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A NOTE ON SUBTRACTION SEMIGROUPS

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Abstract. In this paper, we define an ideal of a subtraction semigroup and a strong subtraction semigroup and characterizations of ideals is given. We prove that $x \wedge y$ is the greatest lower bound of x and y in subtraction semigroup X. Also we define a congruence relation on a subtraction semigroup and a quotient subtraction semigroup and prove the isomorphisms.

1. Introduction

B. M. Schein [2] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [3] discussed a problem proposed by B. M. Schein [2] concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we define an ideal of a subtraction semigroup and a strong subtraction semigroup and characterizations of ideals is given. We prove that $x \wedge y$ is the greatest lower bound of x and y in subtraction semigroup X. Also we define a congruence relation on a subtraction semigroup and a quotient subtraction semigroup and prove the isomorphisms.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(SA1) $x - (y - x) = x$;
(SA2) $x - (x - y) = y - (y - x)$;
(SA3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following hold:

(S1) $x - 0 = x$ and $0 - x = 0$.

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(S2) $x - (x - y) \leq y$.
(S3) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(S4) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(S5) $x - (x - (x - y)) = x - y$.

By a *subtraction semigroup* we mean an algebra $(X; \cdot, -)$ with two binary operations "$-$" and "$\cdot$"that satisfies the following axioms: for any $x, y, z \in X$,

(SS1) $(X; \cdot)$ is a semigroup;
(SS2) $(X; -)$ is a subtraction algebra;
(SS3) $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$.

A subtraction semigroup is said to be *multiplicatively abelian* if multiplication is commutative.

Example 2.1. Let $X = \{0, 1\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	1
0	0	0
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

It is easy to check that X is a subtraction semigroup.

Lemma 2.2. *Let X be a subtraction semigroup. Then the following hold.*

(1) $x0 = 0$ *and* $0x = 0$
(2) $x \leq y$ *implies* $ax \leq ay$ *and* $xa \leq ya$.
(3) $x(y \wedge z) = xy \wedge xz$ *and* $(x \wedge y)z = xz \wedge yz$

Proof. (1) $x0 = x(0 - 0) = x0 - x0 = 0$ and $0x = (0 - 0)x = 0x - 0x = 0$.

(2) Let $x \leq y$. Then we have $x - y = 0$, and so

$$ax - ay = a(x - y) = a0 = 0.$$

Hence $ax \leq ay$. Similarly, we have $xa \leq ya$.

(3) $x(y \wedge z) = x(y - (y - z)) = xy - x(y - z) = xy - (xy - xz) = xy \wedge xz$. Similarly, we have $(x \wedge y)z = xz \wedge yz$. □

Lemma 2.3. *Let X be a subtraction semigroup. Then $(X; \leq)$ is a poset, where $x \leq y \Leftrightarrow x - y = 0$ for any $x, y \in X$.*

Proof. For any $x \in X$, we have $x \leq x$ since $x - x = 0$. Thus $\leq$ is reflexive.

Let $x, y \in X$ be such that $x \leq y$ and $y \leq x$. Then $x - y = 0$ and $y - x = 0$. Thus by (SA1) and (SA2) and (S1), we have $x = x - (y - x) = x - 0 = x - (x - y) = y - (y - x) = y - 0 = y$. Hence $\leq$ is antisymmetry.

Let $x, y, z \in X$ be such that $x \leq y$ and $y \leq z$. Then by (S4), we have $x - z \leq y - z = 0$. Thus we get $x - z = 0$ by (S1). Hence $\leq$ is transitivity. □

Proposition 2.4. *Let X be a subtraction semigroup. Then for any $x, y \in X$, $x \wedge y$ is the greatest lower bound of x and y.*

Proof. Let $x, y \in X$. Then since $x \wedge y = x - (x - y) = y - (y - x) \leq x, y$ from (S2), $x \wedge y$ is a lower bound of x and y.

If z is a lower bound of x and y, then $z - y = 0$ and $z = x - w$ for some $w \in X$ from (S5), and hence

$$\begin{aligned} z - (x \wedge y) &= z - (x - (x - y)) \\ &= (x - w) - (x - (x - y)) \\ &= (x - (x - (x - y))) - w \\ &= (x - y) - w \quad \text{(from (S5))} \\ &= (x - w) - y \\ &= z - y \\ &= 0. \end{aligned}$$

It follows $z \leq x \wedge y$, and so $x \wedge y$ is the greatest lower bound of x and y. □

3. Ideals of subtraction semigroup

Definition 3.1. Let X be a subtraction semigroup. A subalgebra I of $(X, -)$ is called a *left ideal* of X if $XI \subseteq I$, a *right ideal* if $IX \subseteq I$, and an (*two-sided*) *ideal* if it is both a left and right ideal.

Example 3.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	0
5	0	0	5	0	0	5

It is easy to check that $(X; -, \cdot)$ is a subtraction semigroup. Let $I = \{0, 1, 3, 4\}$. Then I is an ideal of X.

Example 3.3. Let X be a subtraction semigroup and $a \in X$. Then

$$Xa = \{xa \mid x \in X\}$$

is a left ideal of X.

Proof. Let $xa, ya \in Xa$. Then $xa - ya = (x - y)a \in Xa$, and so Xa is a subalgebra of $(X, -)$. Let $xa \in Xa$ and $z \in X$. Then $z(xa) = (zx)a \in Xa$, which shows that $X(Xa) \subseteq Xa$. Therefore, Xa is a left ideal. □

Let S be a subset of a subtraction semigroup X. The *ideal* of X generated by S is the intersection of all ideals in X containing S. The element 1 is called a *unity* in a subtraction semigroup X if $1x = x1 = x$ for all $x \in X$.

Definition 3.4. A *strong subtraction semigroup* is a subtraction semigroup X that satisfies the following condition : for each $x, y \in X$,

$$x - y = x - xy.$$

If X ia a strong subtraction semigroup with a unity 1, then 1 is the greatest element in X since $x - 1 = x - x1 = x - x = 0$ for all $x \in X$.

Example 3.5. In Example 3.2, if $\cdot$ is defined by $x \cdot y = 0$ for all $x, y \in X$, then $x \cdot y \neq x \wedge y$ in general.

Example 3.6. Let $X = \{0, a, b, 1\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

It is easy to check that $(X; -, \cdot)$ is a strong subtraction semigroup with unity 1.

Lemma 3.7. *Let X be a strong subtraction semigroup. Then*

(1) *$xy \leq y$ for all $x, y \in X$,*
(2) *$x \leq y$, $x, y \in X$ if and only if $x \leq xy$.*

Proof. (1) For any $x, y \in X$, $xy - y = xy - (xy)y = xy - x(yy) = x(y - yy) = x(y - y) = 0$.

(2) It is easy to show from the definition of strong subtraction semigroup and the above (1). □

Theorem 3.8. *Let $(X, -, \cdot)$ be a strong subtraction semigroup and I a subalgebra of $(X, -)$. Then the followings are equivalent :*

(1) *I is an ideal in $(X, -, \cdot)$,*
(2) *$y \in I$ and $x \leq y$ imply $x \in I$.*

Proof. Suppose that I is an ideal in X, and let $y \in I$ and $x \leq y$. Then $x = y - w$ for some $w \in X$ from Lemma 2.2 and (S5), and so $x = y - w = y - yw \in I$

Conversely, Suppose that $y \in I$ and $x \leq y$ imply $x \in I$. If $s \in X$ and $a \in I$, then by the Lemma 3.7,(1), $sa \leq a \in I$, hence $sa \in I$. Since $s \leq s$ and $s \leq sa$ from Lemma 3.7,(2), we have

$$as - a = as - (as)a = as - a(sa) = a(s - sa) = a0 = 0,$$

and $as \leq a \in I$, and hence $as \in I$. This completes the proof.

□

Theorem 3.9. *If X is a strong subtraction semigroup, then the principal ideal generated by $a \in X$ is $(a] = \{x \in X \mid x \leq a\}$.*

Proof. Let $x, y \in (a]$. Since $(x - y) - a = (x - a) - y = 0 - y = 0$, $x - y \leq a$ and $x - y \in (a]$, and hence $(a]$ is a subalgebra of X. From the Theorem 3.8, $(a]$ is an ideal in X.

If J is an ideal containing a and $x \in (a]$, then $x \leq a \in J$. Since J is an ideal, $x \in J$ from the Theorem 3.8. Hence $(a] \subseteq J$ and it follows that $(a]$ is the principal ideal generated by a. □

If X is a strong subtraction semigroup with 1, then the principal ideal generated by a is $(a] = Xa$.

Theorem 3.10. *Let X be a strong subtraction semigroup with a unity 1. Then the following are equivalent :*

(1) *I is an ideal in X,*
(2) *$y \in I$ and $x \leq y$ imply $x \in I$.*

Proof. Let I be an ideal in X, and let $y \in I$ and $x \leq y$. Then $x = y - w$ for some $w \in X$, hence $x = y - w = y - yw \in I$

Suppose that $y \in I$ and $x \leq y$ imply $x \in I$. If $x, y \in I$, then $x - y \in I$, since $x - y = x - xy = x(1 - y) \leq x \cdot 1 = x \in I$. Hence I is a subalgebra of X. Let $s \in X$ and $a \in I$. Then

$$as - a = a(s - 1) = a0 = 0$$

and

$$sa - a = (s - 1)a = 0a = 0,$$

hence $as \leq a$ and $sa \leq a$, that is, $IX \subseteq I$ and $XI \subseteq I$. It follows that I is an ideal in X. □

Theorem 3.11. *Let X be a strong subtraction semigroup with 1. Then we have*

$$x \wedge y = xy.$$

Proof. For any $x, y \in X$, we have

$$\begin{aligned} x \wedge y = x - (x - y) &= x - (x - xy) = xy - (xy - x) \\ &= xy - x(y - 1) = xy - x0 \\ &= xy - 0 = xy \end{aligned}$$

□

Corollary 3.12. *If X is a strong subtraction semigroup with 1, then $ss = s$ for all $s \in X$, i.e, X is a multiplicatively abelian idempotent subtraction semigroup.*

Lemma 3.13. *Let X be a strong subtraction semigroup with 1. Then the set*

$$ann(a) = \{x \in X \mid x \wedge a = 0, a \in X\}$$

is an ideal of X.

Proof. Let $x, y \in ann(a)$. Then we have $x \wedge a = xa = 0$ and $y \wedge a = ya = 0$. Hence we get $(x - y) \wedge a = (x - y)a = xa - ya = 0 - 0 = 0$, and so $x - y \in ann(a)$. Also, let $x \in ann(a)$ and $s \in X$. Then we obtain $x \wedge a = xa = 0$, and so, $sx \wedge a = (sx)a = s(xa) = s0 = 0$. Thus $sx \in ann(a)$. Similarly, we have $xs \in ann(a)$. This completes the proof. □

Let X be a strong subtraction semigroup. If $s \leq t$ for all $s, t \in X$, then we have $ann(s) \subseteq ann(t)$.

4. Congruence relation and Isomorphism theorem

In what follows, let X denote a subtraction semigroup unless otherwise specified.

Definition 4.1. Let X be a subtraction semigroup and let ρ be a binary relation on X. Then

(1) ρ is said to be *right* (resp. *left*) *compatible* if whenever $(x, y) \in \rho$ then $(x - z, y - z) \in \rho$ (resp. $(z - x, z - y) \in \rho$) and $(xz, yz) \in \rho$ (resp. $(zx, zy) \in \rho$) for all $x, y, z \in X$;
(2) ρ is said to be *compatible* if $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x - u, y - v) \in \rho$ and $(xu, yv) \in \rho$ for all $x, y, u, v \in X$,
(3) A compatible equivalence relation is called a *congruence relation.*

Using the notion of left (resp. right) compatible relation, we give a characterization of a congruence relation.

Theorem 4.2. *Let X be a subtraction semigroup. Then an equivalence relation ρ on X is congruence if and only if it is both left and right compatible.*

Proof. Assume that ρ is a congruence relation on X. Let $x, y \in X$ be such that $(x, y) \in \rho$. Note that $(z, z) \in \rho$ for all $z \in X$ because ρ is reflexive. It follows from the compatibility of ρ that $(x - z, y - z) \in \rho$ and $(xz, yz) \in \rho$. Hence ρ is right compatible. Similarly, ρ is left compatible.

Conversely, suppose that ρ is both left and right compatible. Let $x, y, u, v \in X$ be such that $(x, y) \in \rho$ and $(u, v) \in \rho$. Then $(x - u, y - u) \in \rho$ and $(xu, yu) \in \rho$. by the right compatibility. Using the left compatibility of ρ, we have $(y - u, y - v) \in \rho$ and $(yu, yv) \in \rho$. It follows from the transitivity of ρ that $(x - u, y - v) \in \rho$ and $(xu, yv) \in \rho$. Hence ρ is congruence. □

For a binary relation ρ on a subtraction semigroup X, we denote

$$x\rho := \{y \in X \mid (x, y) \in \rho\} \text{ and } X/\rho := \{x\rho \mid x \in X\}.$$

Theorem 4.3. *Let ρ be a congruence relation on a subtraction semigroup X. Then X/ρ is a subtraction semigroup under the operations*

$$x\rho - y\rho = (x - y)\rho \text{ and } (x\rho)(y\rho) = (xy)\rho$$

for all $x\rho, y\rho \in X/\rho$.

Proof. Since ρ is a congruence relation, the operations are well-defined. Clearly, $(X/\rho, -)$ is a subtraction algebra and $(X/\rho, \cdot)$ is a semigroup. For every $x\rho, y\rho, z\rho \in X/\rho$, we have

$$\begin{aligned} x\rho(y\rho - z\rho) &= x\rho(y - z)\rho = x(y - z)\rho \\ &= (xy - xz)\rho = (xy)\rho - (xz)\rho \\ &= x\rho y\rho - x\rho z\rho, \end{aligned}$$

and

$$\begin{aligned} (x\rho - y\rho)z\rho &= (x - y)\rho z\rho = ((x - y)z)\rho \\ &= (xz - yz)\rho = (xz)\rho - (yz)\rho \\ &= x\rho z\rho - y\rho z\rho. \end{aligned}$$

Thus X/ρ is a subtraction semigroup. □

Definition 4.4. Let X and X' be subtraction semigroups. A mapping $f : X \to X'$ is called a *subtraction semigroup homomorphism* (briefly, *homomorphism*) if $f(x - y) = f(x) - f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$.

Lemma 4.5. *Let $f : X \to X'$ be a subtraction semigroup homomorphism. Then*
(1) $f(0) = 0$,
(2) $x \le y$ *imply* $f(x) \le f(y)$.
(3) $f(x \wedge y) = f(x) \wedge f(y)$.

Proof. (1). Suppose that x is an element of X. Then

$$f(0) = f(x - x) = f(x) - f(x) = 0$$

(2) Let $x \le y$. Then we have $x - y = 0$. Thus we have

$$0 = f(x - y) = f(x) - f(y),$$

and so $f(x) \le f(y)$.

(3) $f(x \wedge y) = f(x - (x - y)) = f(x) - (f(x) - f(y)) = f(x) \wedge f(y)$. □

Proposition 4.6. *Let $f : X \to X'$ be a subtraction semigroup homomorphism and $J = f^{-1}(0) = \{0\}$. Then $f(x) \le f(y)$ imply $x \le y$.*

Proof. If $f(x) \le f(y)$, then we have $f(x) - f(y) = f(x - y) = 0$, and so $x - y$ is an element of J. Hence $x - y = 0$, and so we obtain $x \le y$. □

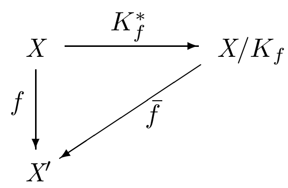
Theorem 4.7. *Let ρ be a congruence relation on a subtraction semigroup X. Then the mapping $\rho^* : X \to X/\rho$ defined by $\rho^*(x) = x\rho$ for all $x \in X$ is a subtraction semigroup homomorphism.*

Proof. Let $x, y \in X$. Then $\rho^*(x - y) = (x - y)\rho = x\rho - y\rho = \rho^*(x) - \rho^*(y)$, and $\rho^*(xy) = (xy)\rho = (x\rho)(y\rho) = \rho^*(x)\rho^*(y)$. Hence ρ^* is a subtraction semigroup homomorphism. □

Theorem 4.8. *Let X and X' be subtraction semigroups and let $f : X \to X'$ be a subtraction semigroup homomorphism. Then the set*

$$K_f := \{(x, y) \in X \times X \mid f(x) = f(y)\}$$

is a congruence relation on X and there exists a unique 1-1 subtraction semigroup homomorphism $\bar{f} : X/K_f \to X'$ such that $\bar{f} \circ K_f^* = f$, where $K_f^* : X \to X/K_f$. That is, the following diagram commute:



Proof. It is clear that K_f is an equivalence relation on X. Let $x, y, u, v \in X$ be such that $(x, y), (u, v) \in K_f$. Then $f(x) = f(y)$ and $f(u) = f(v)$, which imply that

$$f(x - u) = f(x) - f(u) = f(y) - f(v) = f(y - v)$$

and

$$f(xu) = f(x)f(u) = f(y)f(v) = f(yv).$$

It follows that $(x - u, y - v) \in K_f$ and $(xu, yv) \in K_f$. Hence K_f is a congruence relation on X. Let $\bar{f} : X/K_f \to X'$ be a map defined by $\bar{f}(xK_f) = f(x)$ for all $x \in X$. It is clear that $\bar{f}$ is well-defined. For any xK_f, $yK_f \in X/K_f$, we have

$$\begin{aligned}\bar{f}(xK_f - yK_f) &= \bar{f}((x - y)K_f) = f(x - y)\\ &= f(x) - f(y) = \bar{f}(xK_f) - \bar{f}(yK_f)\end{aligned}$$

and

$$\begin{aligned}\bar{f}((xK_f)(yK_f)) &= \bar{f}((xy)K_f) = f(xy)\\ &= f(x)f(y) = \bar{f}(xK_f)\bar{f}(yK_f).\end{aligned}$$

If $\bar{f}(xK_f) = \bar{f}(yK_f)$, then $f(x) = f(y)$ and so $(x, y) \in K_f$, that is, $xK_f = yK_f$. Thus $\bar{f}$ is a 1-1 subtraction semigroup homomorphism. Now let g be a subtraction semigroup homomorphism from X/K_f to X' such that $g \circ K_f^* = f$. Then

$$g(xK_f) = g(K_f^*(x)) = f(x) = \bar{f}(xK_f)$$

for all $xK_f \in X/K_f$. It follows that $g = \bar{f}$ so that $\bar{f}$ is unique. This completes the proof. □

Corollary 4.9. *Let ρ and σ be congruence relations on a subtraction semigroup X such that $\rho \subseteq \sigma$. Then the set*

$$\sigma/\rho := \{(x\rho, y\rho) \in X/\rho \times X/\rho \mid (x, y) \in \sigma\}$$

is a congruence relation on X/ρ *and there exists a* 1-1 *and onto subtraction semigroup homomorphism from* $\frac{X/\rho}{\sigma/\rho}$ *to* X/σ.

Proof. Let $g : X/\rho \to X/\sigma$ be a function defined by $g(x\rho) = x\sigma$ for all $x\rho \in X/\rho$. Since $\rho \subseteq \sigma$, it follows that g is a well-defined onto subtraction semigroup homomorphism. According to Theorem 4.8, it is sufficient to show that $K_g = \sigma/\rho$. Let $(x\rho, y\rho) \in K_g$. Then $x\sigma = g(x\rho) = g(y\rho) = y\sigma$ and so $(x, y) \in \sigma$. Hence $(x\rho, y\rho) \in \sigma/\rho$, and thus $K_g \subseteq \sigma/\rho$.

Conversely, if $(x\rho, y\rho) \in \sigma/\rho$, then $(x, y) \in \sigma$ and so $x\sigma = y\sigma$. It follows that

$$g(x\rho) = x\sigma = y\sigma = g(y\rho)$$

so that $(x\rho, y\rho) \in K_g$. Hence $K_g = \sigma/\rho$, and the proof is complete. □

Theorem 4.10. *Let* I *be an ideal of a subtraction semigroup* X. *Then* $\rho_I := (I \times I) \cup \Delta_X$ *is a congruence relation on* X, *where* $\Delta_X := \{(x, x) \mid x \in X\}$.

Proof. Clearly, ρ_I is reflexive and symmetric. Noticing that $(x, y) \in \rho_I$ if and only if $x, y \in I$ or $x = y$, we know that if $(x, y) \in \rho_I$ and $(y, z) \in \rho_I$ then $(x, z) \in \rho_I$. Hence ρ_I is an equivalence relation on X. Assume that $(x, y) \in \rho_I$ and $(u, v) \in \rho_I$. Then we have the following four cases: (i) $x, y \in I$ and $u, v \in I$; (ii) $x, y \in I$ and $u = v$; (iii) $x = y$ and $u, v \in I$; and (iv) $x = y$ and $u = v$. In either case, we get $x - u = y - v$ or $(x - u, y - v) \in I \times I$, and $xu = yv$ or $(xu, yv) \in I \times I$. Therefore ρ_I is a congruence relation on X. □

Let X be a multiplicatively abelian subtraction semigroup and ρ_X be a binary relation on X defined by

$$(a, b) \in \rho_X \iff \exists u \in X \text{ such that } au = bu. \qquad (*)$$

Clearly, ρ_X is reflexive and symmetric. Let $(a, b), (b, c) \in \rho_X$. Then there exist $u, v \in X$ such that $au = bu$ and $bv = cv$. These imply $a(buv) = (au)(bv) = (bu)(cv) = c(buv)$, whence ρ_X is transitive. Thus ρ_X is an equivalence relation on X.

Theorem 4.11. *Let* X *be a multiplicatively abelian subtraction semigroup and* ρ_X *be a binary relation on* X *defined by* (*). *Then* ρ_X *is a congruence relation on* X, *and* X/ρ_X *is a multiplicatively abelian subtraction semigroup.*

Proof. Let $(a, b), (c, d) \in \rho_X$, Then there exist $u, v \in X$ such that $au = bu$ and $cv = dv$. These imply $(ac)(uv) = (au)(cv) = (bu)(dv) = (bd)(uv)$ and $(a - c)(uv) = auv - cuv = buv - duv = (b - d)uv$, whence $(ac, bd) \in \rho_X$ and $(a - c, b - d) \in \rho_X$. Thus ρ_X is a congruence relation on X, and clearly X/ρ_X is a multiplicatively abelian subtraction semigroup. □

Let X be a multiplicatively abelian subtraction semigroup. Then $(\rho_X)^* : X \to X/\rho_X$ defined by

$$(\rho_X)^*(a) = a\rho_X$$

is a surjective subtraction semigroup homomorphism.

Theorem 4.12. *Let* X *and* X' *be multiplicatively abelian subtraction semigroups with* X/ρ_X *and* X'/ρ'_X, *respectively and* $\phi : X \to X'$ *be a subtraction semigroup homomorphism. Then there exists a unique homomorphism* $\phi/\rho : X/\rho_X \to X'/\rho_{X'}$ *such that* $\phi/\rho \circ (\rho_X)^* = (\rho_{X'})^* \circ \phi$.

Proof. Define $\phi/\rho : X/\rho_X \to X'/\rho_{X'}$ by $\phi/\rho(a\rho_X) = \phi(a)\rho_{X'}$. If $a\rho_X = b\rho_X$, then there exists $u \in X$ such that $au = bu$. Thus $\phi(a)\phi(u) = \phi(b)\phi(u)$ and $(\phi(a), \phi(b)) \in \rho_{X'}$, so $\phi(a)\rho_{X'} = \phi(b)\rho_{X'}$. Therefore ϕ/ρ is well-defined. Next, we prove that ϕ/ρ is a homomorphism. In fact, $\phi/\rho(a\rho_X - b\rho_X) = \phi/\rho((a - b)\rho_X) = \phi(a - b)\rho_{X'} = (\phi(a) - \phi(b))\rho_{X'} = \phi(a)\rho_{X'} - \phi(b)\rho_{X'} = \phi/\rho(a\rho_X) - \phi/\rho(b\rho_X)$ and $\phi/\rho(a\rho_X \cdot b\rho_X) = \phi/\rho((ab)\rho_X) = \phi(ab)\rho_{X'} = (\phi(a) \cdot \phi(b))\rho_{X'} = \phi(a)\rho_{X'} \cdot \phi(b)\rho_{X'} = \phi/\rho(a\rho_X) \cdot \phi/\rho(b\rho_X)$. For any $a \in X$, we have

$(\phi/\rho \circ (\rho_X)^*)(a) = \phi/\rho((\rho_X)^*(a)) = \phi/\rho(a\rho_X) = \phi(a)\rho_{X'} = (\rho_{X'})^*(\phi(a)) = ((\rho_{X'})^* \circ \phi)(a)$. Thus $\phi/\rho \circ (\rho_X)^* = (\rho_{X'})^* \circ \phi$. Finally, if there exists a homomorphism $g : X/\rho_X \to X'/\rho_{X'}$ such that $g \circ (\rho_X)^* = (\rho_{X'})^* \circ \phi$, then $g(a\rho_X) = g((\rho_X)^*(a)) = (g \circ (\rho_X)^*)(a) = ((\rho_{X'})^* \circ \phi)(a) = (\rho_{X'})^*(\phi(a)) = \phi(a)\rho_{X'} = \phi/\rho(a\rho_X)$. Thus $g = \phi/\rho$ and ϕ/ρ is unique. □

It is clear that $Hom(X, X')$ is a semigroup under multiplication defined by $(\phi_1 \cdot \phi_2)(a) = \phi_1(a) \cdot \phi_2(a)$. Likewise $Hom(X/\rho_X, X'/\rho_{X'})$ is a semigroup by Theorem 4.12, we can define a mapping

$$\Phi : Hom(X, X') \to Hom(X/\rho_X, X'/\rho_{X'})$$

by $\Phi(\phi) = \phi/\rho$. Then we have the following theorem.

Theorem 4.13. *Let X and X' be multiplicatively abelian subtraction semigroups with X/ρ_X and $X'/\rho_{X'}$, respectively. Then the above mapping Φ given by $\Phi(\phi) = \phi/\rho$ is a semigroup homomorphism.*

Proof. Let $\phi_1, \phi_2 \in Hom(X, X')$ and $a\rho_X \in X/\rho_X$. Then $((\phi_1 \cdot \phi_2)/\rho)(a\rho_X) = ((\phi_1 \cdot \phi_2)(a))\rho_{X'} = (\phi_1(a) \cdot \phi_2(a))\rho_{X'} = \phi_1(a)\rho_{X'} \cdot \phi_2(a)\rho_{X'} = \phi_1/\rho(a\rho_X) \cdot \phi_2/\rho(a\rho_X) = (\phi_1/\rho \cdot \phi_2/\rho)(a\rho_X)$. Consequently, $(\phi_1 \cdot \phi_2)/\rho = \phi_1/\rho \cdot \phi_2/\rho$. Thus the map

$$\Phi : Hom(X, X') \to Hom(X/\rho_X, X'/\rho_{X'})$$

given by $\Phi(\phi) = \phi/\rho$ is a semigroup homomorphism. □

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