KADEC-KLEE PROPERTY IN MUSIELAK-ORLICZ FUNCTION SPACES EQUIPPED WITH THE LUXEMBURG NORM

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Abstract. In this paper, a criterion for Musielak-Orlicz function spaces equipped with Luxemburg to have Kadec-Klee property are given.

$ 1. *Introduction*

In the following, $(T, \sum, \mu)$ denotes a non-atomic σ-finite separable measure space, R denotes the set of reals, $L^0(\mu)$ denotes the space of all (μ-equivalence classes of) $\sum$-measurable real functions defined on T. Let X be a Banach space and X^* be its dual space. The unit sphere of X is denoted by $S(X)$.

Satisfactory criteria of some geometric properties of Musielak-Orlicz space have been obtained in many papers (see [1], [3], [7] and [8]). But, more important property, namely *Kadec-Klee* property was not characterized. In this paper, we will try to characterize this property.

Definition 1. *A Banach space X is said to be locally uniformly convex if for any sequence $\{x_n\} \subset S(X)$ and some $x \in S(X)$ with $\lim_{n\to\infty} \|x_n + x\| = 2$ there holds $\lim_{n\to\infty} \|x_n - x\| = 0$.*

Definition 2. *A Banach space X is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset S(X)$ with $x_n \xrightarrow{w} x \in S(X)$ we have $x_n \to x$.*

It is clear that a Banach space that is locally uniformly convex has the Kadec-Klee property .

Definition 3. *A map $\Phi : T \times R \to [0, \infty)$ is said to be a Musielak-Orlicz function if it satisfies the following conditions:*

(1) *$\Phi(t, u)$ is vanishing only at zero, convex and even for μ-a.e. $t \in T$;*

(2) *$\Phi(\cdot, u)$ is locally integrable for any $u \in R$;*

Let us first remark that if Φ is a Musielak-Orlicz function then Φ is of the form

$$\Phi(t, u) = \int_0^{|u|} p(t, s) ds,$$

where $p(t, u)$ is the right-hand derivatives of $\Phi(t, u)$ for a fixed $t \in T$.

For any Musielak-Orlicz function Φ, we define its complementary function Ψ in the sense of Young, i.e.,

$$\Psi(t, v) = \sup\{|v| u - \Phi(t, u) : u > 0\}.$$

Given a Musielak-Orlicz function Φ we define on $L^0(\mu)$ a modular I_Φ by the formula

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$$I_\Phi(x) = \int_T \Phi(t, x(t)) d\mu.$$

The Musielak-Orlicz function space generated by a Musielak-Orlicz function Φ is defined to be the set of all $x \in L^0(\mu)$ for which $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ depending on x and it is denoted by $L_\Phi(\mu)$. This space endowed with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_\Phi\left(\frac{x}{k}\right) \le 1\right\}$$

or with the equivalence norm, called the Orlicz norm

$$\|x\|_\Phi^0 = \sup\left\{\int_T x(t)y(t)d\mu : I_\Psi(y) \le 1\right\}.$$

The Amemiya formula for the Orlicz norm is the following:

$$\|x\|_\Phi^0 = \inf\left\{\frac{1}{k}\left(1 + I_\Phi(kx)\right) : k > 0\right\}$$

(see [1] and [2]).

We define the subspace $E_\Phi(\mu)$ of $L_\Phi(\mu)$ by the following formula:

$$E_\Phi(\mu) = \left\{x \in L^0(\mu) : I_\Phi(\lambda x) < \infty \ for \ any \ \lambda > 0\right\}.$$

To simplify notions, we put $L_\Phi = \{L_\Phi(\mu), \|\cdot\|\}$, $E_\Phi = \{E_\Phi(\mu), \|\cdot\|\}$, $L_\Phi^0 = \left\{L_\Phi(\mu), \|\cdot\|_\Phi^0\right\}$ and $E_\Phi^0 = \left\{E_\Phi(\mu), \|\cdot\|_\Phi^0\right\}$.

Definition 4. *We say that a Musielak-Orlicz function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$ for short) if there exist a constant $K \ge 2$, a set T_0 of measure zero and a $\sum$-measurable function $h : T \to (0, \infty)$ such that $\int_T h(t)d\mu < \infty$ and the inequality*

$$\Phi(t, 2u) \le K\Phi(t, u) + h(t)$$

holds for any $u \in R$ and $t \in T \backslash T_0$ ([2] and [1]).

Definition 5. *A Musielak-Orlicz function Φ is called to be strictly convex if $\Phi(t, u)$ is strictly convex for a.e. $t \in T$, i.e.,*

$$\Phi\left(t, \frac{u+v}{2}\right) < \frac{1}{2}\left(\Phi(t, u) + \Phi(t, v)\right)$$

for all $u, v \in R$ and $u \ne v$.

For more details on Musielak-Orlicz space, we refer to [1] , [3] , [4] and [2] .

$2. *RESULTS*

We start with some auxiliary lemmas.

Lemma 1. *E_Φ^0 is separable* (see [1]).

Lemma 2. *$E_\Phi = L_\Phi$ if and only if $\Phi \in \Delta_2$* (see [1]).

Lemma 3. *The modular convergence and the norm convergence are equivalent in L_Φ if and only if $\Phi \in \Delta_2$* (see [1]).

Lemma 4. $L_\Psi = (E_\Phi^0)^*$ (see [1]).

Lemma 5. *L_Φ is a locally uniformly convex if and only if $\Phi \in \Delta_2$ and $\Phi(t,\cdot)$ is strictly convex for a.e. $t \in T$* (see [8]).

Lemma 6. *Let H be a measurable subset of T . If $f(t) > 0$ and $g(t) > 0$ are integrable on H , then for any $\epsilon > 0$ there exist $H_1, H_2 \subset H$ with $\mu H_1 = \mu H_2 = \frac{1}{2}\mu H$ and $H_1 \cap H_2 = \phi$ such that*

$$\left| \int\limits_{H_1} f(t)d\mu + \int\limits_{H_2} g(t)d\mu - \int\limits_{H} \frac{f(t)+g(t)}{2} d\mu \right| < \frac{\epsilon}{2}.$$

Proof. Put

$$e_n^{(1)} = \left\{ t \in H: \ \frac{n-1}{\mu H}\epsilon \le f(t) < \frac{n}{\mu H}\epsilon \right\},$$
$$e_n^{(2)} = \left\{ t \in H: \ \frac{n-1}{\mu H}\epsilon \le g(t) < \frac{n}{\mu H}\epsilon \right\}$$

and

$$e_{n,k} = e_n^{(1)} \cap e_k^{(2)}$$

for $n, k = 1, 2, \cdots$.

Divide $e_{n,k}$ into two subsets $e'_{n,k}$ and $e''_{n,k}$ such that $e_{n,k} = e'_{n,k} \cup e''_{n,k}$, $e'_{n,k} \cap e''_{n,k} = \phi$ and $\mu e'_{n,k} = \mu e''_{n,k}$. Set

$$H_1 = \bigcup_{n,k=1}^{\infty} e'_{n,k}\ , \qquad H_2 = \bigcup_{n,k=1}^{\infty} e''_{n,k}\ .$$

Then $H_1 \cup H_2 = H,\ H_1 \cap H_2 = \phi$ and $\mu H_1 = \mu H_2 = \frac{1}{2}\mu H$. Hence

$$\left| \int\limits_{H_1} f(t)d\mu - \int\limits_{H} \frac{f(t)}{2} d\mu \right| = \frac{1}{2} \left| \int\limits_{H_1} f(t)d\mu - \int\limits_{H_2} f(t)d\mu \right|$$
$$\le \frac{1}{2} \sum_{n,k=1}^{\infty} \left| \int\limits_{e'_{n,k}} f(t)d\mu - \int\limits_{e''_{n,k}} f(t)d\mu \right|$$
$$\le \frac{\epsilon}{2} \sum_{n,k=1}^{\infty} \frac{\mu e'_{n,k}}{\mu H} = \frac{\epsilon}{2}\frac{\mu H}{2}\frac{1}{\mu H} = \frac{\epsilon}{4}.$$

In the same way, we can aslo get

$$\left| \int\limits_{H_1} g(t)d\mu - \int\limits_{H} \frac{g(t)}{2} d\mu \right| < \frac{\epsilon}{4}.$$

So, we have

$$\left| \int\limits_{H_1} f(t)d\mu + \int\limits_{H_2} g(t)d\mu - \int\limits_{H} \frac{f(t)+g(t)}{2} d\mu \right| < \frac{\epsilon}{2}.$$

Theorem. *A Musielak-Orlicz function space L_Φ has the Kadec-Klee property if and only if $\Phi \in \Delta_2$ and Φ is strictly convex .*

Proof. Necessity. Suppose that $\Phi \notin \Delta_2$. By Lemma2 , there is a $x_0 \in S(L^0_\Phi)\backslash E_\Phi$. Hence there exists $\lambda_0 > 0$ such that $I_\Phi(\lambda x_0) = \infty$ when $\lambda > \lambda_0$.

Put $T_n = \{t \in T : |x_0(t)| \leq n\}$. Then $I_\Phi(\lambda x_0 \chi_{T/T_n}) = \infty$ when $\lambda > \lambda_0$. This means that

$$\left\| x_0 \chi_{T \setminus T_n} \right\| \geq \epsilon_0.$$

for any $n \in N$, where $\epsilon_0 = \frac{1}{2\lambda_0}$. For convenience, we put

$$T_n^m = \{t \in T : n \leq |x_0(t)| < m\}.$$

Take $n_0 = 0$. There exists $n_1 \in N$ such that

$$\left\| x_0 \chi_{T_{n_0}^{n_1}} \right\| \geq \frac{\epsilon_0}{2}.$$

Notice that

$$\lim_{m \to \infty} \left\| x_0 \chi_{T_{n_1}^m} \right\| = \left\| x_0 \chi_{T \setminus T_{n_1}} \right\| \geq \epsilon_0.$$

So, there exists $n_2 > n_1$ such that

$$\left\| x_0 \chi_{T_{n_1}^{n_2}} \right\| \geq \frac{\epsilon_0}{2}.$$

In such a way, we get a sequence $\{n_i\}$ of natural numbers such that

$$\left\| x_0 \chi_{T_{n_i}^{n_{i+1}}} \right\| \geq \frac{\epsilon_0}{2}, i = 1, 2, \cdots.$$

Put $x_i = x_0 \chi_{T \setminus T_{n_i}^{n_{i+1}}}$. Then

(1) $\|x_i\| \to \|x_0\|$ as $i \to \infty$.

(2) $x_i \overset{w}{\to} x_0$ as $i \to \infty$. It is well known that for any Musielak-Orlicz function Φ, we have

$$(L_\Phi)^* = L_\Psi^0 + S,$$

where S is the space of all singular functionas over E_Φ , i.e.$\varphi \in S$ if and only if $\langle \varphi, x \rangle = 0$ for any $x \in E_\Phi$ (see [10]).

Look at $x_i - x_0 \in E_\Phi$. We have $\varphi(x_i - x_0) = 0$, where $\varphi \in S$. Let $y \in S(L_\Psi)$. It easily follows from $\int_T x_0(t)y(t)d\mu < \infty$ that $\langle y, x_i - x_0 \rangle = \int_{T_{n_i}^{n_{i+1}}} x_0(t)y(t)d\mu \to 0$ as $i \to \infty$.

(3) $\|x_i - x_0\| = \left\| x_0 \chi_{T_{n_i}^{n_{i+1}}} \right\| \geq \frac{\epsilon_0}{2}, i = 1, 2, \cdots$.

This contradiction shows that $\Phi \in \Delta_2$.

Suppose that if Φ is not strictly convex. Then there exists $T_0 \in \sum$ with $\mu(T_0) > 0$ such that $\Phi(t, \cdot)$ is affine in some intervals if $t \in T_0$. Let (w_i) be the set of all rational numbers. Define

$A_k = \{t \in T_0 : \Phi(t, \cdot)$ is linear on $[a_k, b_k]\}$, where a_k , $b_k \in (w_k)$ is rational numbers for some k .

Since $T_0 = \cup_k A_k$, there exists $l \in N$ such that $\mu(A_l) > 0$. This mean there exist $a, b \in (0, \infty)$ with $a < b$ and $G \subset T$ with $\mu G > 0$ such that $\Phi(t, u)$ is linear on $[a, b]$ for any fixed $t \in G$.

Since $0 < \Phi(t, b - a) < \infty$, there exists $\epsilon > 0$ such that $\mu G_\epsilon < \frac{1}{2}\mu G$, where $G_\epsilon = \{t \in G : \ \Phi(t, b - a) < \epsilon\}$. Put $T_1^0 = G \setminus G_\epsilon$. Then $\mu T_1^0 > 0$. Without loss of generality, we may assume that

$$0 < \int_{T_1^0} \Phi(t, \frac{a + b}{2}) d\mu < 1.$$

Take $c \geq b$ such that

$$\int_{T_1^0} \Phi(t, \frac{a+b}{2})d\mu + \int_{T \backslash T_1^0} \Phi(t,c)d\mu \geq 1.$$

Take a subset $H \subset T \backslash T_1^0$ such that

$$\int_{T_1^0} \Phi(t, \frac{a+b}{2})d\mu + \int_{H} \Phi(t,c)d\mu = 1.$$

By Lemma 6 , there exist $T_1^1, T_2^1 \subset T_1^0$ with $T_1^0 = T_1^1 \cup T_2^1$, $T_1^1 \cap T_2^1 = \phi$ and $\mu T_1^1 = \mu T_2^1$ such that

$$\left| \int_{T_1^1} \Phi(t,a)d\mu + \int_{T_2^1} \Phi(t,b)d\mu - \int_{T_1^0} \frac{\Phi(t,a)+\Phi(t,b)}{2} d\mu \right| < \frac{1}{2}.$$

Suppose that the sequence of sets $\{T_1^{n-1}, T_2^{n-1}, \cdots, T_{2^{n-1}}^{n-1}\}$ is well defined . Every set T_i^{n-1} we divide into two subsets such that $T_i^{n-1} = T_{2i-1}^n \cup T_{2i}^n, T_{2i-1}^n \cap T_{2i}^n = \phi$ and $\mu T_{2i-1}^n = \mu T_{2i}^n$ $(i = 1, 2, \cdots, 2^{n-1})$.

In such a way , we get a partition $\{T_1^n, T_2^n, \cdots, T_{2^n}^n\}$ of T_1^0 with

$$\mu(T_i^n) = 2^{-n}\mu T_1^0 \ , \ T_{2i-1}^n \cap T_{2i}^n = \phi$$

such that

$$\left| \int_{T_{2i-1}^n} \Phi(t,a)d\mu + \int_{T_{2i}^n} \Phi(t,b)d\mu - \int_{T_i^{n-1}} \frac{\Phi(t,a)+\Phi(t,b)}{2} d\mu \right| < \frac{1}{2^{2n-1}}$$

for $i = 1, 2, \cdots, 2^n$. Define

$$x_n = a\chi_{T_{1,n}} + b\chi_{T_{2,n}} + c\chi_H,$$

where $T_{1,n} = \bigcup_{k=1}^{2^{n-1}} T_{2k-1}^n$, $T_{2,n} = \bigcup_{k=1}^{2^n} T_{2k}^n$ $(n = 1, 2, \cdots)$.

Then

$$|I_\Phi(x_n) - 1| = \left| \sum_{k=1}^{2^{n-1}} \left(\int_{T_{2k-1}^n} \Phi(t,a)d\mu + \int_{T_{2k}^n} \Phi(t,b)d\mu \right) + \int_H \Phi(t,c)d\mu - 1 \right|$$

$$= \left| \sum_{k=1}^{2^{n-1}} \left(\int_{T_{2k-1}^n} \Phi(t,a)d\mu + \int_{T_{2k}^n} \Phi(t,b)d\mu - \int_{T_k^{n-1}} \frac{\Phi(t,a)+\Phi(t,b)}{2} d\mu \right) \right|$$

$$\leq \sum_{k=1}^{2^{n-1}} 2^{1-2n} = 2^{-n}$$

for all $n \in N$. Hence $\lim_{n\to\infty} I_\Phi(x_n) = 1$. Notice that $I_\Phi(x_n) \leq \|x_n\|$ when $\|x_n\| \leq 1$ and $I_\Phi(x_n) \geq \|x_n\|$ when $\|x_n\| \geq 1$. Therefore , we have $\lim_{n\to\infty} \|x_n\| = 1$.

Using Lemma 1 and Lemma 4, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $x \in L_\Phi$ for which $\{x_{n_i}\}$ converges weakly star to x , *i.e.*, $x_{n_i} \overset{E_\Psi^0}{\to} x$. Next , we will show

$$x_{n_i} \overset{w}{\to} x.$$

Since $\Phi \in \Delta_2$, we get $L_\Phi = E_\Phi$. Hence $(L_\Phi)^* = L_\Psi^0$. Since $(T, \sum, \mu)$ denotes a non-atomic σ-finite separable measure space, there exists an ascending sequence of set $(T_n)_{n=1}^\infty$ such that $\bigcup_{n=1}^{\infty} T_n = T$. Let $\eta > 0$ be given. Then there exists a $n_0 \in N$ such that $\left\| c\chi_{T\setminus T_n} \right\| < \frac{\eta}{5c\|y\|_\Psi^0}$ when $n > n_0$. For any $y \in L_\Psi^0$, there exists $n_1 > n_0$ such that for $F = \{t \in T_{n_1} : |y(t)| > n_1\}$ we have $\|c\chi_F\| < \frac{\eta}{5c\|y\|_\Psi^0}$. By $\Phi(\cdot, u)$ is locally integrable for any $u \in R$, we have $y\chi_{T_{n_1}\setminus F} \in E_\Psi^0$. Hence there exists $i_0 \in N$ such that

$$\left| \int_T (x_{n_i}(t) - x(t))\, y(t)\chi_{T_{n_1}\setminus F} d\mu \right| < \frac{\eta}{5},$$

when $i > i_0$. So

$$\left| \int_T (x_{n_i}(t) - x(t))\, y(t) d\mu \right| = \left| \int_T (x_{n_i}(t) - x(t))\, y(t)\chi_{T\setminus T_{n_1}} d\mu \right| +$$

$$\left| \int_T (x_{n_i}(t) - x(t))\, y(t)\chi_{T_{n_1}\setminus F} d\mu \right| + \left| \int_T (x_{n_i}(t) - x(t))\, y(t)\chi_F d\mu \right|$$

$$\leq \left| \int_T (x_{n_i}(t) - x(t))\, y(t)\chi_{T_{n_1}\setminus F} d\mu \right| + \left| \int_T x_{n_i}(t)y(t)\chi_{T\setminus T_{n_1}} d\mu \right| +$$

$$\left| \int_T x(t)y(t)\chi_{T\setminus T_{n_1}} d\mu \right| + \left| \int_T x_{n_i}(t)y(t)\chi_F d\mu \right| + \left| \int_T x(t)y(t)\chi_F d\mu \right|$$

$$\leq \frac{\eta}{5} + \left\| x_{n_i}\chi_{T\setminus T_{n_1}} \right\| \|y\|_\Psi^0 + \left\| x\chi_{T\setminus T_{n_1}} \right\| \|y\|_\Psi^0 + \|x_{n_i}\chi_F\| \|y\|_\Psi^0 + \|x\chi_F\| \|y\|_\Psi^0$$

$$\leq \frac{\eta}{5} + \left\| c\chi_{T\setminus T_{n_1}} \right\| \|y\|_\Psi^0 + \left\| c\chi_{T\setminus T_{n_1}} \right\| \|y\|_\Psi^0 + \|c\chi_F\| \|y\|_\Psi^0 + \|c\chi_F\| \|y\|_\Psi^0$$

$$\leq \frac{\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} = \eta,$$

whenever $i > i_0$. This means that $x_{n_i} \overset{w}{\to} x$ as $i \to \infty$.

So, we have $\|x\| \leq \lim_{i\to\infty} \|x_{n_i}\| = 1$. Furthermore, we have$\|x\| = 1$.

In fact, put $y(t) = p(t, a)\chi_{T_1^0} + p(t, c)\chi_H$. Then $y \in L_\Psi^0$ and

$$\|y\|^0 = \|y\|^0 \|x_n\| \geq |\langle x_n, y\rangle| = \left| \int_T x_n(t)y(t)d\mu \right|$$

$$= \int_{T_{n,1}} ap(t, a)d\mu + \int_{T_{n,2}} bp(t, b)d\mu + \int_H cp(t, c)d\mu$$

$$= \int_{T_{n,1}} \Phi(t,a)d\mu + \int_{T_{n,1}} \Psi(t,p(t,a))d\mu + \int_{T_{n,2}} \Phi(t,b)d\mu$$

$$+ \int_{T_{n,2}} \Psi(t,p(t,b))d\mu + \int_{H} \Phi(t,c)d\mu + \int_{H} \Psi(t,p(t,c))d\mu$$

$$= I_\Phi(x_n) + I_\Psi(y) \to 1 + I_\Psi(y) \geq \|y\|_\Psi^0 .$$

This means that $\langle x_n, y\rangle \to \|y\|_\Psi^0$. Hence $\langle x, y\rangle = \|y\|_\Psi^0$, that is $\|x\| \geq 1$.

Obviously, $I_\Phi(x_n - x_m) \geq \inf\left\{\Phi(t, b-a) : \ t \in T_0^1\right\} \frac{\mu T_0^1}{2} \geq \frac{\epsilon \mu T_0^1}{2}$. This assures us that $\{x_{n_i}\}$ is not Cauchy sequence . Hence L_Φ has not the Kadec-Klee property .

This contradiction shows that $\Phi(t,u)$ must be is strictly convex if L_Φ has the Kadec-Klee property .

Sufficiency. Under this conditions, we get that L_Φ is locally uniformly convex thanks to Lemma 5. Of course, L_Φ has then the Kadec-Klee property .

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