle f,g\rangle := \int_\Omega fg\,dm$ for all $f,g\in L^\infty(\Omega)$.

(3) Denote by $L^1(\mathbb{R}^n)$ the Riesz space of all Lebesgue integrable real-valued functions on $\mathbb{R}^n$. Then the mapping $\langle\cdot,\cdot\rangle : L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n)$ defined by $\langle f,g\rangle := f*g$ for all $f,g\in L^1(\mathbb{R}^n)$ is completely proper, where $f*g$ denotes the convolution of $f$ and $g$.

(4) The results that are similar to (1), (2), and (3) hold for the corresponding sequence spaces $(s)$ of all real sequences and $\ell^p$ of all $p$-summable real sequences $(1\le p\le\infty)$.

(5) Let $L$, $M$, and $N$ be Riesz spaces. Assume that $M$ and $N$ are Dedekind complete. Denote by $\mathcal{L}_n(M,N)$ be the Dedekind complete Riesz space of all order continuous, order bounded, linear operators from $M$ into $N$. Similar definitions are made for $\mathcal{L}_n(L,M)$ and $\mathcal{L}_n(L,N)$. Then the mapping $\langle\cdot,\cdot\rangle : \mathcal{L}_n(M,N)\times\mathcal{L}_n(L,M)\to\mathcal{L}_n(L,N)$ defined by $\langle P,Q\rangle := PQ$ for all $P\in\mathcal{L}_n(M,N)$ and $Q\in\mathcal{L}_n(L,M)$ is completely proper.

In [21, Theorem 1.7] it is shown that given quasi-Radon $U$-valued $\sigma$-measure $\mu$ on $X$ and $W$-valued $\sigma$-measure $\nu$ on $Y$ there is a unique quasi-Radon $V$-valued $\sigma$-measure $\lambda$ on $X\times Y$ such that

$$\lambda(A\times B)=\langle\mu(A),\nu(B)\rangle$$

for all $A\in\mathcal{B}(X)$ and $B\in\mathcal{B}(Y)$ in the case that $X$ and $Y$ are locally compact. This measure $\lambda$ is called the *Borel product* of $\mu$ and $\nu$. Other important cases are also discussed in [21].

In this section we shall extend the above result to the case that $X$ and $Y$ are not necessarily locally compact. This will be established by the help of our Riesz representation theorem; see Theorem 1.

Let $\mathcal{C}$ denote the field generated by the measurable rectangles on $X\times Y$, that is, the field generated by sets of the form $A\times B$, $A\in\mathcal{B}(X)$ and $B\in\mathcal{B}(Y)$. If $\mu:\mathcal{B}(X)\to U$ and $\nu:\mathcal{B}(Y)\to W$ are finitely additive set functions, a product of $\mu$ and $\nu$ is the function defined on measurable rectangles $A\times B$ by the formula $\lambda_0(A\times B):=\langle\mu(A),\nu(B)\rangle$. Since every $C\in\mathcal{C}$ can be represented in the form $C=\cup_{i=1}^n(A_i\times B_i)$, where $n\in\mathbb{N}$, $A_i\in\mathcal{B}(X)$, $B_i\in\mathcal{B}(Y)$ $(n=1,2,\dots,n)$ and $\{A_i\times B_i\}_{i=1}^n$ are pairwise disjoint, the set function $\lambda_0$ can be extended to $\mathcal{C}$ by setting $\lambda_0(C):=\sum_{i=1}^n\langle\mu(A_i),\nu(B_i)\rangle$. This definition does not depend on the representation of $C$ and the extension (still denoted by $\lambda_0$) is finitely additive on $\mathcal{C}$.

**Lemma 6.** *Let $X$ and $Y$ be completely regular spaces. Let $\mu$ be a tight $U$-valued $\sigma$-measure on $X$ and $\nu$ a tight $W$-valued $\sigma$-measure on $Y$. Assume that $\langle\cdot,\cdot\rangle$ is completely proper.*

*Then there is a quasi-Radon $V$-valued $\sigma$-measure $\lambda$ on $X \times Y$ such that*

$$\int_{X\times Y} fgd\lambda = \left\langle \int_X fd\mu, \int_Y gd\nu \right\rangle$$

*holds for all $f \in C(X)$ and $g \in C(Y)$.*

*Proof.* Denote by $\Phi$ the set of all $\mathcal{C}$-measurable, simple functions on $X \times Y$. Let $\overline{\Phi}$ be the closure of $\Phi$ in $B(X \times Y)$ with respect to the supremum norm $\|\cdot\|$. Let $\lambda_0 : \mathcal{C} \to V$ be the finitely additive, positive set function constructed above.

For each $h = \sum_{i=1}^n a_i\chi_{C_i} \in \Phi$, where $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$, and $C_1, \dots, C_n$ are pairwise disjoint sets in $\mathcal{C}$, define a positive linear mapping $T_0 : \Phi \to V$ by $T_0(h) := \sum_{i=1}^n a_i\lambda_0(C_i)$. This definition does not depend on the representation of $h$.

Put $e_1 := \mu(X)$, $e_2 := \nu(Y)$, and $e := \lambda_0(X \times Y) = \langle e_1, e_2\rangle$. The principal ideal $V_e$ generated by the element $e$ becomes a Banach lattice with order unit norm $\|u\|_e := \inf\{r > 0 : |u| \le re\}$. Then, $T_0 : (\Phi, \|\cdot\|) \to (V_e, \|\cdot\|_e)$ is a continuous linear mapping, so that it has a unique continuous linear extension $T_1 : (\overline{\Phi}, \|\cdot\|) \to (V_e, \|\cdot\|_e)$. By the same argument in the proof of [21, Lemma 1.3], we have

$$T_1(fg) = \left\langle \int_X fd\mu, \int_Y gd\nu \right\rangle \tag{3}$$

for all $f \in B(X)$ and $g \in B(Y)$.

Since $\overline{\Phi}$ is a majorizing Riesz subspace of $B(X \times Y)$, it follows from the Kantorovič extension theorem [1, Theorem 2.8] that $T_1$ has a positive linear extension $T_2 : B(X\times Y) \to V$. Let $T$ be the restriction of $T_2$ onto the space $C(X \times Y)$. Then it follows from (3) that

$$T(fg) = \left\langle \int_X fd\mu, \int_Y gd\nu \right\rangle \tag{4}$$

for all $f \in C(X)$ and $g \in C(Y)$.

To complete the proof, we show that $T$ satisfies the tightness condition by using the complete properness of $\langle\cdot,\cdot\rangle$. By Lemma 1, the tightness of $\mu$ implies that there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $U$ with $p_\alpha \downarrow 0$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of $X$ such that $\mu(X - K_\alpha) \le p_\alpha$ for all $\alpha \in \Gamma$. Similarly, one can find nets $\{q_\beta\}_{\beta\in\Lambda}$ in $W$ with $q_\beta \downarrow 0$ and $\{L_\beta\}_{\beta\in\Lambda}$ of compact subsets of $Y$ such that $\nu(Y - L_\beta) \le q_\beta$ for all $\beta \in \Lambda$. Put $r_{\alpha,\beta} := \langle p_\alpha, e_2\rangle + \langle e_1, q_\beta\rangle$ for each $(\alpha, \beta) \in \Gamma \times \Lambda$. Since $\Gamma \times \Lambda$ is a directed set under the canonical coordinate-wise ordering, $\{r_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma\times\Lambda}$ is also a net and it is decreasing. It follows from the complete properness of $\langle\cdot,\cdot\rangle$ that $\langle p_\alpha, e_2\rangle \downarrow 0$ and $\langle e_1, q_\beta\rangle \downarrow 0$, so that $r_{\alpha,\beta} \downarrow 0$ [15, Theorem 15.8].

Assume that $(\alpha, \beta) \in \Gamma \times \Lambda$ and $h \in C(X \times Y)$ with $0 \le h \le 1$ and $h(K_\alpha \times L_\beta) = \{0\}$. Then we have

$$h = \chi_{K_\alpha\times L_\beta}h + \chi_{(K_\alpha\times L_\beta)^c}h = \chi_{(K_\alpha\times L_\beta)^c}h \le \chi_{X-K_\alpha} \cdot 1 + 1 \cdot \chi_{Y-L_\beta}.$$

Hence we have

$$\begin{aligned} T(h) &\le T_2\left(\chi_{X-K_\alpha} \cdot 1 + 1 \cdot \chi_{Y-L_\beta}\right) \\ &= T_1\left(\chi_{X-K_\alpha} \cdot 1\right) + T_1\left(1 \cdot \chi_{Y-L_\beta}\right) \\ &= \langle \mu(X - K_\alpha), \nu(Y)\rangle + \langle \mu(X), \nu(Y - L_\beta)\rangle \\ &\le \langle p_\alpha, e_2\rangle + \langle e_1, q_\beta\rangle = r_{\alpha,\beta}, \end{aligned}$$

so that $T$ satisfies the tightness condition. Thus, it follows from Theorem 1 that there is a quasi-Radon $V$-valued $\sigma$-measure $\lambda$ on $X \times Y$ such that

$$T(h) = \int_{X\times Y} h d\lambda \tag{5}$$

for all $h \in C(X \times Y)$. Since $fg \in C(X \times Y)$ whenever $f \in C(X)$ and $g \in C(Y)$, it follows from (4) and (5) that

$$\int_{X\times Y} fg d\lambda = \left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle,$$

and the proof is complete. □

We are now ready to show the existence and uniqueness of a Borel product of $\sigma$-measures on completely regular spaces.

**Theorem 2.** *Let $X$ and $Y$ be completely regular spaces. Let $\mu$ be a quasi-Radon $U$-valued $\sigma$-measure on $X$ and $\nu$ a quasi-Radon $W$-valued $\sigma$-measure on $Y$. Assume that $\langle \cdot, \cdot \rangle$ is completely proper. Then there is a unique quasi-Radon $V$-valued $\sigma$-measure $\lambda$ on $X \times Y$ such that*

$$\lambda(A \times B) = \langle \mu(A), \nu(B) \rangle$$

*for all $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$. Further, we have*

$$\int_{X\times Y} fg d\lambda = \left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle$$

*for all $f \in B(X)$ and $g \in B(Y)$.*

*Proof.* By lemma 6, there is a quasi-Radon $V$-valued $\sigma$-measure $\lambda$ on $X \times Y$ such that

$$\int_{X\times Y} fg d\lambda = \left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle \tag{6}$$

for all $f \in C(X)$ and $g \in C(Y)$.

Fix $f \in C(X)$ with $f \geq 0$. Let $g$ be a positive, bounded, lower semicontinuous real-valued function on $Y$. Then there is an increasing net $\{g_\alpha\}_{\alpha\in\Gamma}$ of continuous functions on $Y$ such that $0 \leq g_\alpha \leq g$ for all $\alpha \in \Gamma$ and $g(x) = \sup_{\alpha\in\Gamma} g_\alpha(x)$ for all $x \in X$. Since $\nu$ is $\tau$-smooth by Lemma 3, it follows from Proposition 1 that $\int_Y g d\nu = \sup_{\alpha\in\Gamma} \int_Y g_\alpha d\nu$. Thus, it follows from the complete properness of $\langle \cdot, \cdot \rangle$ that

$$\left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle = \sup_{\alpha\in\Gamma} \left\langle \int_X f d\mu, \int_Y g_\alpha d\nu \right\rangle. \tag{7}$$

On the other hand, $\{fg_\alpha\}_{\alpha\in\Gamma}$ is also an increasing net of continuous functions on $X \times Y$ such that $0 \leq fg_\alpha \leq fg$ for all $\alpha \in \Gamma$ and $(fg)(x, y) = \sup_{\alpha\in\Gamma}(fg_\alpha)(x, y)$ for all $(x, y) \in X \times Y$. Hence it follows from Proposition 1 and the $\tau$-smoothness of $\lambda$ that

$$\int_{X\times Y} fg d\lambda = \sup_{\alpha\in\Gamma} \int_{X\times Y} fg_\alpha d\lambda. \tag{8}$$

Thus, by (6)–(8) we have

$$\int_{X\times Y} fg d\lambda = \left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle.$$

The same argument as above yields the assertion that

$$\int_{X\times Y} fg d\lambda = \left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle$$

whenever $f$ and $g$ are positive, bounded, lower semicontinuous real-valued functions on $X$ and $Y$, respectively.

Put $\mathcal{B} := \{B \in \mathcal{B}(X) : \lambda(G \times B) = \langle \mu(G), \nu(B) \rangle$ for any open subset $G$ of $X\}$. By the preceding paragraph, $\mathcal{B}$ contains all open subsets of $Y$ and it is easy to check that $\mathcal{B}$ is a Dynkin system. Hence it follows from the Dynkin system theorem that $\mathcal{B}$ contains $\mathcal{B}(Y)$, so that $\lambda(G \times B) = \langle \mu(G), \nu(B) \rangle$ whenever $G$ is an open subset of $X$ and $B \in \mathcal{B}(Y)$.

The same argument as above yields the assertion that $\lambda(A \times B) = \langle \mu(A), \nu(B) \rangle$ whenever $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$, and it follows from a standard argument in measure theory that

$$\int_{X \times Y} fg d\lambda = \left\langle \int_X f d\mu, \int_Y g d\nu \right\rangle$$

whenever $f \in B(X)$ and $g \in B(Y)$.

Finally we prove the uniqueness of $\lambda$. Let $\lambda'$ be a quasi-Radon $V$-valued $\sigma$-measure on $X \times Y$ such that $\lambda'(A \times B) = \langle \mu(A), \nu(B) \rangle$ for all $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$, and we shall prove that $\lambda = \lambda'$ on $\mathcal{B}(X \times Y)$

If $C$ is a finite union of rectangles on $X \times Y$, then $\lambda(C) = \lambda'(C)$ by the preceding paragraph. Since the sets of the form $G \times H$, where $G$ and $H$ are open subsets of $X$ and $Y$ respectively, form a basis of the topology of $X \times Y$, any open subset $J$ of $X \times Y$ can be represented in the form $J = \cup_{\alpha \in \Gamma} J_\alpha$, where $\{J_\alpha\}_{\alpha \in \Gamma}$ is an increasing net of open subsets of $X \times Y$, each of which is a union of finitely many rectangles. Hence it follows from the $\tau$-smoothness of $\lambda$ and $\lambda'$ that

$$\lambda(J) = \sup_{\alpha \in \Gamma} \lambda(J_\alpha) = \sup_{\alpha \in \Gamma} \lambda'(J_\alpha) = \lambda'(J).$$

Put $\mathcal{D} := \{C \in \mathcal{B}(X \times Y) : \lambda(C) = \lambda'(C)\}$. The preceding paragraph implies that $\mathcal{D}$ contains all open subsets of $X \times Y$ and it is easily verified that $\mathcal{D}$ is a Dynkin system. Hence $\mathcal{D}$ contains $\mathcal{B}(X \times Y)$, and this implies the uniqueness of $\lambda$. □

**Definition 3.** The $\sigma$-measure $\lambda$ given in Theorem 2 is called the *Borel product* of $\mu$ and $\nu$, and is denoted by $\mu \times \nu$.

## 5. Joint continuity of the operation making the Borel product

In this section we shall show that the operation of making the Borel product of two $\sigma$-measures is jointly continuous with respect to weak order convergence of measures.

Let $X$ be a topological space and $V$ a Dedekind complete Riesz space. To formulate our results, the following definitions are needed.

**Definition 4.** A net $\{\mu_\alpha\}_{\alpha \in \Gamma}$ of $V$-valued $\sigma$-measures on $X$ is said to *weakly converge in order* to a $V$-valued $\sigma$-measure $\mu$ on $X$, and is denoted by $\mu_\alpha \xrightarrow{wo} \mu$, if $\int_X f d\mu_\alpha \xrightarrow{o} \int_X f d\mu$ for each $f \in C(X)$.

**Definition 5** (Lipecki [14])**.** A $V$-valued $\sigma$-measure $\mu$ on $X$ satisfies the *countable chain condition* (shortly, CCC) if every family $\mathcal{D}$ of pairwise disjoint Borel subsets of $X$ such that $\mu(D) \neq 0$ for all $D \in \mathcal{D}$ is countable.

If $V$ is super Dedekind complete, that is, it is Dedekind complete and every set in $V$ possessing a supremum contains an at most countable subset having the same supremum, then every $V$-valued $\sigma$-measure on $S$ satisfies (CCC). A $V$-valued $\sigma$-measure $\mu$ on $X$ is said to be *dominated* if there is a finitely additive, positive set function $m : \mathcal{B}(X) \to \mathbb{R}$ such that $\mu(A) = 0$ whenever $A \in \mathcal{B}(X)$ and $m(A) = 0$. Every dominated $V$-valued $\sigma$-measure on $X$ also satisfies (CCC).

A Borel subset $A$ of $X$ is called a *$\mu$-continuity set* if $\mu(\partial A) = 0$, where $\partial A$ denotes the boundary of $A$. Denote by $\mathcal{G}_\mu$ the set of all $\mu$-continuity Borel subsets of $X$. The proof of the following lemma is an easy modification of [17, Lemma I.3.2] and so it is omitted.

**Lemma 7.** *Let $X$ be a completely regular space and $V$ a Dedekind complete Riesz space. Let $\mu$ be a $V$-valued $\sigma$-measure on $X$.*

(i) *$\mathcal{G}_\mu$ is a subfield of $\mathcal{B}(X)$.*

(ii) *If $\mu$ satisfies (CCC) then $\mathcal{G}_\mu$ contains a basis of the topology of $X$.*

For a uniform space $X$, denote by $U(X)$ the space of all bounded, uniformly continuous, real-valued functions on $X$. The following is a Riesz space version of the Portmanteau Theorem and was first proved in [3, Theorems 5.2 and 5.6] for a sequence of Dedekind complete Riesz space-valued means on a normal space. See [11, Theorem 7] for the proof of Theorem 3.

**Theorem 3** (The Portmanteau Theorem)**.** *Let $X$ be a completely regular space and $V$ a Dedekind complete Riesz space. Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net of $V$-valued $\sigma$-measures on $X$ which is uniformly order bounded, that is, there is an element $u \in V$ such that $\mu_\alpha(X) \le u$ for all $\alpha \in \Gamma$. Let $\mu$ be a $V$-valued $\sigma$-measure on $X$. Assume that $\mu$ is $\tau$-smooth. Then the following conditions* (i)–(iii) *are equivalent.*

(i) *$\mu_\alpha \xrightarrow{wo} \mu$.*

(ii) *$\mu(G) \le \liminf \mu_\alpha(G)$ for every open subset $G$ of $X$ and $\mu_\alpha(X) \xrightarrow{o} \mu(X)$.*

(iii) *$\limsup \mu_\alpha(F) \le \mu(F)$ for every closed subset $F$ of $X$ and $\mu_\alpha(X) \xrightarrow{o} \mu(X)$.*

*Each of the above conditions implies the condition*

(iv) *$\mu_\alpha(A) \xrightarrow{o} \mu(A)$ for every $\mu$-continuity Borel subset $A$ of $X$.*

*Further, if $\mu$ satisfies (CCC), then all four conditions given above are equivalent.*

*If $X$ is a uniform space, then we can add the following condition equivalent to conditions* (i)–(iii)*.*

(v) *$\int_X f d\mu_\alpha \xrightarrow{o} \int_X f d\mu$ for every $f \in U(X)$.*

With the help of our Riesz space version of the Portmanteau Theorem the following lemma can be proved exactly by the same way as in the scalar case; see [17, Corollary 1 to Theorem I.3.5].

**Lemma 8.** *Let $X$ be a completely regular space and $V$ a Dedekind complete Riesz space. Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net of $V$-valued $\sigma$-measures on $X$ and $\mu$ a $\tau$-smooth $V$-valued $\sigma$-measure on $X$. Let $\mathcal{H}$ be a basis of the topology of $X$ that contains $X$ and is closed under finite intersections. If $\mu_\alpha(H) \xrightarrow{o} \mu(H)$ for each $H \in \mathcal{H}$ then $\mu_\alpha \xrightarrow{wo} \mu$.*

In the rest of the paper, $X$ and $Y$ are completely regular spaces; $U$, $V$, and $W$ are Dedekind complete Riesz spaces with completely proper, bipositive, bilinear mapping $\langle\cdot,\cdot\rangle : U \times W \to V$.

**Lemma 9.** *Let $\{u_\alpha\}_{\alpha\in\Gamma}$ be a net in $U$ and $u \in U$. Let $\{w_\beta\}_{\beta\in\Lambda}$ be a net in $W$ and $w \in W$. Assume that either $\{u_\alpha\}_{\alpha\in\Gamma}$ or $\{w_\beta\}_{\beta\in\Lambda}$ is order bounded. If $u_\alpha \xrightarrow{o} u$ and $w_\beta \xrightarrow{o} w$ then $\langle u_\alpha, w_\beta\rangle \xrightarrow{o} \langle u, w\rangle$.*

*Proof.* We first claim the inequality

$$|\langle u, w\rangle| \le 3\,\langle |u|, |w|\rangle \tag{9}$$

holds for each $u \in U$ and $w \in W$. Indeed, the positivity of $\langle\cdot,\cdot\rangle$ implies that

$$\begin{aligned}\langle u, w\rangle + 3\,\langle |u|, |w|\rangle &\ge \langle u, w\rangle + \langle u, |w|\rangle + \langle |u|, w\rangle + \langle |u|, |w|\rangle \\ &= \langle u + |u|, w + |w|\rangle \ge 0,\end{aligned}$$

so that $\langle u, w\rangle \geq -3\,\langle |u|, |w|\rangle$.

On the other hand, it follows from $\langle |u| - u, w + |w|\rangle \geq 0$ that

$$\begin{aligned}\langle u, w\rangle &\leq \langle |u|, w\rangle + \langle -u, |w|\rangle + \langle |u|, |w|\rangle \\ &\leq \langle |u|, |w|\rangle + \langle |u|, |w|\rangle + \langle |u|, |w|\rangle \\ &= 3\,\langle |u|, |w|\rangle\,.\end{aligned}$$

Thus, the inequality (9) has been proved.

Assume that $\{w_\beta\}_{\beta\in\Lambda}$ is order bounded. Put $w_0 := \sup_{\beta\in\Lambda} |w_\beta|$. By assumption, there are nets $\{p_\alpha\}_{\alpha\in\Gamma}$ in $U$ with $p_\alpha \downarrow 0$ and $\{q_\beta\}_{\beta\in\Lambda}$ in $W$ with $q_\beta \downarrow 0$ such that $|u_\alpha - u| \leq p_\alpha$ for all $\alpha \in \Gamma$ and $|w_\beta - w| \leq q_\beta$ for all $\beta \in \Lambda$. Then it follows from (9) that

$$\begin{aligned}|\langle u_\alpha, w_\beta\rangle - \langle u, w\rangle| &\leq |\langle u_\alpha - u, w_\beta\rangle| + |\langle u, w_\beta - w\rangle| \\ &\leq 3\,\langle |u_\alpha - u|, |w_\beta|\rangle + 3\,\langle |u|, |w_\beta - w|\rangle \\ &\leq 3\,\langle p_\alpha, w_0\rangle + 3\,\langle |u|, q_\beta\rangle\,.\end{aligned}$$

Since $p_\alpha \downarrow 0$ and $q_\beta \downarrow 0$, the complete properness of $\langle\cdot,\cdot\rangle$ implies that $\langle p_\alpha, w_0\rangle \downarrow 0$ and $\langle |u|, q_\beta\rangle \downarrow 0$. Thus, it follows from [15, Theorem 15.8] that $3\,\langle p_\alpha, w_0\rangle + 3\,\langle |u|, q_\beta\rangle \downarrow 0$, so that $\langle u_\alpha, w_\beta\rangle \xrightarrow{o} \langle u, w\rangle$.

The proof is similar in the case that $\{u_\alpha\}_{\alpha\in\Gamma}$ is order bounded. □

The following theorem insists that the operation making the Borel product of two $\sigma$-measures is jointly continuous with respect to weak order convergence of measures. Its proof needs our Riesz space version of the Portmanteau Theorem and Lemmas in this section.

**Theorem 4.** *Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net of quasi-Radon $U$-valued $\sigma$-measures on $X$. Let $\{\nu_\beta\}_{\beta\in\Lambda}$ be a net of quasi-Radon $W$-valued $\sigma$-measures on $Y$. Assume that either the set $\{\mu_\alpha(X)\}_{\alpha\in\Gamma}$ or the set $\{\nu_\beta(Y)\}_{\beta\in\Lambda}$ is order bounded. Let $\mu$ be a quasi-Radon $U$-valued $\sigma$-measure on $X$ and $\nu$ a quasi-Radon $W$-valued $\sigma$-measure on $Y$. Assume that both $\mu$ and $\nu$ satisfy (CCC). Then, $\mu_\alpha \times \nu_\beta \xrightarrow{wo} \mu \times \nu$ whenever $\mu_\alpha \xrightarrow{wo} \mu$ and $\nu_\beta \xrightarrow{wo} \nu$.*

*Proof.* Assume that the set $\{\nu_\beta(Y)\}_{\beta\in\Lambda}$ is order bounded. Denote by $\mathcal{P}$ and $\mathcal{Q}$ the set of all open subsets $G$ of $X$ such that $\mu(\partial G) = 0$ and the set of all open subsets of $Y$ such that $\nu(\partial H) = 0$, respectively. It follows from Lemma 7 that $\mathcal{P}$ contains $X$ and a basis of the topology of $X$. Similarly, $\mathcal{Q}$ contains $Y$ and a basis of the topology of $Y$.

Put $\mathcal{R} := \{G \times H : G \in \mathcal{P}, H \in \mathcal{Q}\}$. Then $\mathcal{R}$ contains $X \times Y$ and a basis of the product topology of $X \times Y$. Further, it is closed under finite intersections. By assumption and Theorem 3 we have $\mu_\alpha(G) \xrightarrow{o} \mu(G)$ for all $G \in \mathcal{P}$ and $\nu_\beta(H) \xrightarrow{o} \nu(H)$ for all $H \in \mathcal{Q}$. Thus, it follows from Lemma 9 that

$$(\mu_\alpha \times \nu_\beta)(G \times H) = \langle \mu_\alpha(G), \nu_\beta(H)\rangle \xrightarrow{o} \langle \mu(G), \nu(H)\rangle = (\mu \times \nu)(G \times H)$$

for all $G \in \mathcal{P}$ and $H \in \mathcal{Q}$. This implies $\mu_\alpha \times \nu_\beta \xrightarrow{wo} \mu \times \nu$ by Lemma 8.

The proof is similar in the case that the set $\{\mu_\alpha(X)\}_{\alpha\in\Gamma}$ is order bounded. □

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