Scientiae Mathematicae Japonicae Online, **Vol. 10**, (2004), 63–64

METRIZABILITY OF k-SPACES

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Received February May 27 2003; revised October 14, 2003

ABSTRACT. In this note, we answer a question about metrizability posted in [YYG].

The following result was given in [YYG]:

Theorem 1 *([YYG] Theorem 4) A regular Fréchet space X is a metrizable space if and only if there is a strongly decreasing g-function on X such that $\{g(n,x) : x \in X\}$ is a CF-family for each $n \in N$ and the following condition (1) is satisfied:*

(1) If $x_n \to p \in X$ and $x_n \in g(n, y_n)$ for each $n \in N$, then $y_n \to p$.

After proving the above theorem, a question was posted :

([YYG] Question 1). *Can the condition " a regular Fréchet space X" in Theorem 1 be changed into "a regular k-space X"?*

In this paper, we answer the above question positively by proving the following theorem:

Theorem 2 *A regular k-space X is a Fréchet space if there is a strongly decreasing g-function on X such that $\{g(n,x) : x \in X\}$ is a CF-family for each $n \in N$ and condition (1) is satisfied.*

In this paper, all spaces are at least T_1 and $\mathbf{N}$ denotes the set of all natural numbers. Recall that a function $g : \mathbf{N} \times X \to 2^X$ is called a *g-function* if $g(n,x)$ is an open neighborhood of x for each $n \in \mathbf{N}$ and $x \in X$. A g-function on X is called a *strongly decreasing* g-function if $\mathrm{Cl}g(n+1,x) \subseteq g(n,x)$ for each $n \in \mathbf{N}$ and $x \in X$. Let $\mathcal{F}$ be a family of subsets in a space X. $\mathcal{F}$ is called a *CF-family* if for each compact subset K in X, $|\{K \cap F : F \in \mathcal{F}\}| < \omega$. The *Arens' space* is the set $S_2 = \{y_{k,i} : i,k \in \mathbf{N}\} \cup \{y_k : k \in \mathbf{N}\} \cup \{y\}$ with the topology as follows: Each $y_{k,i}$ is isolated in S_2. For each $k \in \mathbf{N}$, $\{\{y_{k,i} : j < i \in \mathbf{N}\} \cup \{y_k\} : j \in \mathbf{N}\}$ is an open neighborhood base of y_k, and a subset U of S_2 is a neighborhood of y if and only if y and almost all the y_k, together with a neighborhood of y_k, are contained in U.

Lemma 1 *There is no strongly decreasing g-function on S_2 such that $\{g(n,x) : x \in S_2\}$ is a CF-family for each $n \in \mathbf{N}$ and condition (1) is satisfied.*

Proof: Assume that there is a strongly decreasing g-function on S_2 such that condition (1) is satisfied. We prove $\{g(n,x) : x \in S_2\}$ is not a CF-family for some $n \in \mathbf{N}$.

Claim 1. There exists an $N \in \mathbf{N}$, such that, $y_k \notin g(n, y_{k,i})$ for each $i \in \mathbf{N}$ whenever $k, n \geq N$.

Otherwise, for some $i_0 \in \mathbf{N}$, there are sequences $\{n_j\}, \{k_j\}$ with $n_j, k_j \to \infty$ when $j \to \infty$, such that $y_{k_j} \in g(n_j, y_{k_j,i_0})$ for each j. Since $y_{k_j} \to y$, by condition (1), $y_{k_j,i_0} \to y$, which is impossible.

2000 *Mathematics Subject Classification.* 54E35.

Key words and phrases. g-function, Aren's space, CF-family.

*Supported by the NSFC(Project 10271026) and the Education Department of Jiangsu Province (Project 02KJB110001).

Claim 2. For the N in Claim 1 and each $i \in \mathbf{N}$, there is a $j(i) \in \mathbf{N}$, such that $y_{N,j} \notin g(N+1, y_{N,i})$ when $j > j(i)$.

Otherwise, there is a sequence $\{i_j\}$ with $i_j \to \infty$ when $j \to \infty$, suth that $y_{N,i_j} \in g(N+1, y_{N,i})$, and hence $y_N \in \mathrm{Cl}g(N+1, y_{N,i})$. Since $\{g(n,x) : x \in S_2\}$ is a strongly decreasing g-function on S_2, we have $y_N \in g(N, y_{N,i})$. Which contradicts Claim 1.

Denote $C_N = \{y_{N,i} : i \in \mathbf{N}\} \cup \{y_N\}$. Then C_N is compact in S_2. Take $j_1 = j(1), j_{k+1} = j(k) + 1$ for each $k \in \mathbf{N}$, where the $j(k)$ are in the Claim 2. It follows from the fact $y_{N,j_{k_2}} \notin g(N+1, y_{N,j_{k_1}}) \cap C_N$ whenever $k_1 < k_2$, we can get

$$|\{g(N+1,x) \cap C_N : x \in C_N\}| = \omega,$$

and hence $\{g(N+1,x) : x \in S_2\}$ is not a CF-family.

Proof of Theorem 2:

Assume that X is a k-space with a strongly decreasing g-function such that $\{g(n,x) : x \in X\}$ is a CF-family for each $n \in \mathbf{N}$ and condition (1) is satisfied. Then X contains no closed copy of S_2. Otherwise, $\{g(n,x) \cap S_2 : x \in S_2, n \in \mathbf{N}\}$ will be a strongly decreasing g-function on S_2 such that $\{g(n,x) : x \in S_2\}$ is a CF-family for each $n \in \mathbf{N}$ and condition (1) is satisfied, which contradicts Lemma 1. On the other hand, by Theorem 4.11 of [Gru] , a space with a g-function satisfying condition (1) is a σ-space, since condition (1) is stronger than condition (v)(b) of that Theorem. It is well-known that a σ-space is a space with point G_δ property. By [Ta] Proposition 1.5, a k-space with point G_δ property is a sequential space, and by [Ta] Proposition 1.20, it is a Fréchet space if it contains no closed copy of S_2.

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