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GLOBAL EXISTENCE FOR TREE-GRASS COMPETITION MODEL

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Abstract. We present a tree-grass competition model on the basis of the forest kinematic model due to Kuznetsov-Antonovsky-Biktashev-Aponina [6]. The main purpose of the paper is to construct global solutions and to construct a dynamical system generated by the model equations. By numerical computations, we also show that our model actually admits coexisting solutions of trees and grass.

1 Introduction We want to study the kinematics of forest-grassland system from a viewpoint of competitive system between trees and grass.

Our mathematical model is written as the initial-boundary value problem for a parabolic-ordinary system

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta[w-w_*]_+ - (\lambda g + av^2 + c)u - fu & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial w}{\partial t} = d_w\Delta w - \beta w + \alpha v & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial g}{\partial t} = d_g\Delta g - \mu v g + \gamma(g-\ell)(1-g)g & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial w}{\partial n} = \dfrac{\partial g}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x),\ g(x,0)=g_0(x) & \text{in } \Omega, \end{cases}$$

in a two-dimensional bounded, $\mathcal{C}^2$ or convex domain Ω. Here, the unknown functions $u(x,t)$ and $v(x,t)$ denote tree densities of young and old age classes, respectively, at a position $x\in\Omega$ and at time $t\in[0,\infty)$. The unknown function $w(x,t)$ denotes a density of seeds in the air at $x\in\Omega$ and $t\in[0,\infty)$. Meanwhile, $g(x,t)$ denotes a density of grass at $x\in\Omega$ and $t\in[0,\infty)$.

The third equation in (1.1) describes the kinetics of seeds; $d_w>0$ is a diffusion constant, and $\alpha>0$ and $\beta>0$ are seed production and seed deposition rates, respectively. The first equation describes growth of young age trees; here, $0<\delta\le 1$ is a seed establishment rate, the term $[w-w_*]_+=\max\{w-w_*,0\}$ means that a fixed amount w_* of seeds on the ground are consumed (by animals or birds), $\lambda g+av^2+c$ is a mortality of young age trees which is proportional to the densities g and v^2 with coefficients $\lambda>0$ and $a>0$, $c>0$ being a basic mortality. The second equation describes growth of old age trees; $f>0$ is an aging rate from young age to old age, and $h>0$ is a mortality. Finally, the fourth equation describes growth of grass that is basically given by a reaction-diffusion equation with a diffusion constant $d_g>0$ and with a cubic growth function $\gamma(g-\ell)(1-g)g$, where $0<\ell<1$ is an unstable state and $\gamma>0$ is a reaction rate, the term $-\mu vg$ denotes suppression by the trees

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with a coefficient $\mu > 0$. On w and g, the homogeneous Neumann conditions are imposed on the boundary $\partial\Omega$. Nonnegative initial functions $u_0(x) \geq 0$, $v_0(x) \geq 0$, $w_0(x) \geq 0$ and $g_0(x) \geq 0$ are given in Ω for all unknown functions.

This model is derived by the present authors on the basis of the classical forest kinematic model [6]. The detail of derivation is discussed in Section 2.

First, for suitable initial values (u_0, v_0, w_0, g_0), we construct a unique global solution in the underlying space

$$X = \{(u, v, w, g);\ u \in L_\infty(\Omega),\ v \in L_\infty(\Omega),\ w \in L_2(\Omega),\ g \in L_2(\Omega)\}.$$

As the equations of u and v are an ordinary equation for each $x \in \Omega$, the underlying spaces for u and v must be a Banach algebra. In addition, even if $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$, $u(x,t)$ and $v(x,t)$ of the global solution can tend to a stationary solution $(\overline{u}, \overline{v}, \overline{w}, \overline{g})$ as $t \to \infty$ in which $\overline{u}$ and $\overline{v}$ are discontinuous functions. By this reason, we set $L_\infty(\Omega)$ for the underlying spaces of u and v. Meanwhile, as w and g satisfy a diffusion equation, $u(t)$ and $g(t)$ belong to $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$ for any $t > 0$. In constructing a local solution, we apply the theory of abstract parabolic evolution equations as in [2, 3, 4] (see also [15, Chapter 11]).

Second, after constructing a dynamical system generated by (1.1), we show that there exists a bounded absorbing set (see [14]). This in particular implies that every solution to (1.1) admits a nonempty ω-limit set in a suitable weak topology of X.

Third, by numerical methods, we observe that the model (1.1) includes some solutions showing segregation patterns. Under careful tuning for the parameters in the equations of (1.1), we observe that solutions starting from some class of initial values tend to a stationary solution $(0, 0, 0, 1)$ as $t \to \infty$. Solutions starting another class of initial values tend to a stationary solution of the form $\overline{v}(hf^{-1}, 1, \alpha\beta^{-1}, 0)$, where $\overline{v}$ is a positive solution of the cubic equation

$$ah\overline{v}^3 + [(c+f)h - f\alpha\delta]\overline{v} + f\beta\delta w_* = 0.$$

And solutions starting from the other class of initial values tend to a stationary solution $(\overline{u}, \overline{v}, \overline{w}, \overline{g})$ which is not homogeneous but shows coexistence of trees and grass. As we can see a clear curve which divides Ω into forest and grassland, such a stationary might be called a segregation pattern.

In the forth coming paper [10], the authors will discuss segregation patterns in detail.

2 Kinematics of forests and grasslands Kuznetsov-Antonovsky-Biktashev-Aponina have first presented by their paper [6] a continuous space model describing the kinematics of forests. That is written as

$$(2.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \varphi(v)u - fu & \text{in } \ \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in } \ \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in } \ \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \ \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \text{in } \ \Omega. \end{cases}$$

As seen, the main difference of (2.1) from (1.1) is that the state variables consist of u, v and w and do not include the density of grass $g(x,t)$. As a consequence, the mortality of young age trees is given by a function of the density of old age trees alone, i.e., $\varphi(v)$. As for a typical form of $\varphi(v)$, they proposed $\varphi(v)$ such that

$$\varphi(v) = a(v-b)^2 + c, \tag{2.2}$$

where a, b, $c > 0$ are positive constants (see [6, p. 220]). This means that the mortality takes its minimum when v is an optimal value b.

It is quite reasonable to assume that the higher the density of old age trees is, the higher the mortality of young age trees, because the dense canopy of old age trees admits only a small amount of light transmission and prevents young age trees under it from growing regularly and because the trees which cease growing die at a higher rate. In the meantime, it is very difficult to understand a reason why the mortality $\varphi(v)$ increases as $v \to 0$ for $0 < v < b$. It may be possible to claim that a canopy of suitable density protects young age trees under it by providing them with a comfortable shelter. On the other hand, according to the articles [1, 5, 8, 9], it is known that trees and grass are always in competition. The old age trees prevent grass's growth and conversely the grass prevents seedling's growth. So, we want to explain by tree-grass competition why the mortality of young age trees increases as old age tree's density decreases less than the critical value b. More precisely, we want to present in this paper a tree-grass competition model for the kinematics of forest together with grassland.

As already shown, our mortality function is given by

$$\gamma(g, v) = \lambda g + av^2 + c. \tag{2.3}$$

It is similar to (2.2) for sufficiently large v. But, for small v, the mortality is governed by the density of grass and is actually proportional to it.

In addition, we need to introduce a growth equation for the grassland. As the basic growth equation, we use the usual reaction-diffusion equation

$$\frac{\partial g}{\partial t} = d_g \Delta g + \gamma(g-\ell)(1-g)g$$

including a cubic growth function $\gamma(g-\ell)(1-g)g$. To this equation, we incorporate the effect of competition with trees that is described by $-\mu v g$.

In this way, our tree-grass competition model (1.1) is derived on the basis of the classical model (2.1) due to Kuznetsov-Antonovsky-Biktashev-Aponina just by incorporating newly competition effects between trees and grass and a growth equation of grassland.

3 Preliminary

I) Some inequality. It is easily verified that

$$(g-\ell)(1-g)g^6 \le \frac{1-\ell}{6}\left(1-g^6\right) \qquad \text{for } 0 \le g < \infty. \tag{3.1}$$

Indeed, we have

$$(g-\ell)g^6 < \frac{1-\ell}{6}\left(1+g+g^2+\cdots+g^5\right) \qquad \text{for } 0 \le g < 1.$$

Meanwhile,

$$(g-\ell)g^6 > \frac{1-\ell}{6}\left(1+g+g^2+\cdots+g^5\right) \qquad \text{for } 1 < g < \infty.$$

II) Function Spaces. Let Ω is a bounded, $\mathcal{C}^2$ or convex domain in $\mathbb{R}^2$. For $0 \le s \le 2$, $H^s(\Omega)$ denotes the complex Sobolev space, its norm being denoted by $\|\cdot\|_{H^s}$ (see [13, Chap. 1]). For $0 \le s_0 \le s \le s_1 \le 2$, $H^s(\Omega)$ coincides with the complex interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$, where $s = (1-\theta)s_0 + \theta s_1$, and among their norms the estimate

$$\|\cdot\|_{H^s} \le C\|\cdot\|_{H^{s_0}}^{1-\theta}\ \|\cdot\|_{H^{s_1}}^{\theta} \tag{3.2}$$

holds true. When $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $\frac{1}{p} = \frac{1-s}{2}$, with continuous embedding

$$\|\cdot\|_{L^p} \le C_s\|\cdot\|_{H^s}. \tag{3.3}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $2 \le q < \infty$ with the estimate

$$\|\cdot\|_{L^q} \le C_{pq}\|\cdot\|_{H^1}^{1-\frac{p}{q}}\|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{3.4}$$

where $1 \le p < q < \infty$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|\cdot\|_{\mathcal{C}} \le C_s\|\cdot\|_{H^s}. \tag{3.5}$$

III) Linear Operators. Consider a sesquilinear form given by

$$a(u,v) = d\int_\Omega \nabla u\cdot\nabla\overline{v}\,dx + c\int_\Omega u\overline{v}\,dx, \qquad u,\ v\in H^1(\Omega),$$

d and c being positive constants. From this form, one can define a realization Λ of the Laplace operator $-d\Delta + c$ in the space $L_2(\Omega)$ under the homogeneous Neumann conditions on the boundary $\partial\Omega$ (see [12, Chap. VI]).

The realization Λ is a positive definite self-adjoint operator of $L_2(\Omega)$, i.e., $\Lambda \ge c$. When Ω is bounded and, convex or $\mathcal{C}^2$, its domain is characterized by

$$\mathcal{D}(\Lambda) = H^2_N(\Omega) \equiv \{u\in H^2(\Omega);\ \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}. \tag{3.6}$$

For $0 < \theta < 1$, the fractional powers Λ^θ of Λ are defined and also are positive definite self-adjoint in $L_2(\Omega)$. As shown in [15, Sec. 16.4], their domains are characterized by

$$\mathcal{D}(\Lambda^\theta) = \begin{cases} H^{2\theta}(\Omega), & \text{when } 0 \le \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega) \equiv \{u\in H^{2\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{3}{4} < \theta \le 1. \end{cases} \tag{3.7}$$

In addition, the following equivalence estimates

$$C^{-1}\|\Lambda^\theta\cdot\|_{L^2} \le \|\cdot\|_{H^{2\theta}} \le C\|\Lambda^\theta\cdot\|_{L^2} \tag{3.8}$$

hold true with some constant $C > 0$.

Furthermore, let $e^{-t\Lambda}$ $(0 \le t < \infty)$ denote the semigroup generated by $-\Lambda$. Then, the positivity $\Lambda \ge c$ implies that

$$\|e^{-t\Lambda}\|_{\mathcal{L}(L_2)} \le e^{-ct}, \qquad 0 \le t < \infty. \tag{3.9}$$

In addition, it is known for $0 < \theta \le 1$ that

$$\|\Lambda^\theta e^{-t\Lambda}\|_{\mathcal{L}(L_2)} \le Ct^{-\theta}, \qquad 0 < t < \infty, \tag{3.10}$$

with some constant $C > 0$.

IV) Evolution Equations. Consider the Cauchy problem for an evolution equation

$$\begin{cases} \dfrac{du}{dt} + \Lambda u = f(t), & 0 < t \le T, \\ u(0) = u_0 \end{cases}$$

in the space $L_2(\Omega)$, Λ being a positive definite self-adjoint operator of $L_2(\Omega)$. Let $f \in \mathcal{C}([0,T]; L_2(\Omega))$ and $u_0 \in L_2(\Omega)$. If $u(t)$ is a strict solution lying in the solution space:

$$u \in \mathcal{C}([0,T]; L_2(\Omega)) \cap \mathcal{C}((0,T]; \mathcal{D}(\Lambda)) \cap \mathcal{C}^1((0,T]; L_2(\Omega)),$$

then $u(t)$ is necessarily represented by the formula

$$u(t) = e^{-t\Lambda} u_0 + \int_0^t e^{-(t-s)\Lambda} f(s)ds, \qquad 0 \le t \le T. \tag{3.11}$$

Next, consider the Cauchy problem of a linear equation

$$\begin{cases} \dfrac{du}{dt} = p(t)u + q(t), & 0 < t \le T, \\ u(0) = u_0 \end{cases}$$

in the space $L_\infty(\Omega)$, $p(t)$ and $q(t)$ being functions such that p, $q \in \mathcal{C}([0,T]; L_\infty(\Omega))$. Then, one can show that, for any initial value $u_0 \in L_\infty(\Omega)$, there exists a unique strict solution $u \in \mathcal{C}^1([0,T]; L_\infty(\Omega))$ and the solution is given by

$$u(t) = e^{\int_0^t p(\tau)d\tau} u_0 + \int_0^t e^{\int_s^t p(\tau)d\tau} q(s)ds, \qquad 0 \le t \le T. \tag{3.12}$$

For the proof, see [15, p. 53].

4 Local solutions

In order to construct local solutions to (1.1), we want to apply the theory of abstract semilinear parabolic evolution equations.

As for the first and second equations of (1.1), we handle them in the space $L_\infty(\Omega)$ because they are ordinary differential equations for each $x \in \Omega$. Meanwhile, as for the third and fourth equations which are diffusion equations, we handle them in the space $L_2(\Omega)$. Thereby, we set the following underlying space

$$X \equiv \left\{ \begin{pmatrix} u \\ v \\ w \\ g \end{pmatrix} ;\ u,\ v \in L_\infty(\Omega) \ \text{and}\ w,\ g \in L_2(\Omega) \right\}. \tag{4.1}$$

In the space X, (1.1) can be formulated as the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0. \end{cases} \tag{4.2}$$

Here, A denotes a closed linear operator of X of the form

$$A \equiv \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & \Lambda_w & 0 \\ 0 & 0 & 0 & \Lambda_g \end{pmatrix} = \operatorname{diag}\{f, h, \Lambda_w, \Lambda_g\}, \tag{4.3}$$

where $\varLambda_w$ (resp. $\varLambda$) is a realization of the Laplace operator $-d_w\varDelta+\beta$ (resp. $-d_g\varDelta+1$) in $L_2(\Omega)$ under the homogeneous Neumann conditions on $\partial\Omega$. The domain of A is given by

$$\mathcal{D}(A)=L_\infty(\Omega)\times L_\infty(\Omega)\times H^2_N(\Omega)\times H^2_N(\Omega), \tag{4.4}$$

because of (3.6). As A is diagonal, A is easily seen to be a sectorial operator of X with angle 0, namely, its spectrum is contained in the half real line $(0,\infty)$ and its resolvent satisfies the estimate $\|(z-A)^{-1}\|_{\mathcal{L}(X)}\le\frac{M}{|z|+1}$ for $z\notin(0,\infty)$ with some constant $M>0$. Consequently, $-A$ generates an analytic semigroup $e^{-tA}\,(0\le t<\infty)$ on X which is represented by $e^{-tA}=\operatorname{diag}\{e^{-tf},e^{-th},e^{-t\varLambda_w},e^{-t\varLambda_g}\}$.

Similarly, for $0<\theta<1$, the fractional power A^θ of A is represented by

$$A^\theta=\operatorname{diag}\{f^\theta,h^\theta,\varLambda_w^\theta,\varLambda_g^\theta\}\ \text{ with }\ \mathcal{D}(A^\theta)=L_\infty(\Omega)\times L_\infty(\Omega)\times\mathcal{D}(\varLambda_w^\theta)\times\mathcal{D}(\varLambda_g^\theta), \tag{4.5}$$

(as for $\mathcal{D}(\varLambda_w^\theta)$ and $\mathcal{D}(\varLambda_g^\theta)$, see (3.7)).

In the meantime, $F(U)$ denotes a nonlinear operator of X of the form

$$F(U)\equiv\begin{pmatrix}\beta\delta[\operatorname{Re}w-w_*]_+-(\lambda g+av^2+c)u\\ fu\\ \alpha v\\ -\mu vg+\gamma(g-\ell)(1-g)g+g\end{pmatrix},\qquad U=\begin{pmatrix}u\\v\\w\\g\end{pmatrix}\in\mathcal{D}(F), \tag{4.6}$$

where $\mathcal{D}(F)=[L_\infty(\Omega)]^4$. In what follows we fix an exponent ϑ arbitrarily so that

$$\tfrac{1}{2}<\vartheta<\tfrac{3}{4}. \tag{4.7}$$

Then, on account of (3.5), (3.8) and (4.5), we see that $\mathcal{D}(A^\vartheta)\subset\mathcal{D}(F)$ with continuous embedding. In addition, since the entries of $F(U)$ are a polynomial of u,v,w,g of at most third order except the term $[\operatorname{Re}w-w_*]_+$ and since $[\operatorname{Re}w-w_*]_+$ is Lipschitz continuous for $w\in\mathbb{C}$, it is directly verified that

$$\|F(U)-F(V)\|_{L_\infty}\le C(\|U\|_{L_\infty}+\|V\|_{L_\infty}+1)^2\|U-V\|_{L_\infty}$$
$$\text{for }\ U={}^t(u_1,v_1,w_1,g_1),\ V={}^t(u_2,v_2,w_2,g_2)\in\mathcal{D}(F),$$

with some constant $C>0$. This then readily implies that

$$\|F(U)-F(V)\|_X\le C(\|A^\vartheta U\|_X+\|A^\vartheta V\|_X+1)^2\|A^\vartheta(U-V)\|_X,\qquad U,\,V\in\mathcal{D}(A^\vartheta). \tag{4.8}$$

Finally, U_0 denotes an initial value which is taken from $\mathcal{D}(A^\vartheta)$.

We can then conclude the following result.

Theorem 4.1. *Under* (4.7) *let* $U_0={}^t(u_0,v_0,w_0,g_0)$ *be in* $\mathcal{D}(A^\vartheta)$, *i.e.,* $u_0,\,v_0\in L_\infty(\Omega)$ *and* $w_0,\,g_0\in H^{2\vartheta}(\Omega)$. *Then,* (4.2) *(and hence* (1.1)*) possesses a unique local solution in the function space:*

$$\begin{cases}u,\,v\in\mathcal{C}([0,T_{U_0}];L_\infty(\Omega))\cap\mathcal{C}^1((0,T_{U_0}];L_\infty(\Omega)),\\ w,\,g\in\mathcal{C}([0,T_{U_0}];H^{2\vartheta}(\Omega))\cap\mathcal{C}^1((0,T_{U_0}];L_2(\Omega))\cap\mathcal{C}((0,T_{U_0}];H^2_N(\Omega)),\\ t^{1-\vartheta}w,\,t^{1-\vartheta}g\in\mathcal{B}((0,T_0];H^2_N(\Omega)).\end{cases} \tag{4.9}$$

Here, $T_{U_0}>0$ *is determined only by the norm*

$$\|u_0\|_{L_\infty}+\|v_0\|_{L_\infty}+\|w_0\|_{H^{2\vartheta}}+\|g_0\|_{H^{2\vartheta}} \tag{4.10}$$

of the initial value U_0.

Proof. The fundamental existence theorem [15, Theorem 4.1] (presented first in [7]) is available. Indeed, (4.8) shows that the structural assumption [15, (4.2)] is fulfilled with $\beta = \eta = \vartheta$. Therefore, it is concluded that (4.2) possesses a unique local solution in the function space:

$$\begin{cases} U \in \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A^\vartheta)) \cap \mathcal{C}^1((0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)), \\ t^{1-\vartheta} U \in \mathcal{B}((0, T_{U_0}]; \mathcal{D}(A)), \end{cases}$$

$T_{U_0} > 0$ being determined by the norm $\|A^\vartheta U_0\|_X$ alone.

Hence, in view of (4.4), each entry of the solution $U(t) = {}^t(u(t), v(t), w(t), g(t))$ belongs to the function space (4.9). From (3.8) and (4.5) it seen that

$$\begin{aligned} C^{-1}(\|u\|_{L_\infty} + \|v\|_{L_\infty} + \|w\|_{H^{2\vartheta}} + \|g\|_{H^{2\vartheta}}) &\le \|A^\vartheta U\|_X \\ \le C(\|u\|_{L_\infty} + \|v\|_{L_\infty} + \|w\|_{H^{2\vartheta}} + \|g\|_{H^{2\vartheta}}), \qquad & U = {}^t(u, v, w, g) \in \mathcal{D}(A^\vartheta). \end{aligned}$$

Hence, T_{U_0} is determined by the norm of (4.10). □

5 Nonnegativity of solutions

We shall next verify that nonnegativity of initial functions implies that of the local solution obtained in Theorem 4.1.

Theorem 5.1. *Under* (4.7) *let* $U_0 = {}^t(u_0, v_0, w_0, g_0) \in \mathcal{D}(A^\vartheta)$ *satisfy* $u_0 \ge 0$, $v_0 \ge 0$, $w_0 \ge 0$ *and* $g_0 \ge 0$ *in* Ω. *Then, the local solution* $U(t) = {}^t(u(t), v(t), w(t), g(t))$ *constructed in Theorem 4.1 is also nonnegative, i.e.,* $u(t) \ge 0$, $v(t) \ge 0$, $w(t) \ge 0$ *and* $g(t) \ge 0$ *in* Ω *for every* $0 < t \le T_{U_0}$.

Proof. We want to introduce an auxiliary problem

$$(5.1) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta[\mathrm{Re}\, w - w_*]_+ - [\lambda\chi(\mathrm{Re}\, g) + av^2 + c]u - fu & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial w}{\partial t} = d_w \Delta w - \beta w + \alpha v & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial g}{\partial t} = d_g \Delta g - \mu v g + \gamma(g - \ell)(1 - g)g & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial g}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x),\ w(x, 0) = w_0(x),\ g(x, 0) = g_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\chi(g')$ is a cutoff function for $-\infty < g' < \infty$ given by

$$\chi(g') = \begin{cases} g' & \text{if } g' \ge 0, \\ 0 & \text{if } g' < 0. \end{cases}$$

Since $\chi(\mathrm{Re}\, g)$ is a uniformly Lipschitz continuous function of $g \in \mathbb{C}$, it is possible to construct a local solution to (5.1) on an interval $[0, \tilde{T}_{U_0}]$ which lies in the same function space as (4.9) and is unique in the function space. Furthermore, the arguments as in the proof of [2, Theorem 4.1] (cf. also [15, Subsec. 11.2.3]) are available to conclude that the local solution satisfies that $u(t) \ge 0$, $v(t) \ge 0$ and $w(t) \ge 0$ in Ω for every $0 < t \le \tilde{T}_{U_0}$.

So, let us here verify that $g(t) \geq 0$ in Ω for every $0 < t \leq \tilde{T}_{U_0}$. First, we notice that, since the function $(u(t), v(t), w(t), \overline{g}(t))$ is also a local solution of (5.1), uniqueness of solution implies that $(u(t), v(t), w(t), \overline{g}(t)) = (u(t), v(t), w(t), g(t))$ for every $0 < t \leq \tilde{T}_{U_0}$. In particular, $g(t)$ is a real valued function of $x \in \Omega$ for each t. Second, in view of this fact, we shall use another cutoff function. Let $H(g)$ be a $\mathcal{C}^{1,1}$ function such that $H(g) = \frac{g^2}{2}$ for $-\infty < g < 0$ and $H(g) = 0$ for $0 \leq g < \infty$. We consider the function

$$\psi(t) = \int_\Omega H(g(x,t))dx, \qquad 0 \leq t \leq \tilde{T}_{U_0}.$$

Clearly, $\psi(t)$ is a nonnegative $\mathcal{C}^1$ function with the derivative

$$\begin{aligned}\psi'(t) = \int_\Omega H'(g(t))\frac{dg}{dt}(t)dx = \int_\Omega & H'(g(t))d_g \Delta g(t)dx \\ &+ \int_\Omega H'(g(t))\left[-\mu v(t) + \gamma(g(t) - \ell)(1 - g(t))\right] g(t)dx.\end{aligned}$$

Consequently, there is a constant $C_U > 0$ depending on $U(t)$ such that

$$\psi'(t) \leq -d_g \int_\Omega H''(g(t))|\nabla g(t)|^2 dx + C_U \int_\Omega H'(g(t))g(t)dx, \qquad 0 < t \leq \tilde{T}_{U_0}.$$

Since $H''(g) \geq 0$ and $H'(g)g = 2H(g)$ for $g \in \mathbb{R}$, it follows that $\psi'(t) \leq 2C_U\psi(t)$. Hence, $0 \leq \psi(t) \leq e^{2C_U t}\psi(0)$ for every $0 < t \leq \tilde{T}_{U_0}$. Finally, $g_0 \geq 0$ implies $\psi(0) = 0$ and hence $\psi(t) = 0$ for every t, i.e., $g(t) \geq 0$ in Ω.

We have thus seen that the local solution to the auxiliary problem (5.1) is nonnegative. This in turn shows that the local solution is as well a local solution of (1.1) (because of $\chi(\operatorname{Re} g(t)) = g(t)$). Uniqueness of local solution for (1.1) then yields that the local solution for (1.1) obtained by Theorem 4.1 coincides with that of (5.1) on the interval $[0, \tilde{T}_{U_0}]$. This means that the assertion of theorem is verified at least on the time interval $[0, \tilde{T}_{U_0}]$.

Consider the time $t_1 = \sup\{0 < t \leq T_{U_0};\ U(s)$ is nonnegative for any $s \in [0,t]\}$. And suppose that $t_1 < T_{U_0}$. Then we can repeat the similar arguments with initial time t_1 and initial value $U_1 = U(t_1)$ to conclude that $U(t)$ is nonnegative for all $t > t_1$ which are sufficiently close to t_1. But this contradicts the definition of the time t_1. Hence, $t_1 = T_{U_0}$. □

6 Global solutions Let us first build up a priori estimates for the local solutions of (1.1).

Proposition 6.1. *Under* (4.7) *let* $0 \leq u_0, v_0 \in L_\infty(\Omega)$ *and* $0 \leq w_0, g_0 \in H^{2\vartheta}(\Omega)$. *Let* $U = (u,v,w,g)$ *denote any local solution of* (1.1) *on an interval* $[0, T_U]$ *such that*

$$\begin{cases} 0 \leq u,\, v \in \mathcal{C}([0,T_U]; L_\infty(\Omega)) \cap \mathcal{C}^1((0,T_U]; L_\infty(\Omega)), \\ 0 \leq w,\, g \in \mathcal{C}([0,T_U]; H^{2\vartheta}(\Omega)) \cap \mathcal{C}^1((0,T_U]; L_2(\Omega)) \cap \mathcal{C}((0,T_U]; H^2_N(\Omega)). \end{cases} \tag{6.1}$$

Then, the estimate

$$\begin{aligned}&\|u(t)\|_{L_\infty} + \|v(t)\|_{L_\infty} + \|w(t)\|_{H^{2\vartheta}} + \|g(t)\|_{H^{2\vartheta}} \\ &\qquad \leq C\left[\|u_0\|_{L_\infty} + \|v_0\|_{L_\infty} + \|w_0\|_{H^{2\vartheta}} + \|g_0\|^2_{H^{2\vartheta}} + 1\right], \qquad 0 \leq t \leq T_U,\end{aligned} \tag{6.2}$$

holds with some constant C *independent of* T_U.

Proof. Throughout the proof, we shall use a universal notation C to denote positive constants which are determined by the constants d_w, d_g, a, c, f, h, α, β, γ, δ, λ, μ, w_* and ℓ and by Ω. So, C may change from occurrence to occurrence.

Step 1. Let us first estimate the norms $\|u(t)\|_{L_2}$, $\|v(t)\|_{L_2}$ and $\|w(t)\|_{L^2}$. Multiply the first equation of (1.1) by u and integrate the product in Ω. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + f\int_\Omega u^2dx &= \beta\delta\int_\Omega [w-w_*]_+u\,dx - \int_\Omega(\lambda g + av^2+c)u^2dx\\ &\le \frac{f}{2}\int_\Omega u^2dx + \frac{(\beta\delta)^2}{2f}\int_\Omega(w^2+w_*^2)dx - a\int_\Omega u^2v^2dx.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_\Omega u^2dx + f\int_\Omega u^2dx \le (\beta\delta)^2f^{-1}\int_\Omega(w^2+w_*^2)dx - 2a\int_\Omega u^2v^2dx. \tag{6.3}$$

Multiply the second equation of (1.1) by v and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2dx + h\int_\Omega v^2dx = f\int_\Omega uv\,dx,$$

or

$$\frac{d}{dt}\int_\Omega v^2dx + 2h\int_\Omega v^2dx = 2f\int_\Omega uv\,dx. \tag{6.4}$$

Finally, multiply the third equation of (1.1) by w and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \beta\int_\Omega w^2dx = -d_w\int_\Omega|\nabla w|^2dx + \alpha\int_\Omega vw\,dx \le \frac{\beta}{2}\int_\Omega w^2dx + \frac{\alpha^2}{2\beta}\int_\Omega v^2dx.$$

Therefore,

$$\frac{d}{dt}\int_\Omega w^2dx + \beta\int_\Omega w^2dx \le \alpha^2\beta^{-1}\int_\Omega v^2dx. \tag{6.5}$$

Introduce here two positive parameters ρ and η; and, multiply the inequalities (6.3) and (6.5) by ρ and η, respectively. Then, summing up the resulting inequalities and the equation (6.4), we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega[\rho u^2+v^2+\eta w^2]dx + \int_\Omega[f\rho u^2+2hv^2+\beta\eta w^2]dx\\ &\le \int_\Omega[\alpha^2\beta^{-1}\eta v^2+(\beta\delta)^2f^{-1}\rho w^2]dx + 2\int_\Omega[f(uv)-a\rho(uv)^2]dx + (\beta\delta w_*)^2f^{-1}|\Omega|\rho.\end{aligned} \tag{6.6}$$

Furthermore, fix $\eta>0$ sufficiently small so that $\alpha^2\beta^{-1}\eta<2h$ and then fix $\rho>0$ sufficiently small so that $(\beta\delta)^2f^{-1}\rho<\beta\eta$. Then, as $f(uv)-a\rho(uv)^2\le f^2(4a\rho)^{-1}$ for u, $v\ge 0$, it follows that

$$\frac{d}{dt}\int_\Omega[\rho u^2+v^2+\eta w^2]dx + \varepsilon\int_\Omega[\rho u^2+v^2+\eta w^2]dx \le [(\beta\delta w_*)^2f^{-1}\rho + f^2(2a\rho)^{-1}]|\Omega|$$

with some constant $\varepsilon>0$. Solving this differential inequality, we conclude that

$$\rho\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 + \eta\|w(t)\|_{L_2}^2 \le C\left[e^{-\varepsilon t}(\rho\|u_0\|_{L_2}^2+\|v_0\|_{L_2}^2+\eta\|w_0\|_{L_2}^2)+1\right],$$

or

$$\text{(6.7)}\quad \|u(t)\|_{L_2}^2+\|v(t)\|_{L_2}^2+\|w(t)\|_{L_2}^2 \le C_1\left[e^{-\varepsilon t}(\|u_0\|_{L_2}^2+\|v_0\|_{L_2}^2+\|w_0\|_{L_2}^2)+1\right], \qquad 0\le t\le T_U.$$

Step 2. The estimate (6.7) directly implies the estimate of $\|w(t)\|_{H^{2\vartheta}}$. In fact, it is known by (3.11) that $w(t)$ is represented by

$$w(t)=e^{-t\Lambda_w}w_0+\int_0^t e^{-(t-s)\Lambda_w}\alpha v(s)ds,$$

where Λ_w is a realization of $-d_w\Delta+\beta$ in $L_2(\Omega)$ under the homogeneous Neumann conditions on $\partial\Omega$ and where $e^{-t\Lambda_w}$ is the semigroup on $L_2(\Omega)$ generated by $-\Lambda_w$. Operating Λ_w^ϑ, we have

$$\Lambda_w^\vartheta w(t)=e^{-t\Lambda_w}[\Lambda_w^\vartheta w_0]+\int_0^t \Lambda_w^\vartheta e^{-\frac{t-s}{2}\Lambda_w}e^{-\frac{t-s}{2}\Lambda_w}\alpha v(s)ds.$$

Therefore, by (3.9) and (3.10),

$$\|\Lambda_w^\vartheta w(t)\|_{L_2}\le Ce^{-\beta t}\|\Lambda_w^\vartheta w_0\|_{L_2}+C\int_0^t (t-s)^{-\vartheta}e^{-\frac{\beta}{2}(t-s)}ds\max_{0\le s\le t}\|v(s)\|_{L_2}.$$

Since

$$\int_0^t (t-s)^{-\vartheta}e^{-\frac{\beta}{2}(t-s)}ds=\int_0^t \sigma^{-\vartheta}e^{-\frac{\beta}{2}\sigma}d\sigma<\int_0^\infty \sigma^{-\vartheta}e^{-\frac{\beta}{2}\sigma}d\sigma=\left(\frac{2}{\beta}\right)^{1-\vartheta}\Gamma(1-\vartheta),$$

we obtain by (6.7) that

$$\|\Lambda_w^\vartheta w(t)\|_{L_2}\le C\left[e^{-\frac{\beta}{2}t}\|\Lambda_w^\vartheta w_0\|_{L_2}+\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{L_2}+1\right].$$

Hence, in view of (3.8),

$$\text{(6.8)}\quad \|w(t)\|_{H^{2\vartheta}}\le C_2\left[e^{-\frac{\beta}{2}t}\|w_0\|_{H^{2\vartheta}}+\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{L_2}+1\right], \qquad 0\le t\le T_U.$$

Step 3. In view of (3.5), we see from (6.8) that

$$\|w(t)\|_{L_\infty}\le C[\|u_0\|_{L_2}+\|v_0\|_{L_2}+\|w_0\|_{H^{2\vartheta}}+1], \qquad 0\le t\le T_U.$$

By use of this, let us estimate the norms $\|u(t)\|_{L_\infty}$ and $\|v(t)\|_{L_\infty}$.

First, apply the formula (3.12) to the first equation of (1.1). Then, we have

$$u(t)=e^{-\int_0^t[\lambda g(s)+av(s)^2+c+f]ds}u_0+\int_0^t e^{-\int_s^t[\lambda g(\tau)+av(\tau)^2+c+f]d\tau}\beta\delta[w(s)-w_*]_+ds$$

in the space $L_\infty(\Omega)$. Therefore,

$$\|u(t)\|_{L_\infty}\le e^{-(c+f)t}\|u_0\|_{L_\infty}+\beta\delta\int_0^t e^{-(c+f)(t-s)}[\|w(s)\|_{L_\infty}+w_*]ds.$$

Hence,

$$\|u(t)\|_{L_\infty} \le C_3 \left[e^{-(c+f)t}\|u_0\|_{L_\infty} + \|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^{2\vartheta}} + 1\right], \quad 0 \le t \le T_U. \tag{6.9}$$

Second, the similar arguments yield from the second equation of (1.1) the estimate

$$\|v(t)\|_{L_\infty} \le C_4 \left[e^{-ht}\|v_0\|_{L_\infty} + \|u_0\|_{L_\infty} + \|v_0\|_{L_2} + \|w_0\|_{H^{2\vartheta}} + 1\right], \qquad 0 \le t \le T_U. \tag{6.10}$$

Of course, (6.9) is used for estimating the integral $\int_0^t e^{-h(t-s)}u(s)ds$ in $L_\infty(\Omega)$.

Step 4. The estimates for the norms of $g(t)$ are carried out quite analogously. Let us first estimate the norm $\|g(t)\|_{L_6}$.

Multiply the fourth equation of (1.1) by $g(t)^5$ and integrate the product in Ω. Then, after some calculations, we have

$$\frac{1}{6}\frac{d}{dt}\int_\Omega g^6 dx = -5d_g \int_\Omega g^4|\nabla g|^2 dx - \mu\int_\Omega vg^6 dx + \gamma\int_\Omega (g-\ell)(1-g)g^6 dx.$$

In view of (3.1),

$$\frac{1}{6}\frac{d}{dt}\int_\Omega g^6 dx + \frac{\gamma(1-\ell)}{6}\int_\Omega g^6 dx \le \frac{\gamma(1-\ell)}{6}\int_\Omega dx = \frac{\gamma(1-\ell)}{6}|\Omega|.$$

Solving this differential inequality, we obtain that

$$\|g(t)\|_{L_6}^6 \le e^{-\gamma(1-\ell)t}\|g_0\|_{L_6}^6 + |\Omega|, \qquad 0 \le t \le T_U. \tag{6.11}$$

Step 5. Regarding $g(t)$ as the solution to a linear evolution equation (i.e., the fourth equation of (1.1)), we describe $g(t)$ by the integral

$$g(t) = e^{-tA_g}g_0 + \int_0^t e^{-(t-s)A_g}\left[-\mu v(s) + \gamma(g(s)-\ell)(1-g(s)) + 1\right]g(s)ds$$

in $L_2(\Omega)$ (due to (3.11)), where A_g is a realization of $-d_g\Delta + 1$ in $L_2(\Omega)$ under the homogeneous Neumann conditions on $\partial\Omega$. Then, the similar arguments as in Step 2 yield that

$$\|A_g^\vartheta g(t)\|_{L_2} \le C\left[e^{-t}\|A_g^\vartheta g_0\|_{L_2} + \max_{0\le s\le t}\left\{\|v(s)g(s)\|_{L_2} + \|(1+g(s))^2 g(s)\|_{L_2}\right\}\right].$$

Hence we obtain by (6.10) and (6.11) that

$$\|A_g^\vartheta g(t)\|_{L_2} \le C\left[e^{-t}\|A_g^\vartheta g_0\|_{L_2} + \|u_0\|_{L_\infty} + \|v_0\|_{L_\infty} + \|w_0\|_{H^{2\vartheta}} + \|g_0\|_{L_6}^3 + 1\right],$$

or due to (3.8),

$$\begin{aligned}\|g(t)\|_{H^{2\vartheta}} \le C_5\big[e^{-t}\|g_0\|_{H^{2\vartheta}} + \|u_0\|_{L_\infty} + \|v_0\|_{L_\infty} \\ + \|w_0\|_{H^{2\vartheta}} + \|g_0\|_{L_6}^3 + 1\big], \qquad 0 \le t \le T_U.\end{aligned} \tag{6.12}$$

Combing (6.8), (6.9), (6.10) and (6.12), we conclude the desired estimate (6.2). □

As an immediate consequence of the a priori estimates above, we can prove existence and uniqueness of global solution for the problem (1.1).

Theorem 6.1. *Let ϑ be as in* (4.7)*, and let* $0 \le u_0,\ v_0 \in L_\infty(\Omega)$ *and* $0 \le w_0,\ g_0 \in H^{2\vartheta}(\Omega)$*. Then,* (1.1) *possesses a unique global solution in the function space:*

$$\begin{cases} 0 \le u,\ v \in \mathcal{C}([0,\infty); L_\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L_\infty(\Omega)), \\ 0 \le w,\ g \in \mathcal{C}([0,\infty); H^{2\vartheta}(\Omega)) \cap \mathcal{C}^1((0,\infty); L_2(\Omega)) \cap \mathcal{C}((0,\infty); H^2_N(\Omega)). \end{cases}$$

Of course, the global solution satisfies all the estimates (6.7)$\sim$(6.12) *on the interval* $[0,\infty)$*.*

Proof. First, by Theorems 4.1 and 5.1, there exists a unique local solution (u, v, w, g) to (1.1) on an interval $[0, T_{U_0}]$ which is nonnegative.

Second, consider any local solution of (1.1) on an interval $[0, T_U]$ in the function space (6.1). Then, Proposition 6.1 provides that the norm $\|A^\vartheta U(T_U)\|_X$ is estimated by $\|A^\vartheta U_0\|_X$ alone (independently of the end time T_U). We can then apply Theorems 4.1 and 5.1 with initial time T_U and initial value $U(T_U)$ and know that the solution (u, v, w, g) can be uniquely extended as a nonnegative local solution on an interval $[0, T_U + \tau]$, where $\tau > 0$ depends on the norm $\|A^\vartheta U(T_U)\|_X$ and hence depends only on the norm $\|A^\vartheta U_0\|_X$.

Thus, we have verified that any local solution on $[0, T_U]$ in the function space (6.1) can always be extended as a nonnegative local solution on a longer interval $[0, T_U + \tau]$ with a fixed length $\tau > 0$. This evidently means that the assertion of theorem is true. □

Let us finally observe Lipschitz continuity of solutions $U(t)$ in the initial values U_0. Let B be a bounded set of $\mathcal{D}(A^\vartheta)$ such that

$$B_R = \{U_0 \in \mathcal{D}(A^\vartheta); \|A^\vartheta U_0\|_X \le R \ \text{ and } \ U_0 \ge 0\} \tag{6.13}$$

with radius $R > 0$. Then, there exists a unique global solution to (1.1) for each $U_0 \in B$. As a direct consequence of [15, Theorem 4.3], we observe the following result.

Proposition 6.2. *Let $U(t)$ (resp. $V(t)$) denote the global solution to* (1.1) *for initial value $U_0 \in B_R$ (resp. $V_0 \in B_R$). Then, for each fixed time $T > 0$, there exists some constants $C_{R,T} > 0$ depending on R and T alone such that*

$$\|A^\vartheta[U(t) - V(t)]\|_X \le C_{R,T}\|A^\vartheta[U_0 - V_0]\|_X \quad \text{for any } \ 0 \le t \le T. \tag{6.14}$$

7 Dynamical system This section is devoted to constructing a dynamical system generated by the problem (1.1). As for the phase space we set

$$K = \{U_0 \in \mathcal{D}(A^\vartheta);\ U_0 \ge 0\} \ \subset \ \mathcal{D}(A^\vartheta),$$

K being a metric space equipped with the distance induced by the norm $\|A^\vartheta \cdot \|_X$.

As shown by Theorem 6.1, for each $U_0 \in K$, there exists a unique global solution $U(t; U_0)$ of (1.1) with values in K. Therefore, we can define a nonlinear semigroup $\{S(t)\}_{0 \le t < \infty}$ acting on K by the formula $S(t)U_0 = U(t; U_0)$. As shown by Proposition 6.2, $U_0 \mapsto U(t; U_0)$ is locally Lipschitz continuous from K into itself. Furthermore, according to (6.14), the Lipschitz constant is uniform on any bounded set B_R of K and on any finite interval $[0, T]$. It then easily follows that the mapping $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into K, namely, $S(t)$ is a continuous semigroup on K. Hence, (1.1) generates a dynamical system $(S(t), K, \mathcal{D}(A^\vartheta))$.

The a priori estimates (6.7)$\sim$(6.12) we have established in the proof of Proposition 6.2 provide existence of a bounded absorbing set of K.

Theorem 7.1. *The dynamical system $(S(t), K, \mathcal{D}(A^\vartheta))$ possesses a bounded, invariant and absorbing subset $\widetilde{B}$ of K.*

Proof. Let $R > 0$ and let B_R be a bounded subset of the form (6.13). Let $U_0 \in B_R$ be any initial value and put $S(t)U_0 = {}^t(u(t), v(t), w(t), g(t))$.

From (6.7) we see that there is a time $t_1 > 0$ depending only on R such that

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 + \|w(t)\|_{L_2}^2 \le 2C_1, \qquad t_1 \le \forall t < \infty.$$

Apply (6.8) to $w(t)$ but with initial time t_1 and initial value $S(t_1)U_0$. Then,

$$\|w(t)\|_{H^{2\vartheta}} \le C_2 \Big[e^{-\frac{\beta}{2}(t-t_1)} \|w(t_1)\|_{H^{2\vartheta}} \\ + \|u(t_1)\|_{L_2} + \|v(t_1)\|_{L_2} + \|w(t_1)\|_{L_2} + 1 \Big], \qquad t_1 \le \forall t < \infty.$$

From this we see that there is a time $t_2 > t_1$ depending only on R such that

$$\|w(t)\|_{H^{2\vartheta}} \le C_2 \left[\sqrt{3}\sqrt{2C_1} + 2 \right], \qquad t_2 \le \forall t < \infty.$$

We repeat the similar arguments by using (6.9)$\sim$(6.12) to see ultimately that there is a time $T_R > 0$ depending only on R such that

$$\|A^\vartheta S(t)U_0\|_X \le \widetilde{C}, \qquad t_R \le \forall t < \infty,$$

here $\widetilde{C} > 0$ is a suitable universal constant determined by C_1, C_2, C_3, C_4, C_5 alone.

Set $B \equiv B_{\widetilde{C}} = \{U \in K;\ \|A^\vartheta U\|_X \le \widetilde{C}\}$. Then, as shown above, B is an absorbing set of $(S(t), K, \mathcal{D}(A^\vartheta))$. Since B itself is a bounded subset of K, there is a time $t_{\widetilde{C}}$ such that $S(t)B \subset B$ for all $t \ge t_{\widetilde{C}}$. We then set

$$\widetilde{B} = \bigcup_{0 \le t < \infty} S(t)B = \bigcup_{0 \le t \le t_{\widetilde{C}}} S(t)B.$$

It is clear that $\widetilde{B}$ is an invariant set. Since $B \subset \widetilde{B}$, $\widetilde{B}$ is also an absorbing set. Proposition 6.1 means that $\widetilde{B}$ is a bounded subset. Hence, $\widetilde{B}$ is a subset to be constructed. □

Let us now consider the ω-limit set. For each global solution $S(t)U_0$, its ω-limit set is usually defined by

$$\omega(U_0) = \bigcap_{0 \le t < \infty} \overline{\{S(\tau)U_0;\ t \le \tau < \infty\}} \qquad \text{(closure in the topology of } K).$$

In the present case, however, the trajectory $\{S(t)U_0;\ 0 \le t < \infty\}$ is not necessarily a relatively compact set of K. So, $\omega(U_0)$ may be an empty set in general. So, we will introduce another ω-limit set with respect to some weak topology of K.

We introduce the weak* topology of K: a sequence $\{(u_n, v_n, w_n, g_n)\}$ in K is said to be weak* convergent to $(\overline{u}, \overline{v}, \overline{w}, \overline{g})$ as $n \to \infty$ if

$$\begin{cases} u_n \to \overline{u} \ \text{ and } \ v_n \to \overline{v} & \text{weak* in } \ L_\infty(\Omega), \\ w_n \to \overline{w} \ \text{ and } \ g_n \to \overline{g} & \text{weakly in } \ H^{2\vartheta}(\Omega). \end{cases}$$

The weak* ω-limit set of $S(t)U_0$ is then defined by

$$\text{w}^*\text{-}\omega(U_0) = \left\{ \overline{U} \in K;\ \exists t_n \nearrow \infty \ \text{ such that } \ S(t_n)U_0 \to \overline{U} \ \text{ in weak* topology} \right\}. \tag{7.1}$$

Theorem 7.2. *For each $U_0 \in K$,* w*-$\omega(U_0)$ *is not an empty set.*

Proof. Put $S(t)U_0 = {}^t(u(t), v(t), w(t), g(t))$. Since $\{(u(t), v(t)); 0 \leq t < \infty\}$ is a bounded subset of $L_\infty(\Omega) \times L_\infty(\Omega)$, the Banach-Alaoglu theorem [11, p. 65] guarantees the trajectory $S(t)U_0$ to contain a sequence $(u(t_n), v(t_n))$, where $t_n \nearrow \infty$, which converges to $(\overline{u}, \overline{v})$ in the weak* topology of $L_\infty(\Omega) \times L_\infty(\Omega)$. It is easy to see that $\overline{u} \geq 0$ and $\overline{v} \geq 0$ in Ω. On the other hand, as $H^{2\vartheta}(\Omega)$ is a Hilbert space, $(w(t_n), g(t_n))$ contains a subsequence $(w(t_{n'}), g(t_{n'}))$ which is convergent to $(\overline{w}, \overline{g})$ in the weak topology of $H^{2\vartheta}(\Omega) \times H^{2\vartheta}(\Omega)$. It is clear that $\overline{w} \geq 0$ and $\overline{g} \geq 0$ in Ω. Hence, as $n' \to \infty$, $S(t_{n'})U_0$ is weak* convergent to $\overline{U} = {}^t(\overline{u}, \overline{v}, \overline{w}, \overline{g}) \in K$. Then, by the definition (7.1), we conclude that $\overline{U}$ belongs to w*-$\omega(U_0)$. □

8 Numerical Examples We conclude this paper by presenting some numerical results. These results show that our problem (1.1) can actually admit some coexisting solutions of trees and grass together with the boundary which divides forest and grassland.

Throughout the numerical computations, the domain is set as $\Omega = (0, 1) \times (0, 1)$. The constants in (1.1) are fixed as $d_w = 0.1$, $d_g = 1 \times 10^{-6}$, $a = 1$, $c = 0$, $f = 1$, $h = 0.5$, $\alpha = \beta = 1$, $\gamma = 40$, $\delta = 1$, $\lambda = 9$, $\mu = 50$, $w_* = 0.1$ and $\ell = 0.1$.

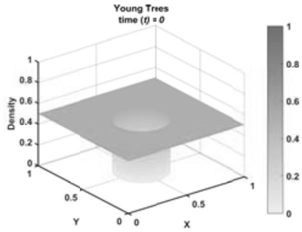
As in Figure 1, the initial functions $u_0(x)$, $v_0(x)$, $w_0(x)$ and $g_0(x)$ are taken as

$$u_0(x),\ v_0(x)\ \text{and}\ w_0(x) \equiv \begin{cases} 0 & \text{for } x \in B(x_0; r), \\ 0.5 & \text{for } x \in \Omega - B(x_0; r), \end{cases}$$

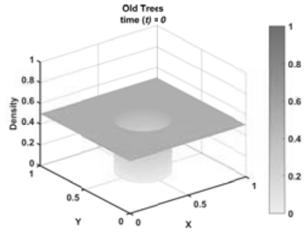
$$g_0(x) \equiv \begin{cases} 0.1 & \text{for } x \in B(x_0; r), \\ 0 & \text{for } x \in \Omega - B(x_0; r), \end{cases}$$

where x_0 denotes the central point $(0.5, 0.5)$ of Ω and $0 < r < 0.5$ denotes a radius of disk to be adjusted in our simulations. Starting from such initial functions, computations are continued until the approximate solution is stabilized numerically (almost $T = 1000$).

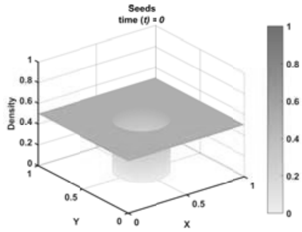
When $r = 0.1$, the solution tends to a state of homogeneous forest and no grass, see Figure 2. When $r = 0.2$ the solution tends to a coexisting state of trees and grass, see Figure 3. Finally, when $r = 0.3$, the solution tends to a state of homogeneous grass and no trees, see Figure 4.



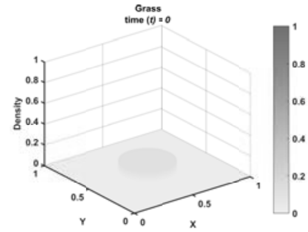
(a) Graph of young trees



(b) Graph of old trees

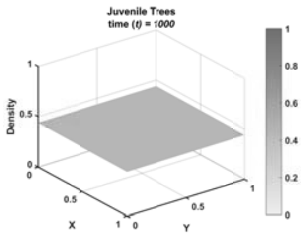


(c) Graph of seeds

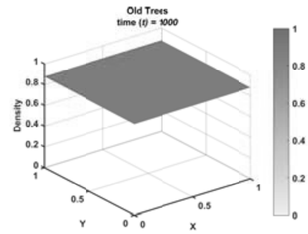


(d) Graph of grass

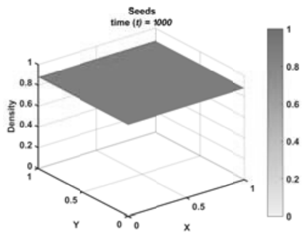
Fig. 1: Initial function.



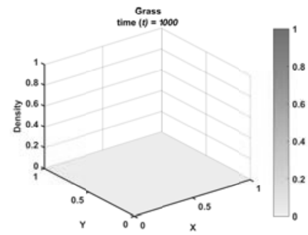
(a) Graph of young trees



(b) Graph of old trees

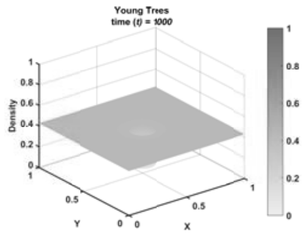


(c) Graph of seeds

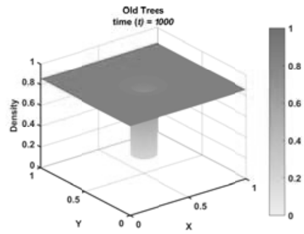


(d) Graph of grass

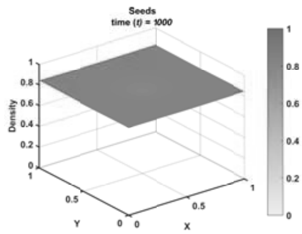
Fig. 2: When $r = 0.1$, the solution tends to a state of homogeneous forest and no grass.



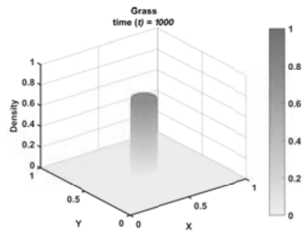
(a) Graph of young trees



(b) Graph of old trees

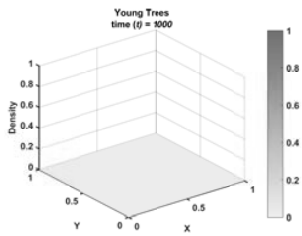


(c) Graph of seeds

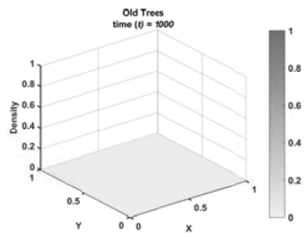


(d) Graph of grass

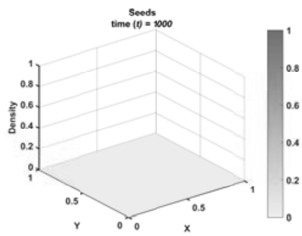
Fig. 3: When $r = 0.2$ the solution tends to a coexisting state of trees and grass.



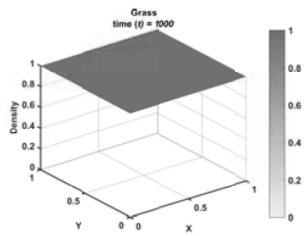
(a) Graph of young trees



(b) Graph of old trees



(c) Graph of seeds



(d) Graph of grass

Fig. 4: When $r = 0.3$, the solution tends to a state of homogeneous grass and no trees.

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TREE-GRASS SEGREGATION PATTERNS

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ABSTRACT. In the preceding paper [21], we have introduced a tree-grass competition model for describing the kinematics of forest-grassland system and have found that the model admits some solutions showing coexistence of forest and grassland. The purpose of the present paper is then to investigate the boundary curves which partition forest patches and grassland patches. Through the investigations, we want to clarify the properties of segregation patterns of tree-grass coexistence in terms of forest connectivity. As it is very difficult to handle the very model equations in [21], we will make a reduction of the full model by extremely restricting the range where the parameters of equations can vary.

1 Introduction In the preceding paper [21], we have introduced a tree-grass competition model for describing the kinematics of forest-grassland system from a viewpoint of competitive system between trees and grass. We have also found after proving global existence of solutions that the model admits some solutions showing coexistence of forest and grassland which are partitioned each other by some clear boundary curves.

The purpose of the present paper is then to investigate the boundary curves partitioning forest and grassland, in other words, the properties of segregation patterns of trees and grass. It is, however, very difficult to handle the very model equations [21, (1.1)], for the dynamics of solutions change drastically depending on the parameters contained in the equations and the model equations actually contain so various parameters. Before investigating the segregation patterns, we want to make some restrictions on the parameters as follows.

First, we will consider an extreme case when the reaction rates μ and γ in the equation of grass of [21, (1.1)] are sufficiently large with respect to the diffusion rate d_g and when they are even sufficiently large with respect to the reaction rates f and h of the equation of old age trees. In such a case, as discussed in the next section, the model equations can reasonably be reduced into a smaller model. As a matter of fact, the reduced model coincides with the classical kinematic model of forest presented by Kuznetsov-Antonovsky-Biktashev-Aponina [11].

Second, we will choose only the mortality h of old age trees as a tuning parameter of our investigations fixing other parameters suitably. The reduced model given by (2.3) below with (2.4)-(2.5) coincides with the classical kinematic model of forest for which an extensive study has already been made, see [12, 17, 20], including the series of papers [1, 2, 3]. Among others, as reviewed in Section 3, the papers [1, 2, 3] clarified that the parameter h plays an important role for determining the dynamics of solutions.

By numerical computations, we shall find that three types of tree-grass segregation patterns, namely, high-connectivity forests, intermediate-connectivity forests and low connectivity forests, are created depending mainly on the mortality of old age trees. We shall also find some very interesting link between the characters of forest connectivity and the

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instability-dimension of a unique homogeneous stationary solution, which is always unstable, showing coexistence of trees and grass (see Remark 3.1).

Those types of habitat patterns are actually observed in the real world by means of satellite imagery, which is a conventional remote-sensing method. For instance, we can find those patterns in a 100km height view of the Black Forest, Schwarzwald, Germany (see Google Earth). As in [7], monitoring data are statistically processed in order to investigate characters of habitat patterns. Devia-Murthya-Debnatha-Jhaa [4] reported that the forest connectivity is playing an important role of regulating its ecological factors such as species level biodiversity, wildlife movement, seed dispersion and so on.

2 Some Reduction of Tree-Grass Competition Model Let us argue a reduction of the original tree-grass competition model introduced in our preceding paper [21].

We begin with recalling the following tree-grass interaction system:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \beta\delta w - (\lambda g + av^2 + c)u - fu & \text{in } \Omega \times (0,\infty), \\
\dfrac{\partial v}{\partial t} = fu - hv & \text{in } \Omega \times (0,\infty), \\
\dfrac{\partial w}{\partial t} = d_w \Delta w - \beta w + \alpha v & \text{in } \Omega \times (0,\infty), \\
\dfrac{\partial g}{\partial t} = d_g \Delta g - \mu v g + \gamma \left(1 - \frac{g}{K}\right) g & \text{in } \Omega \times (0,\infty), \\
\dfrac{\partial w}{\partial n} = \dfrac{\partial g}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x),\ g(x,0) = g_0(x) & \text{in } \Omega,
\end{cases}
\tag{2.1}
$$

in a two-dimensional bounded, $\mathcal{C}^2$ or convex domain Ω. Here, the unknown functions $u(x,t)$ and $v(x,t)$ denote tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and at time $t \in [0,\infty)$. The unknown function $w(x,t)$ denotes a density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$. Meanwhile, $g(x,t)$ denotes a density of grass at $x \in \Omega$ and $t \in [0,\infty)$. The third equation describes the kinetics of seeds; $d_w > 0$ is a diffusion constant, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates, respectively. The first equation describes growth of young age trees; here, $0 < \delta \le 1$ is a seed establishment rate, $\lambda g + av^2 + c$ is a mortality of young age trees which is proportional to the densities g and v^2 with coefficients $\lambda > 0$ and $a > 0$, $c > 0$ being a basic mortality. The second equation describes growth of old age trees; $f > 0$ is an aging rate from young age to old age, and $h > 0$ is a mortality. Finally, the fourth equation describes growth of grass that is basically given by a reaction-diffusion equation with a diffusion constant $d_g > 0$ and with a Fisher growth function $\gamma\left(1 - \frac{g}{K}\right) g$, where $\gamma > 0$ is a reaction rate and K is ground's capacity for grass, the term $-\mu v g$ denotes suppression by the trees with a coefficient $\mu > 0$. On w and g, the homogeneous Neumann conditions are imposed on the boundary $\partial\Omega$. Nonnegative initial functions $u_0(x) \ge 0$, $v_0(x) \ge 0$, $w_0(x) \ge 0$ and $g_0(x) \ge 0$ are given in Ω for all unknown functions. (Note that, for simplicity, the constant w_* in [21, (1.1)] was taken as $w_* = 0$ and the cubic growth function for g was replaced by a square growth function of Fisher type.)

We now want to consider the situation that the reaction rates μ and γ are sufficiently large with respect to the diffusion rate d_g. Then, the equation of density $g(x,t)$ of grass can be dominated by the reaction terms and reduced to the ordinary differential equation

$$\frac{\partial g}{\partial t} = \left[-\mu v + \gamma \left(1 - \frac{g}{K}\right)\right] g \qquad \text{in } (0,\infty),$$

for each spatial point $x \in \Omega$. Furthermore, we assume that the reaction rates μ and γ are sufficiently large with respect to the reaction rates f and h appearing in the equation of $v(x,t)$. Then, $g(x,t)$ reaches its stability much faster than $v(x,t)$. By the theory of ordinary differential equations, we observe (v being given) the following dynamics. If $\mu v > \gamma$, then $\frac{\partial g}{\partial t} < 0$ for every $0 < t < \infty$ and g tends to 0 as $t \to \infty$. If $\mu v \le \gamma$, then g tends to $\frac{K}{\gamma}(\gamma - \mu v)$ as $t \to \infty$. That is, g is represented as a function of v in the form

$$g = g(v) \equiv \begin{cases} \frac{K}{\gamma}(\gamma - \mu v) & \text{for} \quad 0 \le v < \frac{\gamma}{\mu}, \\ 0 & \text{for} \quad \frac{\gamma}{\mu} \le v < \infty. \end{cases} \tag{2.2}$$

Let us substitute $g(v)$ defined by (2.2) with the g in the equation for u of (2.1). Then, (2.1) is reduced to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \varphi(v)u - fu & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d_w \Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & \text{in} \quad \Omega, \end{cases} \tag{2.3}$$

where $\varphi(v) = av^2 + \lambda g(v) + c$.

Let us next investigate the behavior of $\varphi(v)$ for $0 \le v < \infty$. By the definition, $g(v)$ is a piecewise linear continuous function of v, therefore $\varphi(v)$ is a piecewise quadratic continuous function. For $\frac{\gamma}{\mu} \le v < \infty$, $\varphi(v) = av^2 + c$. When $a > \frac{K\lambda\mu^2}{2\gamma^2}$, $\varphi(v)$ takes a minimal value in the interval $0 \le v < \frac{\gamma}{\mu}$. Indeed, $\varphi(v)$ is written as

$$\varphi(v) = a\left(v - \frac{K\lambda\mu}{2a\gamma}\right)^2 + \frac{K\lambda(4a\gamma^2 - K\lambda\mu^2)}{4a\gamma^2} + c, \qquad 0 \le v < \frac{\gamma}{\mu}.$$

Meanwhile, when $a \le \frac{K\lambda\mu}{2\gamma^2}$, $\varphi(v)$ is monotonously decreasing in the interval $0 \le v < \frac{\gamma}{\mu}$. Therefore, in this case, $\varphi(v)$ takes a minimal value at $v = \frac{\gamma}{\mu}$ and its value is given as $\varphi\left(\frac{\gamma}{\mu}\right) = \frac{a\gamma^2}{\mu^2} + c$. In this way, $\varphi(v)$ has been seen to have a unique minimal value and to behave as a quadratic function for large variables v, although it is not smooth at the point $v = \frac{\gamma}{\mu}$.

It is then natural to expect that the dynamics of solutions to (2.3) must be quite analogous to that of solutions to the equations due to Kuznetsov-Antonovsky-Biktashev-Aponina [11] in which $\varphi(v)$ is just a quadratic function of the form

$$\varphi(v) = a'(v - b')^2 + c', \qquad 0 \le v < \infty, \tag{2.4}$$

a', b' and c' being some positive constants. In addition, we already know that when $\varphi(v)$ is as in (2.4) the solutions starting from initial functions $v_0(x)$ given in a neighborhood of b' remain in some other neighborhood of b' and perform very interesting asymptotic behavior. By these arguments, we may be allowed to approximate our non smooth quadratic-like function $\varphi(v) = av^2 + \lambda g(v) + c$ as a square function of form (2.4) by setting

$$
\begin{cases} a' = a, \quad b' = \dfrac{K\lambda\mu}{2a\gamma}, \quad c' = \dfrac{K\lambda(4a\gamma^2 - K\lambda\mu^2)}{4a\gamma^2} + c \qquad \text{when} \quad a > \dfrac{K\lambda\mu^2}{2\gamma^2}, \\ \qquad a' = a, \quad b' = \dfrac{\gamma}{\mu}, \quad c' = \dfrac{a\gamma^2}{\mu^2} + c \qquad \text{when} \quad a \le \dfrac{K\lambda\mu^2}{2\gamma^2}. \end{cases} \tag{2.5}
$$

We have thus verified that, when the assumptions on μ, γ, d_g, f and h mentioned above are satisfied, the tree-grass model equations of (2.1) can reasonably be reduced to the model equations of (2.3). Here, the function $\varphi(v)$ is given by a square function of form (2.4) with the coefficients represented by (2.5).

3 Review of Known Analytical Results Let us review the known results for the problem (2.3) which are obtained by the series of papers [1, 2, 3].

I) Global Existence. In order to handle (2.3) analytically, we set an underlying Banach space X by

$$
X \equiv \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} ; \ u, \, v \in L_\infty(\Omega) \ \text{ and } \ w \in L_2(\Omega) \right\}.
$$

Then, (2.3) can be formulated as the Cauchy problem

$$
\begin{cases} \dfrac{dU}{dt} + AU = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{3.1}
$$

in X. Here, A denotes a closed linear operator of X of the form $A \equiv \operatorname{diag}\{f, h, \Lambda\}$, where Λ is a realization of the Laplace operator $-d_w\Delta + \beta$ in $L_2(\Omega)$ under the homogeneous Neumann boundary conditions, and A has the domain $\mathcal{D}(A) = L_\infty(\Omega) \times L_\infty(\Omega) \times H^2_N(\Omega)$, $H^2_N(\Omega)$ standing for the subspace of $H^2(\Omega)$ such that $u \in H^2_N(\Omega)$ if and only if $u \in H^2(\Omega)$ satisfies the homogeneous Neumann boundary conditions on $\partial\Omega$. Moreover, A is easily seen to be a sectorial operator of X with angle 0, namely, its spectrum is contained in the half real line $(0, \infty)$. Consequently, $-A$ generates an analytic semigroup e^{-tA} $(0 \le t < \infty)$ on X; actually, e^{-tA} is given as $e^{-tA} = \operatorname{diag}\{e^{-tf}, e^{-th}, e^{-t\Lambda}\}$.

In the meantime, $F(U)$ denotes a nonlinear operator of X of the form

$$
F(U) \equiv \begin{pmatrix} \beta\delta w - \varphi(v)u \\ fu \\ \alpha v \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(F) = [L_\infty(\Omega)]^3.
$$

Finally, U_0 denotes an initial value which is taken in X.

We can then apply the theory of semilinear abstract parabolic evolution equations (see [18, Chapter 4]). In fact, according to [1, Theorem 5.2], for any $0 \le u_0 \in L_\infty(\Omega)$, $0 \le v_0 \in L_\infty(\Omega)$ and $0 \le w_0 \in H^s(\Omega)$, where $s > 1$, (3.1) and hence (2.3) possesses a unique global solution such that

$$
\begin{cases} 0 \le u, \, v \in \mathcal{C}([0, \infty); L_\infty(\Omega)) \cap \mathcal{C}^1((0, \infty); L_\infty(\Omega)), \\ 0 \le w \in \mathcal{C}([0, \infty); H^s(\Omega)) \cap \mathcal{C}^1((0, \infty); L_2(\Omega)) \cap \mathcal{C}((0, \infty); H^2_N(\Omega)). \end{cases} \tag{3.2}
$$

II) Lyapunov Function.

Furthermore, as verified in [1, Section 7], the function

$$\Psi(U) = \int_\Omega \Big[\frac{\alpha}{2}(fu - hv)^2 + \frac{d_w f\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v) + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw\Big]dx, \quad U \in \mathcal{D}(A),$$

becomes a Lyapunov function for all the solutions of (2.3), where $\Gamma(v) = \int_0^v[\varphi(v)v + fv]dv$ is a fourth order function for $0 \le v < \infty$ due to (2.4). In fact, let $U(t)$ be any solution of (2.3) lying in (3.2). Then, the value $\Psi(U(t))$ is monotonously decreasing for $0 \le t < \infty$. In addition, it holds that

$$\overline{U} \in \mathcal{D}(A) \text{ is a stationary solution (i.e., } A\overline{U} = F(\overline{U})\text{), if and only if } \Psi'(\overline{U}) = 0.$$

In particular, we notice that (2.3) admits no periodic solutions.

III) Asymptotic Behavior of Solutions. In general, when there exists a Lyapunov function for the global solutions, one can prove that the global solutions tend to a stationary solution as $t \to \infty$. In the present case, however, such a convergence is proved only for some special cases. We can analytically claim only that, for any global solution $U(t)$, there exists a temporal sequence $t_n \nearrow \infty$ for which it holds true that

$$\begin{cases} u(t_n) \to \overline{u} & \text{weak* in } L_\infty(\Omega), \\ v(t_n) \to \overline{v} & \text{weak* in } L_\infty(\Omega), \\ w(t_n) \to \overline{w} & \text{strongly in } L_2(\Omega). \end{cases}$$

See [2, Section 4] and [19].

In spite of these analytical results, our numerical computations show that any global solution tends weakly to a stationary solution as $t \to \infty$. Some of them are described in [2, Section 6].

IV) Structure of Stationary Solutions. Now, we are naturally interested in investigating the structure of stationary solutions, namely, $\overline{U}$ satisfying $A\overline{U} = F(\overline{U})$. We can use the theory of stationary solutions to semilinear abstract parabolic evolution equations (see [18, Section 6. 6]).

As a matter of fact, the structure of stationary solutions changes drastically depending on the parameters of the equations. We here want to focus in the case when

$$a'(b')^2 > 3(c' + f). \tag{3.3}$$

In addition, fixing all the parameters except h, we treat h as a control parameter and consider the four critical values $0 < h_* < h_- < h_+ < h^* < \infty$ of h which are defined by

$$h_* = \frac{f\alpha\delta}{a'(b')^2 + c' + f}, \qquad h^* = \frac{f\alpha\delta}{c' + f},$$

and

$$h_\pm = \frac{f\alpha\delta\left\{a'(b')^2 + 3(c' + f) \pm \sqrt{a'(b')^2[a'(b')^2 - 3(c' + f)]}\,\right\}}{2(c' + f)[a'(b')^2 + c' + f]},$$

respectively.

According [3, Section 2], we know under (3.3) the following results.

1) When $0 < h < h_*$, there exist two homogeneous stationary solutions $O = (0, 0, 0)$ and

$$P_+ = \left(\tfrac{h}{f}[b' + \sqrt{D}],\ b' + \sqrt{D},\ \tfrac{\alpha}{\delta}[b' + \sqrt{D}]\right), \quad \text{where } D = \tfrac{f\alpha\delta - (c'+f)h}{a'h}.$$

In this case, there exist no other (inhomogeneous) stationary solutions.

Furthermore, O is unstable and P_+ is stable. So, as $t \to \infty$, the solutions $U(t)$ of (2.3) generally converge to P_+.

2) When $h_* < h < h_-$, there exist three homogeneous stationary solutions O, P_+ and

$$P_- = \left(\tfrac{h}{f}[b' - \sqrt{D}],\ b' - \sqrt{D},\ \tfrac{\alpha}{\delta}[b' - \sqrt{D}]\right). \tag{3.4}$$

In addition, there exist many inhomogeneous stationary solutions.

In this case, O and P_+ are both stable. But P_- is unstable and its dimension of instability is finite.

3) When $h_- < h < h_+$, there exist the three homogeneous stationary solutions O and $P_\pm$ and there exist many inhomogeneous stationary solutions.

As before, O and P_+ are stable and P_- is unstable. But the dimension of instability of P_- is infinite.

4) When $h_+ < h < h^*$, the situation is similar to that of Case 2. Indeed, there exist the three homogeneous stationary solutions O and $P_\pm$ and there exist many inhomogeneous stationary solutions.

As in Case 2, O and P_+ are both stable, and P_- is unstable. The dimension of instability of P_- is finite.

5) When $h^* < h < \infty$, $O = (0, 0, 0)$ is a globally stable stationary solution. That is, as $t \to \infty$, every solution $U(t)$ of (2.3) tends to O; in particular, there exist no other stationary solutions.

Remark 3.1. When $h_* < h < h^*$, we have as seen the unstable homogeneous stationary solution P_- given by (3.4) whose tree density $\overline{v}$ is equal to $b' - \sqrt{D}$. Then, let us observe what a grass density is at $v = \overline{v}$ by mean of the simplified equation (2.2) of g. According to (2.5), if $a > \frac{K\lambda\mu^2}{2\gamma^2}$, then $b' < \frac{\gamma}{\mu}$, *a fortiori*, $\overline{v} < \frac{\gamma}{\mu}$. Hence, (2.2) yields that $g(\overline{v}) > 0$. Similarly, if $a \le \frac{K\lambda\mu^2}{2\gamma^2}$, then $\overline{v} < b' = \frac{\gamma}{\mu}$. Hence, (2.2) again yields that $g(\overline{v}) > 0$.

In this sense, P_- is considered to be a homogeneous stationary state showing coexistence of trees and grass. However, as announced in Cases 2, 3 and 4, such a homogeneous state can never be stable. □

4 Segregation Patterns This section is devoted to presenting our numerical results. Throughout the numerical computations, the plot is set as $\Omega = (0, 1) \times (0, 1)$ and discretized by 1024×1024. We adopted a central differencing scheme for the 2-dimensional space and the implicit method for the time-dependent computation. About the parameters, we refer, except a', to a series of the study on forest ecology [5, 6, 8, 9, 10, 13, 14, 15, 16, 22]. Indeed, those parameter values are listed in Table 1. Especially, we chose a value for a' that satisfies the condition shown by (3.3).

In this case, the four critical values of h are approximately computed as

$$h_* = 0.0012,\ h_- = 0.0026,\ h_+ = 0.0334,\ h^* = 0.0337,$$

respectively. For our numerical computations, we then choose three values h_i $(i = 1, 2, 3)$ of h in such a way that $h_* < h_1 < h_- < h_2 < h_+ < h_3 < h^*$. Indeed,

$$h_1 = 0.0019,\ h_2 = 0.018,\ h_3 = 0.0335.$$

Table 1: Model parameters (symbol, description, value and units)

Symbol	Description	Vaule	Units
d_w	Seeds diffusion rate	0.01	$m\,day^{-1}$
a'	-	20000	-
b'	Optimal seedling density	0.004	-
c'	Natural mortality of seedlings	0.0014	-
f	Growth rate of young trees	0.01	-
α	Producing rate of seeds	0.5	-
β	Implantation rate of seeds	1	-
δ	Surviving rate of seeds	0.0769	-

As for initial states, we want to design them by two manners. The first one is that we place a certain number of circular grass patches onto the homogeneous stable forest P_+. The second one is that we place a certain number of circular tree patches onto the homogeneous stable grassland O. The radii of tree patches and grass patches are both 0.025. Locations of centers of the circular patches are randomly selected. We choose the number of patches which can lead the system to tree-grass coexisting stable states for each value of h mentioned above.

1) When $h = h_1 = 0.0019$, the stable stationary homogeneous solution P_+ is (0.0013, $0.0071, 0.0035$). (A) and (C) of Figure 1 show the final stabilized states of v at $t = 1000$ starting from the initial states where 612 grass patches are placed onto P_+ and 232 tree patches are placed onto O, respectively. Green color stands for habitats of trees, while yellow color stands for vacant area (habitats of grass). Both of these two states have a tree-area ratio rating at 32%. Actually if we place less than 612 grass patches onto P_+ or more than 232 tree patches onto O, then the system finally tends to P_+.

Meanwhile, (B) and (D) of Figure 1 show the final stabilized states of v at $t = 1000$ starting from the initial states where 1400 grass patches are placed onto P_+ and 4 tree patches are placed onto O, respectively. Both of these two states have a tree-area ratio rating at 1%.

In all these states, each of the habitats of trees looks isolated, namely with relatively low spatial connectivity. We will consider these spatial patterns to be low-connectivity forests.

2) When $h = h_2 = 0.018$, the stable stationary homogeneous solution P_+ is (0.0085, $0.0047, 0.0024$). (A) and (C) of Figure 2 show the final stabilized states of v at $t = 1000$ starting from the initial states where 260 grass patches are placed onto P_+ and 580 tree patches are placed onto O, respectively. Both of these two states have a tree-area ratio rating at 59%. Actually if we place less than 260 grass patches onto P_+ or more than 580 tree patches onto O, then the system finally tends to P_+.

Meanwhile, (B) and (D) of Figure 2 show the final stabilized states of v at $t = 1000$ starting from the initial states where 580 grass patches are placed onto P_+ and 260 tree patches are placed onto O, respectively. Both of these two states have a tree-area ratio rating at 34%. Actually if we place more than 580 grass patches onto P_+ or less than 260 tree patches onto O, then the system finally tends to O.

In (A) and (C), habitats of trees are almost connected, but with not very high spatial connectivity. We will consider these spatial patterns to be intermediate-connectivity forests.

3) When $h = h_3 = 0.0335$, the stable stationary homogeneous solution P_+ is (0.0136, $0.0041, 0.0020$). (A) and (C) of Figure 3 show the final stabilized states of v at $t = 1000$

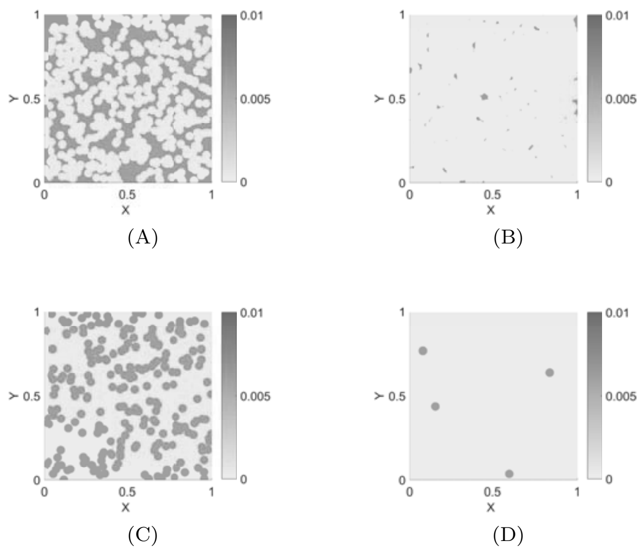

Fig. 1: Graphs of stabilized states of v at $t = 1000$ for $h = 0.0019$.

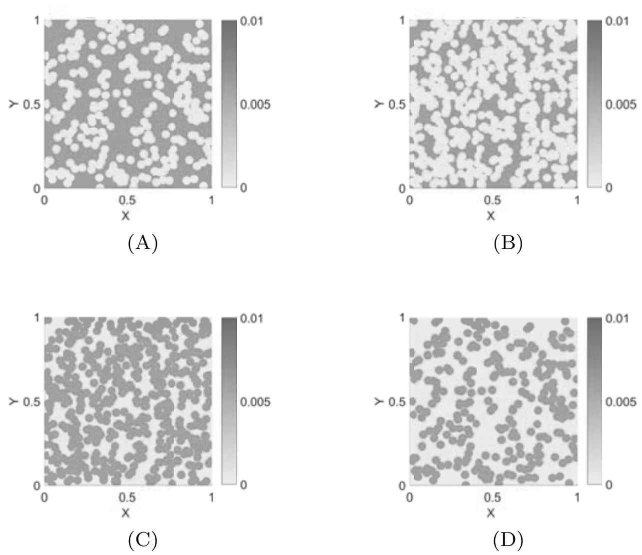

Fig. 2: Graphs of stabilized states of v at $t = 1000$ for $h = 0.018$.

starting from the initial states where 4 grass patches are placed onto P_+ and 1400 tree patches are placed onto O, respectively. Both of these two states have a tree-area ratio rating at 99%.

Meanwhile, (B) and (D) of Figure 3 show the final stabilized states starting from the initial states where 232 grass patches are placed onto P_+ and 612 tree patches are placed onto O, respectively. Both of these two states have a tree-area ratio rating at 68%. Actually if we place more than 232 grass patches onto P_+ or less than 612 tree patches onto O, then the system finally tends to O.

In all these states, habitats of trees look highly connected. We will consider these spatial patterns to be high-connectivity forests.

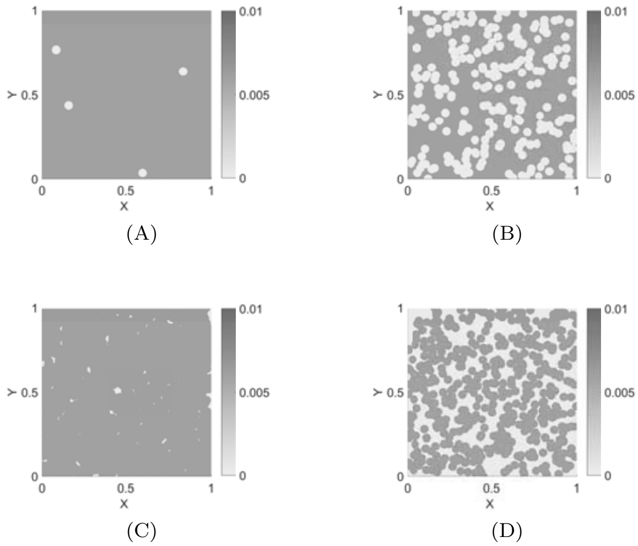

Fig. 3: Graphs of stabilized states of v at $t = 1000$ for $h = 0.0335$.

Our numerical results show a clear correlation between the mortality of old age trees h and the segregation patterns which are exhibiting tree-grass coexistence and are distinguished by different forest connectivity levels. We observe that, in order that tree-grass coexistence takes place, a forest with a relatively high mortality of old age trees needs a relatively high tree-area ratio, and its segregation pattern is of high connectivity. To the contrary, a forest with a relatively low mortality of old age trees needs only a relatively low tree-area ratio, and its segregation pattern is of low connectivity.

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GEOMETRIC DESCRIPTION OF SCHREIER GRAPHS OF B-S GROUPS

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Abstract. Let $BS(1,n) = \langle A, B \mid AB = B^n A \rangle$ be the Baumslag-Solitar group, where $n \geq 2$. This group has the natural action on the real line. In this paper we explicitly construct Schreier coset graphs of the group for stabilizers of all points in the real line under the action. As its consequence, we classify the Schreier coset graphs up to isomorphism, and obtain a relevance to presentations for the stabilizers.

1. Introduction

Let m and n be non-zero integers. The group which has the presentation $\langle A, B \mid AB^m = B^n A \rangle$ is called the *Baumslag-Solitar group* and denoted by $BS(m,n)$. In 1962, G. Baumslag and D. Solitar [1] introduced these groups and showed that $BS(3,2)$ is a non-Hopfian group with one defining relation. It is the first example having such property. Since then these groups have served as a proving ground for many new ideas in combinatorial and geometric group theory (see [2, 3] for examples).

Schreier coset graphs are generalizations of the Cayley graph of a group G, which are constructed for each choice of a subgroup of G and a generating set of G. The detail is given in Section 2. In general, given a group G and its subgroup H, it is difficult to construct the Cayley graph of G or the Schreier coset graph of all left cosets of H in G. However once we have the appropriate Cayley or Schreier graphs, we can use them as discrete models and may learn, from combinatorial and geometric viewpoints, some properties of the original group or its subgroups. Recently, in [5, 6], D. Savchuk constructed Schreier graphs of Thompson's group F from a motivation to study the amenability of the group.

In this paper we focus on the solvable group $BS(1,n)$ for $n \geq 2$. It is known that $BS(1,n)$ is isomorphic to some subgroup G_n with the generator S_n of the affine group $\mathrm{Aff}(\mathbb{R})$ of the real line $\mathbb{R}$, thus it has the natural action on $\mathbb{R}$ (see Section 2 for details). For any $x \in \mathbb{R}$, we explicitly construct the Schreier coset graph $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A, B\}^{\pm})$ for the stabilizer $\mathrm{Stab}_{BS(1,n)}(x)$ of x under the

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action. First, we show that for any $x \in \mathbb{R}$, the Schreier graphs $(\mathrm{Orb}_{G_n}(x), S_n, x)$ and $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A, B\}^{\pm}, \mathrm{Stab}_{BS(1,n)}(x))$ is isomorphic as marked labelled directed graphs, where $\mathrm{Orb}_{G_n}(x)$ is the orbit of x under the natural action on $\mathbb{R}$ (see Proposition 1 below). Hence, in most of this paper we consider the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$. Let $\mathbb{Z}_n^{\omega}$ be the set of all infinite words over the finite group $\mathbb{Z}_n$. The following theorem allows us to understand the structure of the Schreier graphs.

THEOREM 1. *Let $n \geq 2$ and x be a real number represented by $w \in \mathbb{Z}_n^{\omega}$. Then, there exists a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every $v \in V_w$, the subgraph $h^{-1}(v) = (D_v, D_v \times \{b\}^{\pm}, S_n, \alpha|, \beta|, l|)$ is isomorphic to $\Gamma_{\mathbb{Z}}$, where $h^{-1}(v) = (f^{-1}(v), \psi^{-1}(v), S_n, \alpha|, \beta|, l|)$.*

See Definition 3 below for Γ_w and $\Gamma_{\mathbb{Z}}$. As its consequence, we classify the Schreier graphs up to isomorphism.

THEOREM 2. *Let $m, n \geq 2$ with $m \neq n$.*

(1) *For any $x, y \in \mathbb{R}$, the Schreier graph $(\mathrm{Orb}_{G_m}(x), S_m)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ as labelled directed graphs.*
(2) *For any $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_1), S_n, \alpha_1)$ is S_n-isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_2), S_n, \alpha_2)$ as marked labelled directed graphs.*
(3) *For any $q \in \mathbb{Q}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ as labelled directed graphs.*
(4) *Let q_1, $q_2 \in \mathbb{Q}$. Then, the following statements are equivalent.*
    (*a*) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n)$ as labelled directed graphs.*
    (*b*) $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_2)$ *or* $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$.

This result leads to a relevance to presentations for the stabilizers which turn out to be infinite index subgroups in $BS(1,n)$(Theorem 5). Thus we expect that this idea may give a way to investigate infinite index subgroups in a suitable group.

In Section 2, we set up notation and terminology concerning Schreier graphs and Baumslag-Solitar groups. In Section 3, we start to construct Schreier graphs and give a complete description of Schreier graphs of $BS(1,n)$ with respect to any real numbers. In Section 4, we classify them up to isomorphism. In Section 5, by using the Schreier graphs we determine the group structure of the stabilizers and obtain a relevance to presentations for the stabilizers of rational numbers.

2. Schreier graphs and Baumslag-Solitar groups

A *labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l)$ consists of a nonempty set V of vertices, a set E of edges, a set L of labels and three mappings $\alpha : E \to V$,

$\beta : E \to V$, and $l : E \to L$. The vertices $\alpha(e)$ and $\beta(e)$ are called the *initial* and the *terminal vertices* of the edge e, respectively.

A *marked labelled directed graph* denoted by $(V, E, L, \alpha, \beta, l, v_0)$ is a labelled directed graph with a distinguished vertex v_0 called the *marked vertex*.

For $i \in \{1, 2\}$ let $\Gamma_i = (V_i, E_i, L_i, \alpha_i, \beta_i, l_i)$ be a labelled directed graph. Let $f : V_1 \to V_2$, $\psi : E_1 \to E_2 \sqcup V_2$ and $\gamma : L_1 \to L_2$ be maps satisfying the following statements:

(1) If $\psi(e) \in E_2$, then $\alpha_2(\psi(e)) = f(\alpha_1(e))$, $\beta_2(\psi(e)) = f(\beta_1(e))$, and $l_2(\psi(e)) = \gamma(l_1(e)) \in L_2$.
(2) If $\psi(e) \in V_2$, then $\psi(e) = f(\alpha_1(e)) = f(\beta_1(e))$.

The triple (f, ψ, γ) of maps is called the *homomorphism from Γ_1 to Γ_2*. Labelled directed graphs Γ_1 and Γ_2 are *isomorphic* if there exists a homomorphism $(f, \psi, \gamma) : \Gamma_1 \to \Gamma_2$, called *an isomorphism*, such that both f and γ are bijections and ψ is a injection with $\psi(E_1) = E_2$. In particular, if $L_1 = L_2 = L$ and $\gamma = 1_L$, Γ_1 is said to be *L-isomorphic* to Γ_2.

For $i \in \{1, 2\}$ let Γ_i be a marked labelled directed graph. Γ_1 is said to be *isomorphic* to Γ_2 if Γ_1 is isomorphic to Γ_2 as labelled directed graphs and the mapping between vertices preserves the marked vertices.

Let S be a generating set of a group G. The generating set S is *symmetric* if $S = S^{-1}$.

Let G be a group with a symmetric finite generating set S, M be a set and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism, where $\mathrm{Aut}(M)$ is the set of all bijections of M onto itself. The *orbit* of an element m of M is the set $\mathrm{Orb}_G(m) = \{\varphi(g)(m) \mid g \in G\}$. The *stabilizer* of an element m of M is the subgroup $\mathrm{Stab}_G(m) = \{g \in G \mid \varphi(g)(m) = m\}$.

Definition 1. Let G be a group with a symmetric finite generating set S, M be a set and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. The *Schreier graph* denoted by (M, S, φ) is a labelled directed graph $(M, M \times S, S, \alpha, \beta, l)$ such that $\alpha(m, s) = m$, $l(m, s) = s$, and $\beta(m, s) = \varphi(s)(m)$. The *Schreier graph with a marked vertex* denoted by (M, S, φ, m_0) is a Schreier graph with a marked vertex $m_0 \in M$.

Let G be a group with a symmetric finite generating set S, H be a subgroup of G and G/H be the set of all left cosets of H in G. The *Schreier coset graph* denoted by $(G/H, S)$ is a Schreier graph $(G/H, S, \varphi_H)$ where $\varphi_H : G \to \mathrm{Aut}(G/H)$ is the usual left action on G/H.

Remark 1. For $i \in \{1, 2\}$ let G_i be a group with a symmetric finite generate-ing set S_i. The Schreier graph (M_1, S_1, φ_1) is isomorphic to (M_2, S_2, φ_2) as labelled directed graphs if and only if there exist bijections $f : M_1 \to M_2$ and $\gamma : S_1 \to S_2$ such that $\varphi_1(s) = f^{-1}\varphi_2(\gamma(s))f$ for all $s \in S_1$. In particular, if $S_1 = S_2 = S$, (M_1, S, φ_1) is S-isomorphic to (M_2, S, φ_2) as labelled directed graphs if and only if there exists a bijection $f : M_1 \to M_2$ such that $\varphi_1(s) = f^{-1}\varphi_2(s)f$ for all $s \in S$.

The next proposition will help us to describe Schreier graphs explicitly in the later sections.

PROPOSITION 1. *Let G be a group with a symmetric finite generating set S, M be a set, $x_0 \in M$, and $\varphi : G \to \mathrm{Aut}(M)$ be a homomorphism. Then the Schreier graph $(\mathrm{Orb}_G(x_0), S, \varphi, x_0)$ with the marked vertex x_0 is S-isomorphic to the Schreier coset graph $(G/H, S, H)$ with the marked vertex $H = \mathrm{Stab}_G(x_0)$ as marked labelled directed graphs.*

PROOF. Define $f : G/H \to \mathrm{Orb}_G(x_0)$ by $f(gH) = \varphi(g)(x_0)$. Since $g^{-1}g' \in H = \mathrm{Stab}_G(x_0)$ implies $\varphi(g)(x_0) = \varphi(g')(x_0)$, its map is well-defined. Clearly f is a bijection. Since $f(\varphi_H(s)(gH)) = f(sgH) = \varphi(sg)(x_0) = \varphi(s)\varphi(g)(x_0) = \varphi(s)(f(gH))$, we have $\varphi_H(s) = f^{-1}\varphi(s)f$ for all $s \in S$, which is the desired conclusion by Remark 1. □

Let m and n be nonzero integers. The group with the presentation $\langle A, B \mid AB^m = B^nA\rangle$ is called the *Baumslag-Solitar group* and it is denoted by $BS(m,n)$. For any $n \geq 2$, $BS(1,n)$ has a geometric representation. That is, we define two affine maps a and b of the real line $\mathbb{R}$ by $a(x) = nx$ and $b(x) = x + 1$ respectively. Let $n \geq 2$, $S_n = \{a, b\}^{\pm}$ and $G_n = \langle S_n\rangle$ be the subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$. Then there exists the isomorphism $h_n : BS(1,n) \to G_n$ with $h_n(A) = a$ and $h_n(B) = b$ (see [4, p.100]). Thus, $BS(1,n)$ has the natural left action $\varphi_n : BS(1,n) \to G_n \hookrightarrow \mathrm{Aff}(\mathbb{R}) \hookrightarrow \mathrm{Aut}(\mathbb{R})$. By [4, p.102], we note that

$$(*)_n \qquad G_n = \{g : \mathbb{R} \to \mathbb{R} \mid g(x) = n^i x + j/n^k,\ i, j, k \in \mathbb{Z}\}.$$

3. Schreier graphs of all real numbers

Let $x \in \mathbb{R}$ and $\phi_x : G_n \to \mathrm{Aut}(\mathrm{Orb}_{G_n}(x))$ be the usual left action. By the isomorphism h_n and Proposition 1, the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x, x)$ and the Schreier coset graph $(BS(1,n)/\mathrm{Stab}_{BS(1,n)}(x), \{A, B\}^{\pm}, \mathrm{Stab}_{BS(1,n)}(x))$ with the marked vertexes are isomorphic, so we will consider the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x)$ for each $x \in \mathbb{R}$. For simplicity of notation, we write g and $(\mathrm{Orb}_{G_n}(x), S_n)$ instead of $\phi_x(g)$ and the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n, \phi_x)$, respectively.

REMARK 2. For any $x \in \mathbb{R}$ and any $f \in \mathrm{Stab}_{G_n}(x)$ with $f \neq 1_{\mathbb{R}}$, $bfb^{-1} \notin \mathrm{Stab}_{G_n}(x)$. Thus $\mathrm{Stab}_{G_n}(x)$ is not a normal subgroup of G_n.

We notice that the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is S_n-isomorphic to the Cayley graph of $BS(1,n)$ relative to the generators $\{A, B\}^{\pm}$ by the above since the stabilizer $\mathrm{Stab}_{BS(1,n)}(\alpha)$ is trivial. However in this section we construct the Schreier graphs $(\mathrm{Orb}_{G_n}(q), S_n)$ for rational numbers q and will compare those descriptions in the later section (see Theorem 4). Therefore we employ the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$. We construct the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ by an arrangement of elements in the orbit $\mathrm{Orb}_{G_n}(\alpha)$. The construction of the Cayley graph of $BS(1,n) \cong G_n$ given in [4] depends on the fact that the word problem for $BS(1,n)$ is solvable.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the finite group with the additive group structure. The set of all finite words over $\mathbb{Z}_n$ and the set of all infinite words over $\mathbb{Z}_n$ are denoted by $\mathbb{Z}_n^*$ and $\mathbb{Z}_n^\omega$ respectively. Let $\widetilde{\mathbb{Z}_n} = \mathbb{Z}_n^* \setminus \{\varepsilon\}$, where ε denotes the *empty word*. For every word $w = w_1 w_2 \ldots w_k$ in $\mathbb{Z}_n^*$, the *length* of w, denoted by $|w|$, is the number k. Note that $|\varepsilon|$ is zero.

Definition 2. An element w of $\mathbb{Z}_n^\omega$ is called a *rational element* in $\mathbb{Z}_n^\omega$ if there exist $u \in \mathbb{Z}_n^*$ and $v \in \widetilde{\mathbb{Z}_n}$ such that

(1) $w = uv^\infty$,
(2) $v \neq t^k$ whenever $k \geq 2$ and $t \in \widetilde{\mathbb{Z}_n}$, and
(3) $u_{|u|} \neq v_{|v|}$ whenever $u \neq \varepsilon$.

Then, we say that the pair (u, v) of words *satisfies* (A). An element w of $\mathbb{Z}_n^\omega$ which is not rational is called an *irrational element* in $\mathbb{Z}_n^\omega$. Let $x \in \mathbb{R}$. Then, there exists $w \in \mathbb{Z}_n^\omega$ such that $x - \lfloor x \rfloor = \sum_{i \geq 1} w_i / n^i$, where $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$. We say that x is *represented by* $w \in \mathbb{Z}_n^\omega$. It is easy to see that x is a rational number if and only if it is represented by a rational element in $\mathbb{Z}_n^\omega$.

Lemma 1. *Let $x, x' \in \mathbb{Z}_n^*$ and y be an irrational element of $\mathbb{Z}_n^\omega$ with $xy = x'y$. Then $x = x'$.*

Proof. Without loss of generality, we can assume that $|x| \leq |x'|$. By assumption, $y_{|x'|-|x|+j} = y_j$ for each $j \geq 1$. Since y is an irrational element in $\mathbb{Z}_n^\omega$, $|x'| = |x|$. Therefore, $x = x'$. $\square$

Lemma 2. *Suppose that pairs (x, y) and (x', y') of words satisfy (A). Then $xy^\infty = x'y'^\infty$ if and only if $x = x'$ and $y = y'$.*

Proof. Suppose that $xy^\infty = x'y'^\infty$. It suffices to show that $x = x'$ and $y = y'$. First we show that $|x| = |x'|$. On the contrary, suppose that $|x| < |x'|$. For any $k \geq 1$, there exists a unique $\underline{k} \in \{1, \ldots, |y|\}$ such that $k \equiv \underline{k} \mod |y|$. Then

$$x'_{|x'|} = (x'y'^\infty)_{|x'|} = (xy^\infty)_{|x'|} = (y^\infty)_{|x'|-|x|} = y_{\underline{|x'|-|x|}}.$$

On the other hand,

$$y'_{|y'|} = (x'y'^\infty)_{|x'|+|y'|(|y|/g)} = (xy^\infty)_{|x'|+|y|(|y'|/g)} = (y^\infty)_{|x'|-|x|+|y|(|y'|/g)} = y_{\underline{|x'|-|x|}},$$

where $g = \gcd(|y'|, |y|)$. Since $x' \neq \varepsilon$, by the assumption of x', we see $x'_{|x'|} \neq y'_{|y'|}$, a contradiction. Thus $|x| = |x'|$. Hence we have that $x = x'$ and $y^\infty = y'^\infty$.

Next we show that $|y| = |y'|$. On the contrary, suppose that $|y| < |y'|$. There exist $\alpha \in \mathbb{Z}$ and $\beta \geq 0$ such that $|y'|\alpha + |y|\beta = g$. For any $i \geq 1$

$$(y'^{\infty})_{i+g} = (y'^{\infty})_{i+|y'|\alpha+|y|\beta} = (y'^{\infty})_{i+|y|\beta} = (y^{\infty})_{i+|y|\beta} = (y^{\infty})_i = (y'^{\infty})_i.$$

Since y'^{∞} has the period g, y' has the period $g \leq |y| < |y'|$. This contradicts the assumption of y'. Since $|y| = |y'|$, we conclude $y = y'$. □

LEMMA 3. *Let $x, y \in \widetilde{\mathbb{Z}_n}$. Suppose that $x_{|x|} = y_{|y|}$ and the word y satisfies the condition (2) in Definition 2. Then $xy^{\infty} = y^{\infty}$ if and only if $|x| \equiv 0 \bmod |y|$ and $x = y^{|x|/|y|}$.*

PROOF. Suppose that $xy^{\infty} = y^{\infty}$. It suffices to show that $|x| \equiv 0 \bmod |y|$ and $x = y^{|x|/|y|}$. Let $m \geq 0$ and $1 \leq r \leq |y|$ such that $|x| = |y|m + r$. Then for any $i \geq 1$

$$\begin{aligned}(y^{\infty})_{i+r} = (xy^{\infty})_{|x|+i+r} = (xy^{\infty})_{|x|+i+r+|y|m} = (xy^{\infty})_{|x|+i+|x|} &= (y^{\infty})_{i+|x|} \\ &= (xy^{\infty})_{i+|x|} \\ &= (y^{\infty})_i.\end{aligned}$$

Thus y^{∞} has the period r and $(y_1 \dots y_{|y|})^{\infty} = y^{\infty} = (y_1 \dots y_r)^{\infty}$. Since (ε, y) and $(\varepsilon, y_1 \dots y_r)$ satisfy (A), by Lemma 2, we have $|y| = r$. Therefore $|x| \equiv 0 \mod |y|$. Moreover, since $(xy^{\infty})_i = (y^{\infty})_i$ for all $1 \leq i \leq |x|$, we have $x = y^{|x|/|y|}$. □

Let $\sigma : \mathbb{Z}_n^{\omega} \to \mathbb{Z}_n^{\omega}$ be the sift map defined by $\sigma(w_1 w_2 w_3 \dots) = w_2 w_3 w_4 \dots$ Write $\sigma^{k-1} = \underbrace{\sigma\sigma\cdots\sigma}_{k-1}$ for each $k \geq 1$, where σ^0 is the identity map. We note that $\sigma^{k-1}(w)_i = w_{k-1+i}$ for any $k, i \geq 1$ and each $w \in \mathbb{Z}_n^{\omega}$.

LEMMA 4. *Let (x, y) be a pair of words satisfying (A). Then for $|x| \leq j < j'$, $\sigma^j(xy^{\infty}) = \sigma^{j'}(xy^{\infty})$ if and only if $j' - j \equiv 0 \mod |y|$.*

PROOF. For any $k \geq 1$, there exists a unique $\underline{k} \in \{1, \dots, |y|\}$ such that $k \equiv \underline{k} \mod |y|$. Then

$$\sigma^j(xy^{\infty}) = \sigma^{j-|x|}(y^{\infty}) = (y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\, \sigma^{j'-|x|}(y^{\infty}), \text{ and}$$

$$\sigma^{j'}(xy^{\infty}) = \sigma^{j'-|x|}(y^{\infty}).$$

Thus $\sigma^j(xy^{\infty}) = \sigma^{j'}(xy^{\infty})$ if and only if $(y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\, \sigma^{j'-|x|}(y^{\infty}) = \sigma^{j'-|x|}(y^{\infty})$. By Lemma 3, $(y_{\underline{j-|x|+1}} \cdots y_{\underline{j'-|x|}})\, \sigma^{j'-|x|}(y^{\infty}) = \sigma^{j'-|x|}(y^{\infty})$ if and only if $j' - j \equiv 0 \mod |y|$. □

For any $v \in \mathbb{Z}_n^\omega$ and any $t \in \mathbb{Z}_n$, set $D_v = \mathbb{Z} + \sum_{i\geq 1} v_i/n^i \subset \mathbb{R}$, and $D_v^t = n\mathbb{Z} + t + \sum_{i\geq 1} v_i/n^i \subset \mathbb{R}$. Note that $0 \leq \sum_{i\geq 1} v_i/n^i \leq 1$ and $D_v = \bigsqcup_{t\in X} D_v^t$.

LEMMA 5. *Let y and y' be irrational elements in $\mathbb{Z}_n^\omega$. Then, the following statements are equivalent.*

(1) $D_y \cap D_{y'} \neq \emptyset$.
(2) $\sum_{i\geq 1} y_i/n^i = \sum_{i\geq 1} y'_i/n^i$.
(3) $y = y'$.

PROOF. It suffices to show that (2) implies (3). On the contrary, suppose that there exists $i \geq 1$ such that $y_i \neq y'_i$. Let $i_0 = \min\{i \mid y_i \neq y'_i\}$. Then,

$$y_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y_i/n^i = y'_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y'_i/n^i.$$

Without loss of generality, we can assume that $y_{i_0} < y'_{i_0}$. Since y and y' are irrational elements,

$$1/n^{i_0} < y'_{i_0}/n^{i_0} - y_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} y'_i/n^i = \sum_{i\geq i_0+1} y_i/n^i < 1/n^{i_0},$$

a contradiction. □

LEMMA 6. *Let (x, y) and (x', y') be pairs of words satisfying (A) such that $\min\{|y|, |y'|\} \geq 2$ whenever $y \neq y'$. Then, the following statements are equivalent.*

(1) $D_{xy^\infty} \cap D_{x'y'^\infty} \neq \emptyset$.
(2) $\sum_{i\geq 1} (xy^\infty)_i/n^i = \sum_{i\geq 1} (x'y'^\infty)_i/n^i$.
(3) $xy^\infty = x'y'^\infty$.

PROOF. Suppose that $\sum_{i\geq 1} (xy^\infty)_i/n^i = \sum_{i\geq 1} (x'y'^\infty)_i/n^i$. It suffices to prove that $xy^\infty = x'y'^\infty$. On the contrary, suppose that there exists $i \geq 1$ such that $(xy^\infty)_i \neq (x'y'^\infty)_i$. Let $i_0 = \min\{i \mid (xy^\infty)_i \neq (x'y'^\infty)_i\}$. Then

$$(xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (xy^\infty)_i/n^i = (x'y'^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i.$$

Without loss of generality, we can assume that $(xy^\infty)_{i_0} < (x'y'^\infty)_{i_0}$.

If $\min\{|y|, |y'|\} \geq 2$, or if $y = y' \in \{1, \ldots, n-2\}$, then we have

$$1/n^{i_0} < (x'y'^\infty)_{i_0}/n^{i_0} - (xy^\infty)_{i_0}/n^{i_0} + \sum_{i\geq i_0+1} (x'y'^\infty)_i/n^i = \sum_{i\geq i_0+1} (xy^\infty)_i/n^i < 1/n^{i_0},$$

a contradiction.

If $y = y' = 0$, then $i_0 \leq |x'|$. Then

$$1/n^{i_0} \leq (x'y'^{\infty})_{i_0}/n^{i_0} - (xy^{\infty})_{i_0}/n^{i_0} + \sum_{i \geq i_0+1} (x'y'^{\infty})_i/n^i = \sum_{i \geq i_0+1} (xy^{\infty})_i/n^i < 1/n^{i_0},$$

a contradiction.

If $y = y' = n-1$, then $i_0 \leq |x|$. Then

$$1/n^{i_0} < (x'y'^{\infty})_{i_0}/n^{i_0} - (xy^{\infty})_{i_0}/n^{i_0} + \sum_{i \geq i_0+1} (x'y'^{\infty})_i/n^i = \sum_{i \geq i_0+1} (xy^{\infty})_i/n^i \leq 1/n^{i_0},$$

a contradiction. Therefore $xy^{\infty} = x'y'^{\infty}$. $\square$

The proof of the following lemma is immediate, so the details are left to the reader.

LEMMA 7. *Let $v \in \mathbb{Z}_n^{\omega}$ and $t \in \mathbb{Z}_n$. Then,*

(*a*) $a(D_v) = D_{\sigma(v)}^{v_1}$, $a^{-1}(D_v^t) = D_{tv}$, $a^{-1}(D_v) = \bigsqcup_{t \in \mathbb{Z}_n} D_{tv}$,

(*b*) $b^{\pm 1}(D_v^t) = D_v^{t \pm 1}$, *and* $b^{\pm 1}(D_v) = D_v$.

DEFINITION 3. Let $w \in \mathbb{Z}_n^{\omega}$. Set $V_w = \{u\sigma^j(w) \mid j \geq 0, u \in \mathbb{Z}_n^*\}$, $E_w = V_w \times (\{a\} \sqcup \mathbb{Z}_n)$, and $L_w = \{a\}^{\pm}$. Define $\alpha_w : E_w \to V_w$, $\beta_w : E_w \to V_w$ and $l_w : E_w \to L_w$ by $\alpha_w(v,a) = \alpha_w(v,k) = v$, $\beta_w(v,a) = \sigma(v)$, $\beta_w(v,k) = kv$, $l_w(v,a) = a$ and $l_w(v,k) = a^{-1}$ for each $v \in V_w$ and each $k \in \mathbb{Z}_n$. The labelled directed graph $(V_w, E_w, L_w, \alpha_w, \beta_w, l_w)$ and the Schreier graph $(\mathbb{Z}, \{\pm 1\}, \phi)$ will be denoted by Γ_w and $\Gamma_{\mathbb{Z}}$ respectively, where $\phi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ is the usual action.

LEMMA 8. (1) *If w is an irrational element in $\mathbb{Z}_n^{\omega}$, then*

$$V_w = \bigsqcup_{j \geq 1} \{\sigma^j(w)\} \sqcup \bigsqcup_{u \in \mathbb{Z}_n^*} \{uw\} \sqcup \bigsqcup_{j \geq 1, s \in \mathbb{Z}_n^*, t \in \mathbb{Z}_n, t \neq w_j} \{st\sigma^j(w)\}.$$

(2) *If $w = uv^{\infty}$ is a rational element in $\mathbb{Z}_n^{\omega}$ as in Definition 2, then*

$$V_w = \bigsqcup_{|u| \leq j < |u|+|v|} \{\sigma^j(w)\} \sqcup \bigsqcup_{|u| < j \leq |u|+|v|, s \in \mathbb{Z}_n^*, t \in \mathbb{Z}_n, t \neq w_j} \{st\sigma^j(w)\}.$$

PROOF. By Lemmas 2 and 4, we can easily show (2). Thus we prove (1). Let $j, j' \geq 1$, $u, u' \in \mathbb{Z}_n^*$, and $t, t' \in \mathbb{Z}_n$ with $t \neq w_j$ and $t' \neq w_{j'}$. It suffices to show the following statements:

(*a*) $j = j'$ whenever $\sigma^j(w) = \sigma^{j'}(w)$.
(*b*) $u = u'$ whenever $uw = u'w$.
(*c*) $u = u'$, $t = t'$, and $j = j'$ whenever $ut\sigma^j(w) = u't'\sigma^{j'}(w)$.
(*d*) $\sigma^j(w) \neq uw$.

(e) $\sigma^j(w) \neq ut'\sigma^{j'}(w)$.
(f) $uw \neq u't\sigma^j(w)$.

The statements (b) and (d) directly follow from Lemma 1.

Suppose that $ut\sigma^j(w) = u't'\sigma^{j'}(w)$ and $j \leq j'$. Since $\sigma^j(w) = w_{j+1}\dots w_{j'}\sigma^{j'}(w)$, by Lemma 1, we have $utw_{j+1}\dots w_{j'} = u't'$. Since $t' \neq w_{j'}$, we see $j = j'$, thus $u = u'$ and $t = t'$, which proves (c). Similarly, we can show (a).

If $j \geq j'$, by Lemma 1, $ut'\sigma^{j'}(w) = ut'w_{j'+1}\dots w_j\sigma^j(w) \neq \sigma^j(w)$. Suppose that $j \leq j'$ and $\sigma^j(w) = ut'\sigma^{j'}(w)$. Since $\sigma^j(w) = w_{j+1}\dots w_{j'}\sigma^{j'}(w)$, $w_{j+1}\dots w_{j'}\sigma^{j'}(w) = ut'\sigma^{j'}(w)$. Hence by Lemma 1 $w_{j+1}\dots w_{j'} = ut'$. Thus $w_{j'} = t'$, a contradiction, and (e) is proved.

Since $w_j \neq t$, $uw_1\dots w_j \neq u't$. By Lemma 1, $uw = uw_1\dots w_j\sigma^j(w) \neq u't\sigma^j(w)$, which proves (f). □

LEMMA 9. *Let $n \geq 2$ and $x \in \mathbb{R}$ represented by $w \in \mathbb{Z}_n^\omega$. Then,* $\mathrm{Orb}_{G_n}(x) = \bigsqcup_{v\in V_w} D_v$.

PROOF. By Lemmas 5,6 and 8, $\bigcup_{v\in V_w} D_v = \bigsqcup_{v\in V_w} D_v$. Thus it suffices to show that $\mathrm{Orb}_{G_n}(x) = \bigcup_{v\in V_w} D_v$. Since $x \in D_w \subset \bigcup_{v\in V_w} D_v$, by Lemma 7,

$$\mathrm{Orb}_{G_n}(x) \subset \bigcup_{g\in G_n}\bigcup_{v\in V_w} g(D_v) = \bigcup_{v\in V_w} D_v.$$

Let $j \geq 0$ and $u \in \mathbb{Z}_n^*$. It suffices to show that $D_{u\sigma^j(w)} \subset \mathrm{Orb}_{G_n}(x)$. We have

$$\begin{aligned}
D_{u\sigma^j(w)} &= \mathbb{Z} + \sum_{i\geq 1}(u\sigma^j(w))_i/n^i \\
&= \mathbb{Z} + \sum_{i=1}^{|u|} u_i/n^i + \sum_{l\geq j+1} w_l/n^{l-j+|u|} \\
&= \mathbb{Z} + \sum_{i=1}^{|u|} u_i/n^i + n^{j-|u|}(\sum_{l\geq 1} w_l/n^l - \sum_{l=1}^{j} w_l/n^l) \\
&= \mathbb{Z} + n^{-|u|}(\sum_{i=1}^{|u|} n^{|u|-i}u_i - \sum_{i=1}^{j} n^{j-i}w_i + n^j(x - \lfloor x\rfloor)) \\
&= \{b^k a^{-|u|} b^{(\sum_{i=1}^{|u|} n^{|u|-i}u_i - \sum_{i=1}^{j} n^{j-i}w_i)} a^j b^{-\lfloor x\rfloor}(x) \mid k \in \mathbb{Z}\} \subset \mathrm{Orb}_{G_n}(x).
\end{aligned}$$

□

THEOREM 3. *Let $n \geq 2$ and x be a real number represented by $w \in \mathbb{Z}_n^\omega$. Then, there exists a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every*

$v \in V_w$, the subgraph $h^{-1}(v) = (D_v, D_v \times \{b\}^{\pm}, S_n, \alpha|, \beta|, l|)$ is isomorphic to $\Gamma_{\mathbb{Z}}$, where $h^{-1}(v) = (f^{-1}(v), \psi^{-1}(v), S_n, \alpha|, \beta|, l|)$.

PROOF. It suffices to find a homomorphism $h = (f, \psi, \gamma) : (\mathrm{Orb}_{G_n}(x), S_n) \to \Gamma_w$ such that for every $v \in V_w$, the subgraph $h^{-1}(v)$ is isomorphic to $\Gamma_{\mathbb{Z}}$. By Lemmas 8 and 9, for any $y \in \mathrm{Orb}_{G_n}(x)$, there exists a unique $v_y \in V_w$ and $k \in \mathbb{Z}_n$ such that $y \in D_{v_y}^k \subset D_{v_y}$. Thus, we can define $f : \mathrm{Orb}_{G_n}(x) \to V_w$, $\psi : \mathrm{Orb}_{G_n}(x) \times S_n \to E_w \sqcup V_w$ and $\gamma : S_n \to L_w$ by $f(y) = v_y$, $\psi(y, a) = (f(y), a)$, $\psi(y, a^{-1}) = (f(y), k)$, $\psi(y, b) = f(y)$, $\psi(y, b^{-1}) = f(y)$, $\gamma(a) = a$, $\gamma(a^{-1}) = a^{-1}$, $\gamma(b) = a$, and $\gamma(b^{-1}) = a^{-1}$. □

4. Classification of Schreier graphs

In this section we classify Schreier graphs described in the previous section.

LEMMA 10. *Let $v \in \widetilde{\mathbb{Z}_n}$. For $i \geq 1$ set $W_i = b^{-(v^\infty)_i} a$ and $Z_i = b^{(v^\infty)_i} a$. Then, for every $k \geq 1$, $W_k \cdots W_1$ and $Z_k \cdots Z_1$ are nontrivial affine maps with the slopes n^k such that*

$$(W_k \cdots W_1)(\sum_{j\geq 1} (v^\infty)_j / n^j) = \sum_{j \geq 1} (v^\infty)_{k+j} / n^j \text{ and}$$

$$(Z_k \cdots Z_1)(-\sum_{j\geq 1} (v^\infty)_j / n^j) = -\sum_{j \geq 1} (v^\infty)_{k+j} / n^j.$$

PROOF. The proof is by induction on k. The affine map W_1 has the slope n such that

$$\begin{aligned} W_1(\sum_{j\geq 1}(v^\infty)_j/n^j) = b^{-(v^\infty)_1}\, a\,(\sum_{j\geq 1}(v^\infty)_j/n^j) &= b^{-(v^\infty)_1}\,((v^\infty)_1 + \sum_{j\geq 2}(v^\infty)_j/n^{j-1}) \\ &= \sum_{j\geq 1}(v^\infty)_{1+j}/n^j. \end{aligned}$$

Assume the formula holds for $k-1$, we have

$$\begin{aligned} (W_k W_{k-1} \cdots W_1)\,(\sum_{j\geq 1}(v^\infty)_j/n^j) &= W_k(\sum_{j\geq 1}(v^\infty)_{k-1+j}/n^j) \\ &= b^{-(v^\infty)_k}\, a\,(\sum_{j\geq 1}(v^\infty)_{k-1+j}/n^j) \\ &= b^{-(v^\infty)_k}\,((v^\infty)_k + \sum_{j\geq 2}(v^\infty)_{k-1+j}/n^{j-1}) \\ &= \sum_{j\geq 1}(v^\infty)_{k+j}/n^j \end{aligned}$$

and the affine map $W_k \cdots W_1$ has the slope n^k. Similarly, we can prove it for $Z_k \cdots Z_1$. □

REMARK 3. Let $x, y \in \mathbb{R}$. Then, by Remark 1, Schreier graphs $(\mathrm{Orb}_{G_n}(x), S_n)$ and $(\mathrm{Orb}_{G_n}(y), S_n)$ are isomorphic if and only if there exist two bijections $f : \mathrm{Orb}_{G_n}(x) \to \mathrm{Orb}_{G_n}(y)$ and $\gamma : S_n \to S_n$ such that $\gamma(s)(f(z)) = f(s(z))$ for each $z \in \mathrm{Orb}_{G_n}(x)$ and each $s \in S_n$.

LEMMA 11. *Let $x, y \in \mathbb{R}$. Suppose that the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n\,)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y),\, S_n\,)$ by a bijection $\gamma : S_n \to S_n$. Then*

$$\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}} \text{ in } G_n$$

if and only if

$$\gamma = 1_{S_n} \quad \text{or} \quad \gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}, \text{ and } \gamma(b^{-1}) = b.$$

PROOF. Let $f : \mathrm{Orb}_{G_n}(x) \to \mathrm{Orb}_{G_n}(y)$ be a bijection as in Remark 3. For any $s \in S$ and any $x_0 \in \mathrm{Orb}_{G_n}(x)$, $\gamma(s)\gamma(s^{-1})(f(x_0)) = f(ss^{-1}(x_0)) = f(x_0)$ by Remark 3. Since f is a bijection, $\gamma(s)\gamma(s^{-1}) = 1_{\mathrm{Orb}_{G_n}(y)}$. Since $\gamma(s)\gamma(s^{-1})$ is an affine map, $\gamma(s)\gamma(s^{-1}) = 1_{\mathbb{R}}$, thus $\gamma(s)^{-1} = \gamma(s^{-1}) \in \mathrm{Aff}(\mathbb{R})$.

Suppose that $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}}$ and $\gamma \neq 1_{S_n}$. Since $a(x) = nx$ and $\gamma(b^{-1})$ has the n-th power, $\gamma(b^{-1}) \in \{b\}^{\pm}$.

Suppose that $\gamma(b^{-1}) = b^{-1}$. Then $\gamma(b) = b$. Since $\gamma \neq 1_{S_n}$, we have $\gamma(a) = a^{-1}$. Then $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = a^{-1}bab^{-n} \neq 1_{\mathbb{R}}$, a contradiction. Thus $\gamma(b^{-1}) = b$ and $\gamma(b) = b^{-1}$.

If $\gamma(a) = a^{-1}$, then $\gamma(a^{-1}) = a$ and $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = a^{-1}b^{-1}ab^n \neq 1_{\mathbb{R}}$, a contradiction. Hence $\gamma(a) = a$ and $\gamma(a^{-1}) = a^{-1}$. □

THEOREM 4. *Let $m, n \geq 2$ with $m \neq n$.*

(1) *For any $x, y \in \mathbb{R}$, the Schreier graph $(\mathrm{Orb}_{G_m}(x), S_m)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(y), S_n)$ as labelled directed graphs.*
(2) *For any $\alpha_1, \alpha_2 \in \mathbb{R}\setminus\mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_1), S_n, \alpha_1)$ is S_n-isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha_2), S_n, \alpha_2)$ as marked labelled directed graphs.*
(3) *For any $q \in \mathbb{Q}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is not isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(\alpha), S_n)$ as labelled directed graphs.*
(4) *Let $q_1,\, q_2 \in \mathbb{Q}$. Then, the following statements are equivalent.*
    (a) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n)$ as labelled directed graphs.*
    (b) $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_2)$ *or* $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$.

PROOF. On the contrary, suppose that the Schreier graphs $(\mathrm{Orb}_{G_m}(x), S_m)$ and $(\mathrm{Orb}_{G_n}(y), S_n)$ are isomorphic by bijections $f : \mathrm{Orb}_{G_m}(x) \to \mathrm{Orb}_{G_n}(y)$ and $\gamma : S_m \to$

S_n as in Remark 1. We check at once that $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^m \neq 1_{\mathbb{R}} \in G_n$. By Remark 1, $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^m(f(z)) = f(aba^{-1}b^{-m}(z)) = f(z)$ for each $z \in \mathrm{Orb}_{G_m}(x)$, contradiction, which proves (1). Since $\mathrm{Stab}_{G_n}(\alpha) = 1$ for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, by Proposition 1, the statement (2) is proved.

Let q be a rational number represented by uv^∞ and $x \in \mathbb{R}$ such that the Schreier graph $(\mathrm{Orb}_{G_n}(q), S_n)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(x), S_n)$ as labelled directed graphs by bijections $f : \mathrm{Orb}_{G_n}(q) \to \mathrm{Orb}_{G_n}(x)$ and $\gamma : S_n \to S_n$ as in Remark 3. Let $q_0 = \sum_{j\geq 1}(v^\infty)_j/n^j \in \mathrm{Orb}_{G_n}(q)$. Since $aba^{-1}b^{-n}(q') = q'$ for each $q' \in \mathrm{Orb}_{G_n}(q)$, by Remark 3, we have $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n(f(q')) = f(aba^{-1}b^{-n}(q')) = f(q')$. Hence, $\gamma(a)\gamma(b)\gamma(a^{-1})\gamma(b^{-1})^n = 1_{\mathbb{R}}$. By Lemma 11,

$$\gamma = 1_{S_n} \quad \text{or} \quad \gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}, \text{ and } \gamma(b^{-1}) = b. \tag{E}$$

On the other hand, by Lemma 10, there exists a nontrivial affine map $W_{|v|}\cdots W_1 = c_k \cdots c_1$ such that $c_k \cdots c_1(q_0) = q_0$, where $c_i \in \{a, b^{-1}\}$. By Remark 3, we have $\gamma(c_k)\cdots\gamma(c_1)(f(q_0)) = f(c_k\cdots c_1(q_0)) = f(q_0)$.

(*i*) If $\gamma = 1_{S_n}$, then the nontrivial affine map $c_k \cdots c_1$ fixes both q_0 and $f(q_0)$. Hence, $f(q_0) = q_0$.

(*ii*) If $\gamma(a) = a, \gamma(a^{-1}) = a^{-1}, \gamma(b) = b^{-1}$, and $\gamma(b^{-1}) = b$, then by Lemma 10, $\gamma(c_k)\cdots\gamma(c_1)(-q_0) = Z_{|v|}\cdots Z_1(-q_0) = -q_0$. Since the nontrivial affine map $\gamma(c_k)\cdots\gamma(c_1)$ fixes both $-q_0$ and $f(q_0)$, we have $-q_0 = f(q_0)$.

We start to prove (3). On the contrary, if $x = \alpha \in \mathbb{R} \setminus \mathbb{Q}$, by the above, we see $f(q_0) \in \mathbb{Q}$, a contradiction, which proves (3).

Next we prove (4). Suppose that the statement (*a*) holds, i.e.,$q = q_1$, $x = q_2 \in \mathbb{Q}$ above. If $\gamma = 1_{S_n}$, by (*i*) above, $\mathrm{Orb}_{G_n}(q_1) = \mathrm{Orb}_{G_n}(q_0) = \mathrm{Orb}_{G_n}(q_2)$. If $\gamma \neq 1_{S_n}$, by (*ii*) above, $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(-q_0) = \mathrm{Orb}_{G_n}(q_2)$, which proves (*b*).

Suppose that the statement (*b*) holds. We show that $(\mathrm{Orb}_{G_n}(q_1), S_n)$ and $(\mathrm{Orb}_{G_n}(q_2), S_n)$ are isomorphic. Without loss of generality, we can assume that $\mathrm{Orb}_{G_n}(-q_1) = \mathrm{Orb}_{G_n}(q_2)$. Define $\gamma : S_n \to S_n$ by $\gamma(a) = a$, $\gamma(a^{-1}) = a^{-1}$, $\gamma(b) = b^{-1}$, and $\gamma(b^{-1}) = b$. In addition define $f : \mathrm{Orb}_{G_n}(q_1) \to \mathrm{Orb}_{G_n}(q_2)$ by $f(c_k\cdots c_1(q_1)) = \gamma(c_k)\cdots\gamma(c_1)(-q_1)$, where $c_i \in S_n$. By induction on k, we can show that $(c_k\cdots c_1)(q_1) + (\gamma(c_k)\cdots\gamma(c_1))(-q_1) = 0$ for each $k \geq 1$ and each $c_i \in S_n$. Hence, f is well-defined and an injection. By definition, f is a surjection satisfying that $f(s(z)) = \gamma(s)(f(z))$ for each $z \in \mathrm{Orb}_{G_n}(q_1)$ and each $s \in S_n$. By Remark 3, the Schreier graphs $(\mathrm{Orb}_{G_n}(q_1), S_n)$ and $(\mathrm{Orb}_{G_n}(q_2), S_n)$ are isomorphic by f and γ. □

COROLLARY 1. *Let q_1, q_2 be rational numbers. Then, the following statements are equivalent.*

(*a*) *The Schreier graph $(\mathrm{Orb}_{G_n}(q_1), S_n, q_1)$ is isomorphic to the Schreier graph $(\mathrm{Orb}_{G_n}(q_2), S_n, q_2)$ as marked labelled directed graphs.*

(*b*) $|q_1| = |q_2|$.

Proof. From the latter part of the proof of Theorem 4, we can show that (b) implies (a). Suppose that $(\mathrm{Orb}_{G_n}(q_1), S_n, q_1)$ is isomorphic to $(\mathrm{Orb}_{G_n}(q_2), S_n, q_2)$ by bijections $f : \mathrm{Orb}_{G_n}(q_1) \to \mathrm{Orb}_{G_n}(q_2)$ with $f(q_1) = q_2$ and $\gamma : S_n \to S_n$ as in Remark 3. It suffices to show that $|q_1| = |q_2|$. Let us represent by $uv^\infty \in \mathbb{Z}_n^\omega$ $q_1 \in \mathbb{Q}$. Set $q_0 = \sum_{j\geq 1}(v^\infty)_j/n^j \in \mathrm{Orb}_{G_n}(q_1)$. Then, there exist $d_1, \ldots, d_j \in S_n$ such that $(d_j \cdots d_1)(q_1) = q_0$. From the proof of Theorem 4, the map γ satisfies (E) in the proof of Theorem 4, and the map f satisfies

$$f(q_0) = \begin{cases} q_0 & \text{if } \gamma = 1_{S_n} \\ -q_0 & \text{if } \gamma \neq 1_{S_n}. \end{cases}$$

Moreover, there exist $c_1, \ldots, c_k \in S_n$ such that $(c_k \cdots c_1)(q_0) = q_0$ and $\gamma(c_k)\cdots\gamma(c_1)(f(q_0)) = f(q_0)$. Then

$$(d_j \cdots d_1)^{-1}(c_k \cdots c_1)(d_j \cdots d_1)(q_1) = q_1.$$

By Remark 3

$$\gamma(d_1)^{-1}\cdots\gamma(d_j)^{-1}\gamma(c_k)\cdots\gamma(c_1)\gamma(d_j)\cdots\gamma(d_1)(q_2) = q_2.$$

Thus $\gamma(c_k)\cdots\gamma(c_1)(\gamma(d_j)\cdots\gamma(d_1)(q_2)) = \gamma(d_j)\cdots\gamma(d_1)(q_2)$.

Suppose that $\gamma = 1_{S_n}$. Then, $(c_k \cdots c_1)((d_j \cdots d_1)(q_2)) = (d_j \cdots d_1)(q_2)$. Since the nontrivial affine map $c_k \cdots c_1$ fixes both $q_0 = (d_j \cdots d_1)(q_1)$ and $(d_j \cdots d_1)(q_2)$, $(d_j \cdots d_1)(q_1) = (d_j \cdots d_1)(q_2)$. We conclude that $q_1 = q_2$.

Suppose that $\gamma \neq 1_{S_n}$. By Remark 3, $\gamma(d_j)\cdots\gamma(d_1)(q_2) = (\gamma(d_j)\cdots\gamma(d_1))(f(q_1)) = f((d_j \cdots d_1)(q_1)) = f(q_0) = -q_0 = -(d_j \cdots d_1)(q_1)$. Since the map γ satisfies (E) in the proof of Theorem 4, by induction on j, we can show $q_1 = -q_2$. □

5. Applications

First we determine the group structure of stabilizers for all rational numbers by using the Schreier graphs described in the previous section. The proof of next proposition allows us to understand a word stood for a generator as well as the group structure. We note that the the stabilizer $\mathrm{Stab}_{G_n}(q)$ is an infinite index subgroup of G_n since the orbit $\mathrm{Orb}_{G_n}(q)$ is an infinite set.

Proposition 2. *Let $n \geq 2$ and q be a rational number represented by $uv^\infty \in \mathbb{Z}_n^\omega$. Then, there exists $f \in \mathrm{Aff}(\mathbb{R})$ such that $f(x) = n^{|v|}(x - q) + q$ for each $x \in \mathbb{R}$, and $\mathrm{Stab}_{G_n}(q) = \langle f\rangle \cong \mathbb{Z}$.*

PROOF. For $i \geq 1$ set $\widetilde{W_i} = b^{-(uv^\infty)_i}a$. By Lemma 10 we have

$$\begin{aligned}\widetilde{W}_{|u|+|v|}\cdots\widetilde{W}_{|u|+1}\widetilde{W}_{|u|}\cdots\widetilde{W}_1(b^{-\lfloor q\rfloor}(q)) &= \widetilde{W}_{|u|+|v|}\cdots\widetilde{W}_{|u|+1}(\sum_{i\geq 1}(v^\infty)_i/n^i)\\ &= W_{|v|}\cdots W_1(\sum_{i\geq 1}(v^\infty)_i/n^i)\\ &= \sum_{i\geq 1}(v^\infty)_i/n^i\\ &= \widetilde{W}_{|u|}\cdots\widetilde{W}_1\,(b^{-\lfloor q\rfloor}(q)).\end{aligned}$$

Set $f = b^{\lfloor q\rfloor}\widetilde{W}_1^{-1}\cdots\widetilde{W}_{|u|}^{-1}\widetilde{W}_{|u|+|v|}\cdots\widetilde{W}_{|u|+1}\widetilde{W}_{|u|}\cdots\widetilde{W}_1 b^{-\lfloor q\rfloor}$. Then, f is an affine map with the slope $n^{|v|}$ such that $f(q) = q$. Hence $\langle f\rangle < \mathrm{Stab}_{G_n}(q)$.

Let $g \in \mathrm{Stab}_{G_n}(q)$. By $(*)_n$, there exists $i \in \mathbb{Z}$ such that $g(x) = n^i(x-q)+q$ for any $x \in \mathbb{R}$. If $|v| = 1$, f has the slope n, thus $g = f^i$. Hence, we may assume that $|v| \geq 2$. On the contrary, suppose that there exist $h \in \mathrm{Stab}_{G_n}(q) \setminus \langle f\rangle$, $0 < r < |v|$, $j \in \mathbb{Z}$, and $k \geq 0$ such that $h(x) = n^r x + j/n^k$ and $h(q) = q$. Then, we have

$$q = \frac{-j}{n^k(n^r-1)}.$$

There exist $m \geq 0$ and $z = z_1z_2\dots z_r \in \widetilde{\mathbb{Z}_n}$ with $z \neq (n-1)^r$ such that

$$|j| = \left(\sum_{i=0}^{r-1}(n-1)n^i\right)m + \sum_{i=0}^{r-1} z_{r-i}n^i = n^r\left(m\sum_{i=1}^{r}\frac{n-1}{n^i} + \sum_{i=1}^{r}\frac{z_i}{n^i}\right).$$

Since

$$\frac{n^r}{n^r-1} = \sum_{j\geq 0}\left(\frac{1}{n^r}\right)^j,$$

we have

$$qn^k = m + \sum_{i\geq 1}\frac{(z^\infty)_i}{n^i} \quad \text{or} \quad qn^k = -(m+1) + \sum_{i\geq 1}\frac{(\overline{z}^\infty)_i}{n^i},$$

where $\overline{z} = (n-1-z_1)\dots(n-1-z_r) \in \widetilde{\mathbb{Z}_n}$. Thus, qn^k has a repeating part whose length is the period of z^∞. However,

$$qn^k = \left(\lfloor q\rfloor + \sum_{i\geq 1}\frac{(uv^\infty)_i}{n^i}\right)n^k = \left(\lfloor q\rfloor n^k + \sum_{i=1}^{k}(uv^\infty)_i n^{k-i}\right) + \sum_{i\geq 1}\frac{(uv^\infty)_{i+k}}{n^i},$$

which contradicts (2) in Definition 2. $\square$

Next we introduce the definition of being isomorphic in presentations for subgroups in order to translate the graphical expression of the Schreier graphs into the algebraic expression of subgroups. Consequently, we get a relevance to presentations for the stabilizers from the previous result about the classification of the Schreier graphs (see Theorem 5).

For $i \in \{1, 2\}$, let G_i be a group with a generating set T_i. Let $T_i^{-1} = \{t^{-1} \mid t \in T_i\}$ and $T_i^{\pm} = T_i \cup T_i^{-1}$. We assume that

$$(*) \qquad t \in T_i \cap T_i^{-1} \text{ if and only if } t \in T_i, \quad t^2 = 1.$$

For $i \in \{1, 2\}$ let $X_i = \{\, x_t \mid t \in T_i \,\}$. Set $X_i^{-1} = \{x_t^{-1} \mid t \in T_i\}$, where x_t^{-1} denotes a new symbol corresponding to the element x_t. We assume that $X_i \cap X_i^{-1} = \emptyset$ and that the expression $(x_t^{-1})^{-1}$ denotes the element x_t. For $i \in \{1, 2\}$ the free group with the basis X_i is denoted by $F(X_i)$, and for a subset R_i of $F(X_i)$ the normal closure of the set R_i in $F(X_i)$ is denoted by $\langle\langle R_i \rangle\rangle$. Let G_i be the group with the presentation $\langle X_i \mid R_i \rangle$ with respect to the epimorphism $\psi_i : F(X_i) \to G_i$ given by $\psi_i(x_t) = t$.

Definition 4. For $i \in \{1, 2\}$, let H_i be a subgroup of G_i. H_1 and H_2 are *isomorphic in presentations* $\langle X_1 | R_1 \rangle$ *and* $\langle X_2 | R_2 \rangle$ *respectively* if there exists a bijection $\gamma : X_1^{\pm} \to X_2^{\pm}$ with $\gamma(x_t^{-1}) = \gamma(x_t)^{-1}$ such that $\widetilde{\gamma}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $\widetilde{\gamma}(\langle\langle R_1 \rangle\rangle) = \langle\langle R_2 \rangle\rangle$, where $\widetilde{\gamma} : F(X_1) \to F(X_2)$ is defined by $\widetilde{\gamma}(x_{t_1}^{\varepsilon_1} \cdots x_{t_k}^{\varepsilon_k}) = \gamma(x_{t_1})^{\varepsilon_1} \cdots \gamma(x_{t_k})^{\varepsilon_k}$ for $\varepsilon_i \in \{\pm 1\}$. Then, $\widetilde{\gamma}$ is an isomorphism and $H_1 \cong H_2$. Conversely, if there exists an isomorphism $\widetilde{\gamma} : F(X_1) \to F(X_2)$ such that $\widetilde{\gamma}(K_1) = K_2$ for each $K_i \in \{\psi_i^{-1}(H_i), \mathrm{Ker}\psi_i, X_i^{\pm}\}$, then $\gamma = \widetilde{\gamma}|_{X_1^{\pm}}$ satisfies the above condition.

Proposition 3. *Let* $\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i)$ *and* $\Gamma_i' = (F(X_i)/\psi_i^{-1}(H_i), X_i^{\pm}, \psi_i^{-1}(H_i))$ *be Schreier coset graphs for* $i \in \{1, 2\}$. *Then, the following statements are equivalent.*

(a) Γ_1 *is isomorphic to* Γ_2 *as marked labelled directed graphs by a bijection* $\gamma : T_1^{\pm} \to T_2^{\pm}$ *such that* $\gamma(t^{-1}) = \gamma(t)^{-1}$ *for every* $t \in T_1$.

(b) Γ_1' *is isomorphic to* Γ_2' *as marked labelled directed graphs by a bijection* $\gamma' : X_1^{\pm} \to X_2^{\pm}$ *with* $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ *for every* $x_t \in X_1$ *satisfying the condition*

$$(B) \qquad \psi_1(x_t)^2 = 1_{G_1} \text{ if and only if } \psi_2(\gamma'(x_t))^2 = 1_{G_2}.$$

Proof. Let $\varphi_i : G_i \to \mathrm{Aut}(G_i/H_i)$ and $\varphi_i' : F(X_i) \to \mathrm{Aut}(F(X_i)/\psi_i^{-1}(H_i))$ be the usual left actions for $i \in \{1, 2\}$. We define $\Psi_i : F(X_i)/\psi_i^{-1}(H_i) \to G_i/H_i$ by $\Psi_i(y\,\psi_i^{-1}(H_i)) = \psi_i(y)H_i$. Since $y^{-1}y' \in \psi_i^{-1}(H_i)$ is equivalent to $\psi_i(y)^{-1}\psi_i(y') \in H_i$, Ψ_i is well-defined and an injection. Since ψ_i is a surjection, Ψ_i is also a surjection.

Suppose that the statement (a) holds. Let $f : G_1/H_1 \to G_2/H_2$ be a bijection between vertices such that $f(H_1) = H_2$ and $f\varphi_1(t) = \varphi_2(\gamma(t))f$ for every $t \in T_1$.

Set $f' = \Psi_2^{-1} f \Psi_1 : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$. Clearly f' is bijective with $f'(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$.

Define $\gamma' : X_1^{\pm} \to X_2^{\pm}$ by

$$\gamma'(x_t^{\varepsilon}) = \begin{cases} x_{\gamma(t)}^{\varepsilon} & \text{if } \gamma(t) \in T_2 \text{ and } \varepsilon \in \{\pm 1\}, \\ x_{\gamma(t)^{-1}}^{-\varepsilon} & \text{if } \gamma(t) \notin T_2 \text{ and } \varepsilon \in \{\pm 1\}. \end{cases}$$

Then we have $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$. To show that γ' is bijective, we define $\sigma : X_2^{\pm} \to X_1^{\pm}$ by

$$\sigma(x_t^{\varepsilon}) = \begin{cases} x_{\gamma^{-1}(t)}^{\varepsilon} & \text{if } \gamma^{-1}(t) \in T_1 \text{ and } \varepsilon \in \{\pm 1\}, \\ x_{\gamma^{-1}(t)^{-1}}^{-\varepsilon} & \text{if } \gamma^{-1}(t) \notin T_1 \text{ and } \varepsilon \in \{\pm 1\}. \end{cases}$$

Then

$$\sigma\,\gamma'(x_t^{\varepsilon}) = \begin{cases} \sigma(x_{\gamma(t)}^{\varepsilon}) & \text{if } \gamma(t) \in T_2 \text{ and } \varepsilon \in \{\pm 1\}, \\ \sigma(x_{\gamma(t)^{-1}}^{-\varepsilon}) & \text{if } \gamma(t) \notin T_2 \text{ and } \varepsilon \in \{\pm 1\}. \end{cases}$$

If $\gamma(t) \in T_2$, $\gamma^{-1}(\gamma(t)) = t \in T_1$. If $\gamma(t) \notin T_2$, $\gamma^{-1}(\gamma(t)^{-1}) = \gamma^{-1}(\gamma(t^{-1})) = t^{-1} \notin T_1$ by $(*)$. Since $\gamma(t^{-1}) = \gamma(t)^{-1}$, we have $\gamma^{-1}(s^{-1}) = \gamma^{-1}(s)^{-1}$. Hence we have

$$\sigma\,\gamma'(x_t^{\varepsilon}) = \begin{cases} x_t^{\varepsilon} & \text{if } \gamma(t) \in T_2 \text{ and } \varepsilon \in \{\pm 1\}, \\ x_t^{\varepsilon} & \text{if } \gamma(t) \notin T_2 \text{ and } \varepsilon \in \{\pm 1\}, \end{cases}$$

thus $\sigma\,\gamma' = 1_{X_1^{\pm}}$. The similar argument gives $\gamma'\sigma = 1_{X_2^{\pm}}$. Thus γ' is a bijection.

Since $\psi_2(\gamma'(x_t)) = \gamma(t)$ and $t^2 = 1_{G_1}$ if and only if $\gamma(t)^2 = 1_{G_2}$, we have $\psi_1(x_t)^2 = 1_{G_1}$ if and only if $\psi_2(\gamma'(x_t))^2 = 1_{G_2}$, which establishes (B).

Since $\Psi_1 \varphi_1'(x_t) = \varphi_1(t)\Psi_1$ and $\Psi_2\varphi_2'(\gamma'(x_t)) = \varphi_2(\gamma(t))\Psi_2$, we have

$$\begin{aligned} \varphi_2'(\gamma'(x_t)) f' \varphi_1'(x_t)^{-1} = \varphi_2'(\gamma'(x_t)) \Psi_2^{-1} f \Psi_1 \varphi_1'(x_t)^{-1} &= \Psi_2^{-1} \varphi_2(\gamma(t)) f \varphi_1(t)^{-1} \Psi_1 \\ &= \Psi_2^{-1} f \Psi_1 \\ &= f'. \end{aligned}$$

By Remark 1 we obtain (b).

Suppose that the statement (b) holds. Let $f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$ be a bijection between vertices such that $f'(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $f'\varphi_1'(x_t) = \varphi_2'(\gamma'(x_t))f'$ for every $x_t \in X_1$. Set $f = \Psi_2 f' \Psi_1^{-1} : G_1/H_1 \to G_2/H_2$. Clearly f is bijective with $f(H_1) = H_2$.

Define $\gamma : T_1^{\pm} \to T_2^{\pm}$ by $\gamma(t^{\varepsilon}) = \psi_2(\gamma'(x_t^{\varepsilon}))$ for each $t \in T_1$ and $\varepsilon \in \{\pm 1\}$. First we show that γ is well-defined. Suppose that $t_1^{\varepsilon_1} = t_2^{\varepsilon_2}$. If $\varepsilon_1 = \varepsilon_2$ and $t_1 = t_2$, then $\psi_2(\gamma'(x_{t_1}^{\varepsilon_1})) = \psi_2(\gamma'(x_{t_2}^{\varepsilon_2}))$. If $\varepsilon_1 \neq \varepsilon_2$, then $t_1 = t_2$. Since $\psi_2(\gamma'(x_{t_j}))^2 = 1_{G_2}$

by (B), $\psi_2(\gamma'(x_{t_1}^{\varepsilon_1})) = \psi_2(\gamma'(x_{t_1}^{-\varepsilon_1})) = \psi_2(\gamma'(x_{t_2}^{\varepsilon_2}))$. Thus γ is well-defined. Then we have $\gamma(t^{-1}) = \gamma(t)^{-1}$. Next we show that γ is bijective. We define $\rho : T_2^{\pm} \to T_1^{\pm}$ by $\rho(t^{\varepsilon}) = \psi_1(\gamma'^{-1}(x_t^{\varepsilon}))$ for each $t \in T_2$ and $\varepsilon \in \{\pm 1\}$. Since γ' satisfies the condition (B), $\psi_2(x_t)^2 = 1_{G_2}$ if and only if $\psi_1(\gamma'^{-1}(x_t))^2 = 1_{G_1}$. Hence ρ is well-defined. We can easily see that $\gamma\rho = 1_{T_2^{\pm}}$ and $\rho\gamma = 1_{T_1^{\pm}}$. Hence γ is a bijection.

Since $\Psi_1\varphi_1'(x_t) = \varphi_1(t)\Psi_1$ and $\Psi_2\varphi_2'(\gamma'(x_t)) = \varphi_2(\gamma(t))\Psi_2$,

$$\begin{aligned}\varphi_2(\gamma(t))f\varphi_1(t)^{-1} = \varphi_2(\gamma(t))\Psi_2 f'\Psi_1^{-1}\varphi_1(t)^{-1} &= \Psi_2\varphi_2'(\gamma'(x_t))f'\varphi_1'(x_t)^{-1}\Psi_1^{-1} \\ &= \Psi_2 f'\Psi_1^{-1} \\ &= f.\end{aligned}$$

By Remark 1 we obtain (a). □

LEMMA 12. *Let $\Gamma_i = (G_i/H_i, T_i^{\pm}, H_i)$ be Schreier coset graphs for $i \in \{1,2\}$. Then the following statements are equivalent.*

- (a) *Γ_1 is isomorphic to Γ_2 as marked labelled directed graphs by a bijection $\gamma : T_1^{\pm} \to T_2^{\pm}$ satisfying the following condition: for any $t_1, \ldots, t_k \in T_1$ and any $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$,*

$$(C) \qquad t_1^{\varepsilon_1}\cdots t_k^{\varepsilon_k} = 1_{G_1} \quad \textit{if and only if} \quad \gamma(t_1^{\varepsilon_1})\cdots\gamma(t_k^{\varepsilon_k}) = 1_{G_2}.$$

- (b) *H_1 and H_2 are isomorphic in presentations $\langle X_1|R_1\rangle$ and $\langle X_2|R_2\rangle$ respectively.*

PROOF. By Proposition 3, (a) is equivalent to the following statement.

- (a') Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ such that $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ for every $x_t \in X_1$ and

$$(C') \quad \psi_1(x_{t_1}^{\varepsilon_1})\cdots\psi_1(x_{t_k}^{\varepsilon_k}) = 1_{G_1} \text{ if and only if } \psi_2(\gamma'(x_{t_1}^{\varepsilon_1}))\cdots\psi_2(\gamma'(x_{t_k}^{\varepsilon_k})) = 1_{G_2}.$$

In addition we note that the following statements are equivalent.

1. There exists a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ satisfying the condition (C').
2. There exists a group isomorphism $\delta : F(X_1) \to F(X_2)$ such that $\delta(X_1^{\pm}) = X_2^{\pm}$ and $\delta(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle$.

Suppose that the statement (a) holds. By the above, we may suppose that the statement (a') holds, and can take $\widetilde{\gamma'}$ as δ in (2), where $\widetilde{\gamma'} : F(X_1) \to F(X_2)$ given by $\widetilde{\gamma'}(x_{t_1}^{\varepsilon_1}\cdots x_{t_k}^{\varepsilon_k}) = \gamma'(x_{t_1})^{\varepsilon_1}\cdots\gamma'(x_{t_k})^{\varepsilon_k}$. It suffices to prove that $\widetilde{\gamma'}(\psi_1^{-1}(H_1))$ $= \psi_2^{-1}(H_2)$. Let $f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$ be a bijection between vertices which preserves marked vertices. Now, we note that for $i \in \{1,2\}$, $\psi_i^{-1}(H_i) =$

$\{\, l(P) \,|\, P$ is an edge path in Γ_i' from $\psi_i^{-1}(H_i)$ to itself $\}$, where $l(P) = l(e_n) \dots l(e_1)$ whenever $P = e_1 \dots e_n$.

Let $l(P) \in \psi_1^{-1}(H_1)$, where $e_j = (x_{t_{j-1}}^{\varepsilon_{j-1}} \cdots x_{t_1}^{\varepsilon_1} \psi_1^{-1}(H_1),\, x_{t_j}^{\varepsilon_j})$ and $P = e_1 \dots e_n$. Since $x_{t_n}^{\varepsilon_n} \cdots x_{t_1}^{\varepsilon_1} \psi_1^{-1}(H_1) = \beta(e_n) = \psi_1^{-1}(H_1)$, by Remark 1,

$$\begin{aligned}\widetilde{\gamma'}(l(P))\psi_2^{-1}(H_2) = \gamma'(x_{t_n}^{\varepsilon_n}) \cdots \gamma'(x_{t_1}^{\varepsilon_1}) f'(\psi_1^{-1}(H_1)) &= f'(x_{t_n}^{\varepsilon_n} \cdots x_{t_1}^{\varepsilon_1} \psi_1^{-1}(H_1)) \\ &= f'(\psi_1^{-1}(H_1)) \\ &= \psi_2^{-1}(H_2).\end{aligned}$$

Thus we have $\widetilde{\gamma'}(\psi_1^{-1}(H_1)) \subset \psi_2^{-1}(H_2)$. Similarly $\widetilde{\gamma'}^{-1}(\psi_2^{-1}(H_2)) \subset \psi_1^{-1}(H_1)$, which proves $\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$.

Suppose that the statement (b) holds. There exists a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$ with $\gamma'(x_t^{-1}) = \gamma'(x_t)^{-1}$ such that $\widetilde{\gamma'}\,(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$ and $\widetilde{\gamma'}(\langle\langle R_1\rangle\rangle) = \langle\langle R_2\rangle\rangle$, which establishes (2). Define $f' : F(X_1)/\psi_1^{-1}(H_1) \to F(X_2)/\psi_2^{-1}(H_2)$ by $f'(g\psi_1^{-1}(H_1)) = \widetilde{\gamma'}(g)\psi_2^{-1}(H_2)$. Since $g_2^{-1}g_1 \in \psi_1^{-1}(H_1)$ is equivalent to $\widetilde{\gamma'}(g_2^{-1}g_1) \in \widetilde{\gamma'}(\psi_1^{-1}(H_1)) = \psi_2^{-1}(H_2)$, f' is well-defined and an injection. Since $\widetilde{\gamma'}$ is a surjection, f' is also a surjection. Since

$$\begin{aligned}f'\varphi_1'(x_t)(g\psi_1^{-1}(H_1)) = f'(x_t g\psi_1^{-1}(H_1)) &= \widetilde{\gamma'}(x_t g)\psi_2^{-1}(H_2) \\ &= \widetilde{\gamma'}(x_t)\widetilde{\gamma'}(g)\psi_2^{-1}(H_2) \\ &= \varphi_2'(\gamma'(x_t)) f'(g\psi_1^{-1}(H_1)),\end{aligned}$$

we have $f'\varphi_1'(x_t) = \varphi_2'(\gamma'(x_t))f'$ for every $x_t \in X_1$. Thus Γ_1' is isomorphic to Γ_2' as marked labelled directed graphs by a bijection $\gamma' : X_1^{\pm} \to X_2^{\pm}$, which establishes (a'), i.e., (a). □

By Lemmas 11 and 12, Corollary 1, (1) in Theorem 4 and the isomorphism h_n, we obtain the following theorem.

THEOREM 5. *Let $m, n \geq 2$ and q_1, $q_2 \in \mathbb{Q}$. Then the following statements are equivalent.*

(a) $\mathrm{Stab}_{BS(1,m)}(q_1)$ *and* $\mathrm{Stab}_{BS(1,n)}(q_2)$ *are isomorphic in presentations $BS(1,m)$ and $BS(1,n)$ respectively.*

(b) $m = n$ *and* $|q_1| = |q_2|$.

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EXTENSIONS OF ANDO-HIAI INEQUALITY WITH NEGATIVE POWER

Dedicated to the 100th anniversary of the birth of
the late Professor Masahiro Nakamura

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Abstract. The Ando-Hiai inequality says that if $A\#_\alpha B \le 1$ for a fixed $\alpha \in [0,1]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\alpha B^r \le 1$ for $r \ge 1$, where $\#_\alpha$ is the α-geometric mean defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. In this note, we generalize it as follows: If $A\natural_\alpha B \le 1$ for a fixed $\alpha \in [-1,0]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\beta B^s \le 1$ for $r \in [0,1]$ and $s \in [\frac{-2\alpha r}{-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$ and $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. As an application, we pose operator inequalities of type of Furuta inequality and grand Furuta inequality. For instance, if $A \ge B > 0$, then $A^{-r}\natural_{\frac{1+r}{p+r}} B^p \le A$ holds for $p \le -1$ and $r \in [-1,0]$, where $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

1 Introduction

Throughout this note, an operator A means a bounded linear operator acting on a complex Hilbert space H. An operator A is positive, denoted by $A \ge 0$, if $(Ax,x) \ge 0$ for all $x \in H$. We denote $A > 0$ if A is positive and invertible. The α-geometric mean $\#_\alpha$ is defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $A > 0$ and $B \ge 0$.

A log-majorization theorem due to Ando-Hiai [1] is expressed as follows: For $\alpha \in [0,1]$ and positive definite matrices A and B,

$$(A\#_\alpha B)^r \succ_{(\log)} A^r\#_\alpha B^r \quad (r \ge 1).$$

The core in the proof is that $A\#_\alpha B \le 1$ implies $A^r\#_\alpha B^r \le 1$ for $r \ge 1$. It holds for positive operators A, B on a Hilbert space, and is called the Ando-Hiai inequality,

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simply (AH). Afterwards, it is generalized to two variable version: If $A\#_\alpha B \le 1$ for $\alpha \in [0,1]$ and positive operators A, B, then $A^r\#_\beta B^s \le 1$ for $r, s \ge 1$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$. It is known that both one-sided versions are equivalent, and that they are alterantive expressions of the Furuta inequality, see [4, 5].

A binary operation $\natural_\alpha$ is defined by the same formula as the α-geometric mean for $\alpha \notin [0,1]$, that is,

$$A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } A, B > 0.$$

Very recently (AH) is extended by Seo [17] and [13] as follows: For $\alpha \in [-1,0]$, $A\natural_\alpha B \le 1$ for $A, B > 0$ implies $A^r\natural_\alpha B^r \le 1$ for $r \in [0,1]$.

In this note, we present two variable version of it, presicely we show that if $A\natural_\alpha B \le 1$ for $\alpha \in [-1,0]$ and positive invertible operators A, B, then $A^r\natural_\beta B^s \le 1$ for $r \in [0,1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$. As an application, we pose operator inequalities of type of Furuta inequality and grand Furuta inequality.

2 Extensions of (AH) with negative power

In the beginning of this section, we mention the following useful identity on the binary operation $\natural$: For $\beta \in \mathbb{R}$ and positive invertible operators X and Y,

$$X\natural_\beta Y = X(X^{-1}\natural_{-\beta}Y^{-1})X. \tag{2.1}$$

Lemma 2.1. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1,0]$ and positive invertible operators A and B, then $A^r\natural_\beta B \le 1$ for $r \in [0,1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)}$.*

Proof. For convenience, we show that if $A^{-1}\natural_\alpha B \le 1$, then $A^{-r}\natural_\beta B \le 1$ for $r \in [0,1]$. Thus the assumption ensures that $C^\alpha \le A$, where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$. Note that $\beta \in [-1,0]$.

Now we first assume that $r = 1-\epsilon \in [\frac{1}{2}, 1]$, i.e., $\epsilon \in [0, \frac{1}{2}]$. Then we have

$$\begin{aligned} A^\epsilon\natural_\beta C &= A^\epsilon(A^{-\epsilon}\#_{-\beta}C^{-1})A^\epsilon \\ &\le A^\epsilon(C^{-\alpha\epsilon}\#_{-\beta}C^{-1})A^\epsilon \\ &= A^\epsilon C^{\alpha(1-2\epsilon)}A^\epsilon \\ &\le A^\epsilon A^{1-2\epsilon}A^\epsilon = A. \end{aligned}$$

Hence it follows that

$$A^{-r}\natural_\beta B = A^{-\frac{1}{2}}(A^\epsilon \natural_\beta C)A^{-\frac{1}{2}} \le A^{-\frac{1}{2}}AA^{-\frac{1}{2}} = 1.$$

In particular, we note that $A^r\natural_\beta B \le 1$ for $r = \frac{1}{2}$, that is, $A^{-\frac{1}{2}}\natural_{\alpha_1} B \le 1$ holds for $\alpha_1 = \frac{\alpha}{2-\alpha}$. Hence it follows from the preceding paragragh that for $r \in [\frac{1}{2}, 1]$,

$$1 \ge (A^{-\frac{1}{2}})^r \natural_{\beta_1} B = A^{-\frac{r}{2}}\natural_{\beta_1} B,$$

where $\beta_1 = \frac{\alpha_1 r}{\alpha_1 r + (1-\alpha_1)} = \frac{\alpha r/2}{\alpha r/2 + (1-\alpha)}$. This means tht the desired inequality holds for $r \in [\frac{1}{4}, \frac{1}{2}]$. Finally we have the conclusion by the induction.

Lemma 2.2. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1, 0]$ and positive invertible operators A and B, then $A\natural_\beta B^s \le 1$ for $s \in [\frac{-2\alpha}{1-\alpha}, 1]$, where $\beta = \frac{\alpha}{\alpha + (1-\alpha)s}$.*

Proof. For convenience, we show that if $A\natural_\alpha B^{-1} \le 1$, then $A\natural_\beta B^{-s} \le 1$ for $s \in [\frac{-2\alpha}{1-\alpha}, 1]$. Thus the assumption is understood as $D^{1-\alpha} \le B$, where $D = B^{\frac{1}{2}}AB^{\frac{1}{2}}$. We first note that $\beta \in [-1, 0]$ by $s \in [\frac{-2\alpha}{1-\alpha}, 1]$. So we put $s = 1 - \epsilon$ for some $\epsilon \in [0, 1 - \frac{-2\alpha}{1-\alpha}]$. Then we have

$$D\natural_\beta B^\epsilon = D(D^{-1}\#_{-\beta}B^{-\epsilon})D \le D(D^{-1}\#_{-\beta}D^{-\epsilon(1-\alpha)})D = D^{1-\alpha} \le B,$$

so that

$$A\natural_\beta B^{-s} = B^{-\frac{1}{2}}(D\natural_\beta B^\epsilon)B^{-\frac{1}{2}} \le B^{-\frac{1}{2}}BB^{-\frac{1}{2}} = 1.$$

Theorem 2.3. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1, 0]$ and positive invertible operators A and B, then $A^r\natural_\beta B^s \le 1$ for $r \in [0, 1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$.*

Proof. Suppose that $A\natural_\alpha B \le 1$. Then Lemma 2.1 says that $A^r\natural_\gamma B \le 1$ for $r \in [0, 1]$, where $\gamma = \frac{\alpha r}{\alpha r + (1-\alpha)}$. Next we apply Lemma 2.2 to this obtained inequality. Then we have

$$1 \ge A^r \natural_{\frac{\gamma}{\gamma + (1-\gamma)s}} B^s = A^r \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s$$

for $s \in [\frac{-2\gamma}{1-\gamma}, 1] = [\frac{-2\alpha r}{1-\alpha}, 1]$.

As a special case $s = r$ in the above, we obtain Seo's original extension of (AH) because $\beta = \alpha$ (by $s = r$) and $r \in [\frac{-2\alpha r}{1-\alpha}, 1]$.

Corollary 2.4. *If $A\natural_\alpha B \le 1$ for $\alpha \in [-1, 0]$ and positive invertible operators A and B, then $A^r\natural_\beta B^r \le 1$ for $r \in [0, 1]$.*

Remark 2.5. We here consider the condition $s \in [\frac{-2\alpha}{1-\alpha}, 1]$ in Lemma 2.2. In particular, take $\alpha = -1$. Then the assumption $A\natural_\alpha B \le 1$ means that $B \ge A^2$, and $\beta = \frac{\alpha}{\alpha+(1-\alpha)s} = \frac{1}{1-2s}$. Though $s = 1$ in this case by $s \in [\frac{-2\alpha}{1-\alpha}, 1]$, the inequality in Lemma 2.2 still holds for $s \in [\frac{3}{4}, 1]$. We use the formula $X\natural_\gamma Y = Y\natural_{1-\gamma}X = Y(Y^{-1}\natural_{\gamma-1}X^{-1})Y$. Note that $-\beta \in [1, 2]$. Therefore we have

$$\begin{aligned} A\natural_\beta B^s &= A(A^{-1}\natural_\beta B^{-s})A = AB^{-s}(B^s\#_{\beta-1}A)B^{-s}A \\ &\le AB^{-s}(B^s\#_{-\beta-1}B^{\frac{1}{2}})B^{-s}A = AB^{-1}A \le AA^{-2}A = 1. \end{aligned}$$

On the other hand, it is false for $s \in [0, \frac{1}{4}]$. Note that $\beta = \frac{1}{1-2s} \in [1, 2]$. Suppose to the contrary that $A\natural_\beta B^s \le 1$ holds under the assumption $B \ge A^2$. Then it follows that $1 \le A\natural_\beta B^s = B^s(B^{-s}\#_{\beta-1}A^{-1})B^s$ and so

$$B^{-2s} \ge B^{-s}\#_{\beta-1}A^{-1} \ge B^{-s}\#_{\beta-1}B^{-\frac{1}{2}} = B^{-2s},$$

so that $B = A^2$ follows, which is imposible in general.

3 Operator inequalities of Furuta type In this section, we discuss representations of Furuta type associtated with extensions of Ando-Hiai inequality obtained in the preceding section. For convenience for readers, we cite the Furuta inequality which is a remarkable and amazing extension of Löwner-Heinz inequality (LH) in [?], [?] and [?], i.e., if $A \ge B \ge 0$, then $A^\alpha \ge B^\alpha$ for $\alpha \in [0, 1]$.

Furuta Inequality (FI)

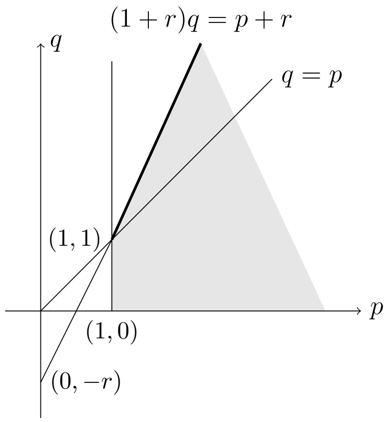
If $A \ge B \ge 0$, then for each $r \ge 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \ge (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \ge (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \ge 0$ and $q \ge 1$ with $(1+r)q \ge p+r$.



Related to Furuta inequality, see [2], [3], [6], [8], [9] and [18].

Especially the optimal case $(1+r)q = p+r$ is the most important, which is realized as a beautiful formula by the use of the α-geometric mean:

If $A \geq B \geq 0$, then for each $r \geq 0$

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq A$$

holds for $p \geq 1$.

More precisely, the conclusion in above is improved by

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \ (\leq A)$$

holds for $p \geq 1$, due to Kamei [12].
The following inequality is led by Lemma 2.1.

Theorem 3.1. *If $A \geq B > 0$, then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$$

holds for $p \leq -1$ and $r \in [-1, 0]$.

Proof. As in the proof of Lemma 2.1, it says that if $A^{-1} \natural_\alpha B \leq 1$, then $A^{-r} \natural_\beta B \leq 1$ for $r \in [0,1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)}$. Thus the assumption is that $C^\alpha \leq A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. So we put $B_1 = C^\alpha \leq A$, and moreover $p = \frac{1}{\alpha}$, $r_1 = r - 1$. Then $p \leq -1$ and $r_1 \in [-1, 0]$ and $\beta = \frac{1+r_1}{p+r_1}$. Moreover the conclusion is rephrased as

$$A^{-r+1} \natural_\beta C \leq A, \ \text{ or } A^{-r_1} \natural_{\frac{1+r_1}{p+r_1}} {B_1}^p \leq A.$$

Now the Furuta inequality was generalized to so-called "grand Furuta inequality" by the appearence of Ando-Hiai inequality, which is due to Furuta [10], see also [5] and [6].

Grand Furuta inequality (GFI) *If $A \geq B > 0$ and $t \in [0,1]$, then*

$$[A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

As a matter of fact, (GFI) interpolates (FI) with (AH), presicely

$$\text{(GFI) for } t = 1,\ r = s \iff \text{(AH)}$$

$$\text{(GFI) for } t = 0,\ (s = 1) \iff \text{(FI)}.$$

As well as (FI), (GFI) has also mean theoretic expression as follows:
If $A \geq B > 0$ and $t \in [0, 1]$, then

$$A^{-r+t} \ \#_{\frac{1-t+r}{(p-t)s+r}} \ (A^t \ \#_s \ B^p) \leq A$$

holds for $r \geq t$ and $p, s \geq 1$.

In succession with the above discussion, Theorem 2.3 gives us the following inequality of (GFI)-type.

Theorem 3.2. *If $A \geq B > 0$, then*

$$A^{-r+1} \natural_{\frac{r}{r+(p-1)s}} (A \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, 1]$ and $s \in [\frac{-2r}{p-1}, 1]$.

Proof. Theorem 2.3 says that if $A^{-1} \natural_\alpha B \leq 1$, then $A^{-r} \natural_\beta B^s \leq 1$ for $r \in [0, 1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$. So the assumption is that $B_1 = C^\alpha \leq A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. On the other hand, the conclusion is that, putting $\alpha = \frac{1}{p}$,

$$1 \geq A^{-r} \ \natural_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} \ B^s = A^{-r} \ \natural_{\frac{r}{r+(p-1)s}} (A^{-\frac{1}{2}} {B_1}^p A^{-\frac{1}{2}})^s$$

or equivalently

$$A \geq A^{-r+1} \ \natural_{\frac{r}{r+(p-1)s}} (A \#_s {B_1}^p).$$

Furthermore, from the viewpoint of (GFI), the following generalization is expected:

Conjecture 3.3. *If $A \geq B > 0$ and $t \in [0, 1]$, then*

$$A^{-r+t} \ \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ and $s \in [\frac{-2r}{p-t}, 1]$.

At present, we can prove it under a restriction:

Theorem 3.4. *If* $A \geq B > 0$ *and* $t \in [0,1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,t]$ *and* $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$.

Proof. First of all, we note that $-1 \leq \frac{1-t+r}{r+(p-t)s} \leq 0$. Hence we have

$$A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} (A^{-t} \#_s B^{-p}) \leq A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} B^{-(p-t)s-t} \leq A^{2(r-t)+1}.$$

The second inequality in above is shown as follows: The exponent $-(p-t)s-t$ of B is nonnegative by $\frac{-t}{p-t} \leq s$. Thus, if $-(p-t)s-t \leq 1$, the second inequality holds. On the other hand, if $-(p-t)s-t \geq 1$, then the Furuta inequality assures that

$$(A^{\frac{t-r}{2}} B^{(-p+t)s-t} A^{\frac{t-r}{2}})^{\frac{1-t+r}{(-p+t)s-r}} \leq A^{1-t+r},$$

or equivalently

$$A^{r-t} \#_{\frac{1-t+r}{(-p+t)s-r}} B^{(-p+t)s-t} \leq A^{2(r-t)+1}.$$

Hence, noting that $X \natural_{-q} Y = X(X^{-1} \natural_q Y^{-1})X$, it follows that

$$\begin{aligned} A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) &= A^{-r+t} \{A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} (A^{-t} \#_s B^{-p})\} A^{-r+t} \\ &\leq A^{-r+t} A^{2(r-t)+1} A^{-r+t} = A. \end{aligned}$$

Remark. On $\gamma = \max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}$ in the statement, $\gamma = \frac{-2r-(1-t)}{p-t}$ is equivalent to the condition $t - r \leq \frac{1}{2}$, which appears in Theorem 3.4.

The following two theorems show that Theorem 3.4 is true at the critical points $s = \frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}$.

Theorem 3.5. *If* $A \geq B > 0$ *and* $t \in [0,1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,t]$ *and* $s = \frac{-2r-(1-t)}{p-t}$.

Proof. First of all, we note that $\frac{1-t+r}{r+(p-t)s} = -1$ and $X \natural_{-1} Y = XY^{-1}X$. Therefore the conclusion is arranged as

$$A^{-r+t} \natural_{-1} (A^t \#_s B^p) \leq A,$$

$$A^{-r+t}(A^{-t}\#_s B^{-p})A^{-r+t} \le A$$

and so

$$A^{-t}\#_s B^{-p} \le A^{1+2r-2t}. \quad (*)$$

To prove this, we recall the Furuta inequality, i.e., if $A \ge B \ge 0$, then

$$(A^{\frac{t}{2}}B^P A^{\frac{t}{2}})^{\frac{1}{q}} \le A^{\frac{P+t}{q}}$$

holds for $t, P \ge 0$ and $q \ge 1$ with $(1+t)q \ge P+t$. Taking $P = -p$ and $q = \frac{1}{s}$, the required condition $(1+t)q \ge P+t$ is enjoyed and we obtain

$$(A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}})^s \le A^{1+2r-t},$$

which is equivalent to (*).

In succession to Theorem 3.5, the other case $s = \frac{-t}{p-t}$ can be proved:

Theorem 3.6. *If $A \ge B > 0$ and $t \in [0,1]$, then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \le A$$

holds for $p \le -1$, $r \in [0,t]$ and $s = \frac{-t}{p-t}$.

Since we have only to consider the case $\frac{-t}{p-t} < \frac{-2r-(1-t)}{p-t}$ by the above theorems, that is, $0 \le t-r < \frac{1}{2}$ can be assumed as cited in Remark of Theorem 3.4, we have

$$\frac{1-t+r}{r+(p-t)s} = 1 - \frac{1}{t-r} < -1.$$

As a special case, we take $t = \frac{2}{3}$, $r = \frac{1}{3}$ and $p = -2$. Then $s = \frac{1}{4}$ and $\frac{1-t+r}{r+(p-t)s} = -2$. Hence the statement in this case is arranged as follows:
If $A \ge B > 0$, then

$$A^{\frac{1}{3}} \natural_{-2} (A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2}) \le A$$

holds? It is proved by using Furuta inequality twice: First of all, since $A \ge B > 0$, (FI) ensures that

$$(A^{\frac{1}{3}}B^2 A^{\frac{1}{3}})^{\frac{5}{8}} \le A^{\frac{5}{3}}.$$

So we have

$$
\begin{aligned}
A^{\frac{1}{3}} \natural_{-2}(A^{\frac{2}{3}}\#_{\frac{1}{4}}B^{-2}) &= A^{\frac{1}{6}}(A^{-\frac{1}{6}}(A^{\frac{2}{3}} \#_{\frac{1}{4}}B^{-2})A^{-\frac{1}{6}})^{-2}A^{\frac{1}{6}} \\
&= A^{\frac{1}{6}}(A^{\frac{1}{6}}(A^{-\frac{2}{3}} \#_{\frac{1}{4}}B^{2})A^{\frac{1}{6}})^{2}A^{\frac{1}{6}} \\
&= A^{\frac{1}{6}}(A^{-\frac{1}{3}} \#_{\frac{1}{4}}A^{\frac{1}{6}}B^{2}A^{\frac{1}{6}})^{2}A^{\frac{1}{6}} \\
&= A^{\frac{1}{6}}(A^{-\frac{1}{6}}(A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{4}}A^{-\frac{1}{6}})^{2}A^{\frac{1}{6}} \\
&= (A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{4}}A^{-\frac{1}{3}}(A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{4}} \\
&\leq (A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{1}{2}-\frac{1}{8}} \\
&\leq (A^{\frac{1}{3}}B^{2}A^{\frac{1}{3}})^{\frac{3}{8}} \\
&\leq A,
\end{aligned}
$$

as desired.

To prove Theorem 3.6, we cite a lemma obtained by the Furuta inequality.

Lemma 3.7. *If $A \geq B > 0$, $t \geq 0$ and $p \leq -1$, then*

$$(A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}})^{\frac{1+t}{-p+t}} \leq A^{1+t};$$

in particular, $(A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}})^{s} \leq A^{t}$ holds for $s = \frac{t}{-p+t}$.

To show Theorem 3.6, we reformulate it as follows:

Theorem 3.8. *If $A \geq B > 0$, $t \geq \frac{c-1}{c+1}$ for some $c \geq 2$, $1 \geq t > r \geq 0$ with $t - r = \frac{1}{c+1}$ and $p \leq -1$, then*

$$A^{\frac{1}{c+1}} \natural_{-c} (A^{t}\#_{s}B^{p}) \leq A$$

holds for $s = \frac{t}{-p+t}$.

Proof. Put $\alpha = t-r$. Then $\alpha = \frac{1}{c+1} < \frac{1}{2}$, $c = \frac{1-\alpha}{\alpha}$ and the assumption $t \geq \frac{c-1}{c+1}$ means $\alpha(c-1) \leq t$, which plays a role when we use the Löwner-Heinz inequality in the below. We put $X = A^{\frac{t}{2}}B^{-p}A^{\frac{t}{2}}$ and $Y = A^{-\frac{r}{2}}X^{s}A^{-\frac{r}{2}}$. Then $A^{\frac{1}{c+1}} \natural_{-c} (A^{t}\#_{s}B^{p}) = A^{\frac{\alpha}{2}}Y^{c}A^{\frac{\alpha}{2}}$, and $X^{\frac{s}{t}} = X^{\frac{1}{-p+t}} \leq A$, in particular, $X^{s} \leq A^{t}$ and $X^{\frac{st'}{t}} \leq A^{t'}$ for $0 \leq t' \leq 1+t$ by Lemma 3.7.

(1) First we suppose that $2n \leq c < 2n+1$ for some n, i.e., $c = 2n+\epsilon$ for some $\epsilon \in [0,1)$. Since $\alpha(c-2) \leq t-\alpha = r$ by $\alpha(c-1) \leq t$, we have $\alpha\epsilon \leq \alpha(2(n-1)+\epsilon) = \alpha(c-2) \leq r$ and so

$$-1 \leq \frac{\alpha\epsilon - r}{t} \leq \frac{\alpha(2(n-k)+\epsilon) - r}{t} \leq 0$$

for $k = 1, 2, \cdots, n$. Noting that $0 \le 2s + [\alpha(2(n-1)+\epsilon) - r]\frac{s}{t} \le \frac{1+t}{-p+t}$ by $\frac{c-1}{c+1} \le 1$, it follows that

$$\begin{aligned}
Y^c &= Y^n Y^\epsilon Y^n = Y^n (A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}})^\epsilon Y^n \\
&\le Y^n (A^{-\frac{r}{2}} A^t A^{-\frac{r}{2}})^\epsilon Y^n = Y^n A^{\alpha\epsilon} Y^n \quad \text{by } X^s \le A^t \text{ and (LH)} \\
&= Y^{n-1} A^{-\frac{r}{2}} X^s A^{\alpha\epsilon - r} X^s A^{-\frac{r}{2}} Y^{n-1} \\
&\le Y^{n-1} A^{-\frac{r}{2}} X^{2s + (\alpha\epsilon - r)\frac{s}{t}} A^{-\frac{r}{2}} Y^{n-1} \quad \text{by } X^s \le A^t,\ \frac{\alpha\epsilon - r}{t} \in [-1, 0] \\
&\le Y^{n-1} A^{2t + \alpha\epsilon - 2r} Y^{n-1} \quad \text{by putting } t' = 2t + \alpha\epsilon - r \le 1 + t \\
&= Y^{n-1} A^{\alpha(2+\epsilon)} Y^{n-1} \\
&\le Y^{n-2} A^{\alpha(4+\epsilon)} Y^{n-2} \\
&\cdots \\
&\le Y A^{\alpha(2(n-1)+\epsilon)} Y \\
&\le A^{\alpha(2n+\epsilon)} \\
&= A^{\alpha c}.
\end{aligned}$$

Hence we have

$$A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}} \le A^{\alpha c + \alpha} = A,$$

as desired.

(2) Next we suppose that $2n+1 \le c < 2n+2$ for some n, i.e., $c = 2n+1+\epsilon$ for some $\epsilon \in [0, 1)$. For this case, we prepare the inequality

$$Y^{1+\epsilon} \le A^{\alpha(1+\epsilon)}.$$

It is proved as follows:

$$\begin{aligned}
Y^{1+\epsilon} &= (A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}})^{1+\epsilon} \\
&= A^{-\frac{r}{2}} X^{\frac{s}{2}} (X^{\frac{s}{2}} A^{-r} X^{\frac{s}{2}})^\epsilon X^{\frac{s}{2}} A^{-\frac{r}{2}} \\
&\le A^{-\frac{r}{2}} X^{\frac{s}{2}} (X^{\frac{s}{2}} X^{-\frac{sr}{t}} X^{\frac{s}{2}})^\epsilon X^{\frac{s}{2}} A^{-\frac{r}{2}} \\
&= A^{-\frac{r}{2}} X^{s + (s - \frac{sr}{t})\epsilon} A^{-\frac{r}{2}} \\
&\le A^{-\frac{r}{2}} A^{t + \alpha\epsilon} A^{-\frac{r}{2}} = A^{\alpha(1+\epsilon)}.
\end{aligned}$$

Now, if $n = 0$, i.e., $c = 1 + \epsilon$, then

$$A^{\frac{\alpha}{2}} Y^{1+\epsilon} A^{\frac{\alpha}{2}} \le A^{\frac{\alpha}{2}} A^{\alpha(1+\epsilon)} A^{\frac{\alpha}{2}} = A^{\alpha(2+\epsilon)} = A.$$

Next, if $c = 2n + 1 + \epsilon$ for some $\epsilon \in [0, 1)$ with $n \neq 0$, then

$$\begin{aligned} Y^c &= Y^n Y^{1+\epsilon} Y^n \leq Y^n A^{\alpha(1+\epsilon)} Y^n \\ &= Y^{n-1} A^{-\frac{r}{2}} X^s A^{\alpha(1+\epsilon)-r} X^s A^{-\frac{r}{2}} Y^{n-1} \\ &\leq Y^{n-1} A^{-\frac{r}{2}} X^{2s+(\alpha(1+\epsilon)-r)\frac{s}{t}} A^{-\frac{r}{2}} Y^{n-1} \\ &\leq Y^{n-1} A^{2t+\alpha(1+\epsilon)-2r} Y^{n-1} \\ &= Y^{n-1} A^{\alpha(3+\epsilon)} Y^{n-1} \\ &\leq Y^{n-2} A^{\alpha(5+\epsilon)} Y^{n-2} \\ &\cdots \\ &\leq Y A^{\alpha(2(n-1)+1+\epsilon)} Y \\ &\leq A^{\alpha(2n+1+\epsilon)} = A^{\alpha c}, \end{aligned}$$

in which $(-1 \leq -r \leq)\ \alpha(2(n-1) + 1 + \epsilon) - r \leq 0$ is required in order to use the Löwner-Heinz inequality. (Fortunately it is assured by the assumption $t \geq \frac{c-1}{c+1}$.) Hence we have

$$A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}} \leq A^{\alpha c + \alpha} = A,$$

as desired.

4 Log-majorization

In this section, we express operator inequalities obtained in Section 2 as log-majorization inequalities.

Theorem 4.1. *For $\alpha \in [-1, 0]$ and positive invertible operators A and B,*

$$(A \natural_\alpha B)^{\frac{rs}{\alpha r + (1-\alpha)s}} \succ_{(\log)} A^r \natural_\beta B^s$$

holds for $r, s \in [0, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$.

Theorem 4.2. *For $\alpha \in [-1, 0]$ and positive invertible operators A and B,*

$$(A \natural_\alpha B)^{\frac{(1-t+r)s}{\alpha r + (1-\alpha t)s}} \succ_{(\log)} A^r \natural_\beta B^s$$

holds for $r, s \in [0, 1]$, where $\beta = \frac{\alpha(1-t+r)}{\alpha r + (1-\alpha t)s}$.

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SEPARATION AXIOMS IN BI-ISOTONIC SPACES

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Abstract. The purpose of this study is to introduce and study the concept of bi-isotonic spaces. In this study, we introduce the notion of the continuous map between bi-isotonic spaces and give the characterizations of bi-isotonic maps. Moreover, we explore the topological concepts of separation axioms in bi-isotonic spaces.

1 Introduction A topological structure on a set is not only defined by the axioms for open sets but also by the collections of closed sets, neighborhood systems, closure operators or interior operators, etc. For instance, Day [6] and Hausdorff [15] developed the topological concepts from the notions of convergence, closure, and neighborhoods. Kuratowski [17] brought a different approach to construct a topological structure on a non-empty set X by defining closure operator $\mathrm{cl} : P(X) \to P(X)$ (where $P(X)$ is the power set of X) with the following properties for all $A, B \in P(X)$;

K0) $\mathrm{cl}(\emptyset) = \emptyset$ (grounded)

K1) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ (Isotony)

K2) $A \subseteq \mathrm{cl}(A)$ (Expansive)

K3) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ (Preservation of binary union)

K4) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (Idempotency).

In this way, the closure operator satisfying the aforementioned axioms allows to define the topological space (X, cl) by taking closed sets as sets such as $\mathrm{cl}(A) = A$. Moreover, Kuratowski extended the topological spaces by removing the axiom $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and defined closure spaces. On the other hand, the approach of Ćech in the definition of closure space excludes the idempotency axiom $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ [5]. In order to avoid confusion on the term of closure space, the terms of Kuratowski closure space and Ćech closure space can be seen in literature. Additionally, Gnilka [8], [9], [10] and Hammer [13], [14] preferred the term extended topological space instead of closure space. The basic concepts of compactness, quasi-metrizability, symmetry, continuity were investigated by the closure operators in these studies. In recent years, Stadler et al. [25], [26], and [27] have revealed a topological approach to chemical organizations, evolutionary theory, and combinatorial chemistry and exposed the relationships between the topological concepts of similarity, neighborhood, connectedness, and continuity with the chemical and biological situations. In these interdisciplinary studies, the authors have considered the basic concepts of the closure and isotonic spaces, such that a closure space (X, cl) satisfying only the grounded and the isotony closure axioms is called an isotonic space. The notions of the connectedness, lower and upper separation axioms in isotonic spaces have been studied

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by Habil and Elzenati [11], [12]. On the other hand, another essential construction in this realm is *Bitopological Spaces* defined by Kelly [16]. There have been a number of longitudinal studies involving bitopological spaces, Wilson [29], Weston [28], and Wiweger [30]. For instance, the separation axioms have been generalized in bitopological spaces and some related characterizations have been given by Lane [18], Marin and Romaguera [19], Murdeshwar and Naimpally [20], Patty [21], Ravi and Thivagar [22], and Reilly [24]. T_0-strongly nodec space has been introduced by using the quotient map in [23]. The definitions and relationships for pairwise T_1, T_2, T_3, $T_{3\frac{1}{2}}$ and T_4-spaces have been presented by Dvalishvili [7]. A great deal of research has been conducted on the bitopological spaces, but few studies have been carried out to discover the biclosure spaces [1]–[4] and to date, none has been discussed bi-isotonic spaces.

2 Bi-Isotonic Spaces

Definition 2.1 *A generalized bi-closure space is a triple* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *where the maps* $\mathrm{cl}_1 : P(X) \to P(X)$ *and* $\mathrm{cl}_2 : P(X) \to P(X)$ *defined on the power set* $P(X)$ *of a non-empty set* X *are two closure operators [1].*

If the closure operators $\mathrm{cl}_1 : P(X) \to P(X)$ and $\mathrm{cl}_2 : P(X) \to P(X)$ are isotonic operators satisfying only the grounded and isotony axioms given by (K0) and (K1), respectively, then the concepts to be studied will be more general than ones given in [1]-[4].

Definition 2.2 *Let* cl_1 *and* cl_2 *be two isotonic operators on* X*, then the triple* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called bi-isotonic space.*

Definition 2.3 *A subset* A *of a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called closed if* $\mathrm{cl}_1\mathrm{cl}_2(A) = A$*. The complement of a closed set is called open.*

Under the light of this definition, the following proposition is obvious.

Proposition 2.4 *A subset* A *of a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is closed if and only if it is a closed subset of* (X, cl_1) *and* (X, cl_2)*.*

In other words, the followings are equivalent in bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$;

i) $\mathrm{cl}_1\mathrm{cl}_2(A) = A$.

ii) $\mathrm{cl}_1(A) = A$ and $\mathrm{cl}_2(A) = A$.

Example 2.5 *Let us consider that*

$$\mathrm{cl}_1(A) = \begin{cases} \emptyset\,, & A = \emptyset \\ (-\infty, a]\,, & \sup A = a \\ \mathbb{R}, & \sup A = \infty \end{cases}$$

and

$$\mathrm{cl}_2(A) = \begin{cases} \emptyset\,, & A = \emptyset \\ [b, \infty)\,, & \inf A = b \\ \mathbb{R}, & \inf A = -\infty \end{cases}$$

be two operators on $\mathbb{R}$ *then the bi-isotonic space* $(\mathbb{R}, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a non-discrete space since* $\mathrm{cl}_1(\mathrm{cl}_2(A)) = \mathbb{R}$ *or* $\mathrm{cl}_1(\mathrm{cl}_2(A)) = \emptyset$ *for all* $A \subseteq \mathbb{R}$*.*

A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is not needed to be a topological space since the finite intersection of closed subsets is not need to be closed.

Example 2.6 *Let* $\mathrm{cl}_1 : P(X) \to P(X)$ *and* $\mathrm{cl}_2 : P(X) \to P(X)$ *be two maps on* $X = \{a,b,c\}$ *satisfying* $\mathrm{cl}_1(\emptyset) = \emptyset$, $\mathrm{cl}_1(\{b\}) = \{b\}$, $\mathrm{cl}_1(\{c\}) = \{c\}$, $\mathrm{cl}_1(\{a,b\}) = \{a,b\}$, $\mathrm{cl}_1(\{b,c\}) = \{b,c\}$, $\mathrm{cl}_1(X) = \mathrm{cl}_1(\{a\}) = \mathrm{cl}_1(\{a,c\}) = X$, *and* $\mathrm{cl}_2(\{b\}) = \{b\}$, $\mathrm{cl}_2(\{c\}) = \{c\}$, $\mathrm{cl}_2(\emptyset) = \emptyset$, $\mathrm{cl}_2(X) = \mathrm{cl}_2(\{a\}) = \mathrm{cl}_2(\{a,b\}) = \mathrm{cl}_2(\{a,c\}) = \mathrm{cl}_2(\{b,c\}) = X$.

In the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *although the subsets* $\{b\}$ *and* $\{c\}$ *are closed,* $\{b,c\}$ *is not closed since* $\mathrm{cl}_1(\{b,c\}) = \{b,c\}$ *and* $\mathrm{cl}_2(\{b,c\}) = X$.

Proposition 2.7 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $A \subseteq X$. *Then*

i) *A is an open set if and only if* $A = X - \mathrm{cl}_1\mathrm{cl}_2(X - A)$,

ii) *If A is an open set and* $A \subseteq G$ *then* $A \subseteq X - \mathrm{cl}_1\mathrm{cl}_2(X - G)$.

Proof.

i) It is obvious from the definition of an open set in bi-isotonic space.

ii) Let A be open and $A \subseteq G$, then $X - G \subseteq X - A$. Thus, $\mathrm{cl}_1(\mathrm{cl}_2(X - G)) \subseteq \mathrm{cl}_1(\mathrm{cl}_2(X - A))$ since the isotonic operators cl_1 and cl_2 have the property (K1). It is clear that $X - \mathrm{cl}_1(\mathrm{cl}_2(X - A)) \subseteq X - \mathrm{cl}_1(\mathrm{cl}_2(X - G))$. The openness of A and the first assertion completes the proof.

The duals of the isotonic operators cl_i on a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ are defined by

$$\mathrm{int}_i : P(X) \to P(X), \mathrm{int}_i(A) = X - (\mathrm{cl}_i(X - A))$$

and called interior operators. In that case, a subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is open if $X - A = \mathrm{cl}_i(X - A)$ or $\mathrm{int}(A) = A$ for all $i \in \{1,2\}$.

The neighborhood operators for $x \in X$ are defined by

$$\nu_i : X \to P(P(X)) \quad , \quad \nu_i(x) = \{N \in P(X) : x \in \mathrm{int}_i(N)\}.$$

Proposition 2.8 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $Y \subseteq X$ *if* $\mathrm{cl}_i^Y(A) = \mathrm{cl}_i(A) \cap Y$ *is satisfied for all* $A \subseteq Y$ *and* $i \in \{1,2\}$, *then the operators* $\mathrm{cl}_i^Y : P(Y) \to P(Y)$ *are isotonic.*

*Proof.*Let us consider the subsets $A, B \subseteq Y$ such that $A \subseteq B$. Then $\mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(B)$ for each $i \in \{1,2\}$ since $\mathrm{cl}_i : P(X) \to P(X)$ are isotonic. Thereby, $\mathrm{cl}_i(A) \cap Y \subseteq \mathrm{cl}_i(B) \cap Y$, that is, $\mathrm{cl}_i^Y(A) \subseteq \mathrm{cl}_i^Y(B)$.

Definition 2.9 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $Y \subseteq X$. *A bi-isotonic space* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *given with induced isotonic operators* cl_1^Y *and* cl_2^Y *is called a subspace of* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$.

Definition 2.10 *If* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *is a subspace of a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, *then the induced interior operators* int_i^Y *and induced neighborhood operators* ν_i^Y *on* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *are defined by*

$$\mathrm{int}_i^Y(A) = Y - \mathrm{cl}_i^Y(Y - A) = Y \cap \mathrm{int}_i(A \cup (X - Y))$$

and

$$\nu_i^Y(A) = \{N \cap Y : N \in \nu_i(A)\},$$

respectively, for any $A \subseteq Y$ *and* $i \in \{1,2\}$.

Proposition 2.11 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space,* Y *be a closed subset of* X*, and* $A \subseteq Y$*.* A *is a closed subset of bi-isotonic space* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *if and only if* A *is a closed subset of bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$*.*

Proof.($\Rightarrow$:) Let A be a closed subset in $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$. In that case, there are closed subsets $\mathrm{cl}_i(A)$ in X for each $i \in \{1,2\}$ such that $A = \mathrm{cl}_i^Y(A) = \mathrm{cl}_i(A) \cap Y$. Thus $A \subseteq \mathrm{cl}_i(A)$. Also $\mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(Y)$, i.e., $\mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(Y) \cap \mathrm{cl}_i(A)$ since $A \subseteq Y$. The closedness of Y gives us $\mathrm{cl}_i(A) \subseteq Y \cap \mathrm{cl}_i(A) = A$. As a consequence $A = \mathrm{cl}_i(A)$ is found.

($\Leftarrow$:) Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space, Y be a closed subset of X, and $A \subseteq Y$. Assume that A is closed in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, then $\mathrm{cl}_i(A) = A$ for each $i \in \{1,2\}$. It is easily seen that $\mathrm{cl}_1^Y(A) = \mathrm{cl}_i(A) \cap Y = A \cap Y = A$. This means that A is closed in $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$.

Example 2.12 *Let us consider the set* $X = \{a,b,c\}$ *and the isotonic operators* $\mathrm{cl}_1 : P(X) \to P(X)$ *and* $\mathrm{cl}_2 : P(X) \to P(X)$*, respectively, defined by* $\mathrm{cl}_1(X) = \mathrm{cl}_1(\{b,c\}) = \mathrm{cl}_1(\{a,c\}) = X$*,* $\mathrm{cl}_1(\emptyset) = \emptyset$*,* $\mathrm{cl}_1(\{a\}) = \{a\}$*,* $\mathrm{cl}_1(\{b\}) = \{b\}$*,* $\mathrm{cl}_1(\{c\}) = \{c\}$*,* $\mathrm{cl}_1(\{a,b\}) = \{a,b\}$ *and* $\mathrm{cl}_2(\emptyset) = \emptyset$*,* $\mathrm{cl}_2(\{a\}) = \{a\}$*,* $\mathrm{cl}_2(\{a,b\}) = \{a,b\}$*,* $\mathrm{cl}_2(X) = \mathrm{cl}_2(\{c\}) = \mathrm{cl}_2(\{b\}) = \mathrm{cl}_2(\{a,c\}) = \mathrm{cl}_2(\{b,c\}) = X$*. If the subset* $Y = \{a,b\}$ *of* X *is considered, then the bi-isotonic subspace* $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ *is constructed by the induced isotonic operators* $\mathrm{cl}_1^Y : P(Y) \to P(Y)$ *and* $\mathrm{cl}_2^Y : P(Y) \to P(Y)$ *such as* $\mathrm{cl}_1^Y(Y) = \mathrm{cl}_1^Y(\{a,b\}) = Y$*,* $\mathrm{cl}_1^Y(\emptyset) = \emptyset$*,* $\mathrm{cl}_1^Y(\{a\}) = \{a\}$*,* $\mathrm{cl}_1^Y(\{b\}) = \{b\}$ *and* $\mathrm{cl}_2^Y(\emptyset) = \emptyset$*,* $\mathrm{cl}_2^Y(\{a\}) = \{a\}$*,* $\mathrm{cl}_2^Y(Y) = \mathrm{cl}_2^Y(\{b\}) = \mathrm{cl}_2^Y(\{a,b\}) = Y$*.*

3 Bi-Continuous Maps in Bi-Isotonic Spaces

Definition 3.1 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be generalized bi-closure spaces. If* $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ *continuous (open, closed or homeomorphism) then* $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *is called* $i-$*continuous (* $i-$*open,* $i-$*closed or* $i-$*homeomorphism).*
Also, the map f *is called bi-continuous if it is* $i-$*continuous map for each* $i \in \{1,2\}$*.*

Example 3.2 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be generalized bi-closure spaces. The identity map* $\mathrm{I} : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (X, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *is bi-continuous if and only if the operator* cl_i *is coarser than* $\mathrm{cl'}_i$ *for each* $i \in \{1,2\}$*, that is,* $\mathrm{cl'}_i(A) \subseteq \mathrm{cl}_i(A)$ *for all* $A \in P(X)$ *and* $i \in \{1,2\}$*.*

Definition 3.3 *Let* (X, cl) *and* $(Y, \mathrm{cl'})$ *be two spaces. A map* $f : X \to Y$ *is continuous if and only if for* $f(\mathrm{cl}(A)) \subseteq \mathrm{cl'}(f(A))$ *all* $A \in P(X)$ *[5].*

Hence by considering the continuity of the maps $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ for each $i \in \{1,2\}$ the following proposition can be given.

Proposition 3.4 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be two generalized bi-closure spaces. A map* $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *is bi-continuous if and only if* $f(\mathrm{cl}_i(A)) \subseteq \mathrm{cl'}_i(f(A))$ *for all* $A \in P(X)$ *and* $i \in \{1,2\}$*.*

Proposition 3.5 *Let* (X, cl) *and* $(Y, \mathrm{cl'})$ *be two isotonic spaces. A map* $f : X \to Y$ *is continuous if and only if* $\mathrm{cl}\left(f^{-1}(B)\right) \subseteq f^{-1}\left(\mathrm{cl'}(B)\right)$ *for all* $B \in P(Y)$ *[27].*

Thus the following proposition is obvious.

Proposition 3.6 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be bi-isotonic spaces. A map* $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *is bi-continuous if and only if* $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl'}_i(B))$ *for all* $B \in P(Y)$ *and* $i \in \{1,2\}$*.*

Proposition 3.7 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *and* $(Z, \mathrm{cl''}_1, \mathrm{cl''}_1)$ *be bi-isotonic spaces.* $f : X \to Y$ *and* $g : Y \to Z$ *are bi-continuous maps, then* $g \circ f : X \to Z$ *is bi-continuous.*

*Proof.*Consider a subset $B \in P(Z)$, then $\mathrm{cl'}_i \left(g^{-1}(B)\right) \subseteq g^{-1}(\mathrm{cl''}_i(B))$ for each $i \in \{1, 2\}$ since g is bi-continuous.
Also, $\mathrm{cl}_i \left(f^{-1}\left(g^{-1}(B)\right)\right) \subseteq f^{-1}\left(\mathrm{cl'}_i\left(g^{-1}(B)\right)\right)$ since $g^{-1}(B) \in P(Y)$ and f is a bi-continuous map.
Moreover, $f^{-1}\left(\mathrm{cl'}_i\left(g^{-1}(B)\right)\right) \subseteq f^{-1}\left(g^{-1}(\mathrm{cl''}_i(B))\right)$ is satisfied. By these last two relations, we get $\mathrm{cl}_i\left(f^{-1}\left(g^{-1}(B)\right)\right) \subseteq f^{-1}\left(g^{-1}(\mathrm{cl''}_i(B))\right)$ and this completes the proof.

Proposition 3.8 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(X, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be bi-isotonic spaces. Then the following conditions (for bi-continuity) are equivalent:*

i) $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *is bi-continuous.*

ii) $f^{-1}(\mathrm{int'}_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ *for all* $B \in P(Y)$ *and* $i \in \{1, 2\}$.

iii) $f^{-1}(B) \in \nu_i(x)$ *provided* $B \in \nu'_i(f(x))$ *for all* $B \in P(Y)$ *and* $i \in \{1, 2\}$.

Proof.(i $\Rightarrow$ ii) Let f be a bi-continuous map. There is the equality

$$f^{-1}(\mathrm{int'}_i(B)) = f^{-1}(Y - \mathrm{cl'}_i(Y - B))$$

for all $B \in P(Y)$ and $i \in \{1, 2\}$ since $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl'}_i(B))$. Hence $X - f^{-1}(\mathrm{cl'}_i(Y - B)) \subseteq X - \mathrm{cl}_i\left(f^{-1}(Y - B)\right)$. Also $X - f^{-1}(\mathrm{cl'}_i(Y - B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ since $X - \mathrm{cl}_i\left(f^{-1}(Y - B)\right) = \mathrm{int}_i\left(f^{-1}(B)\right)$

(ii $\Rightarrow$ iii) Assume that $f^{-1}(\mathrm{int'}_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ for $\forall B \in P(Y)$ and $B \in \nu'_i(f(x))$. In that case, $f(x) \in \mathrm{int}'_i(B)$. Hence $x \in f^{-1}(\mathrm{int'}_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$, that is, $x \in \mathrm{int}_i\left(f^{-1}(B)\right) \Rightarrow f^{-1}(B) \in \nu_i(x)$.

(iii $\Rightarrow$ i) Assume that $f^{-1}(B) \in \nu_i(x)$ when $B \in \nu'_i(f(x))$ for $\forall B \in P(Y)$. If we take $x \in \mathrm{cl}_i\left(f^{-1}(B)\right)$, we find $x \in X - \mathrm{int}_i\left(X - f^{-1}(B)\right)$ since $\mathrm{cl}_i\left(f^{-1}(B)\right) = X - \mathrm{int}_i\left(X - f^{-1}(B)\right)$. Hereby $x \notin \mathrm{int}_i\left(X - f^{-1}(B)\right)$, i.e., $X - f^{-1}(B) \notin \nu_i(x)$. So $f^{-1}(B) \in \nu_i(x)$ is obtained. Also by the hypothesis $B \in \nu'_i(f(x))$, it is easy to derive $(Y - B) \notin \nu'_i(f(x))$ and we have $f(x) \notin \mathrm{int}'_i(Y - B)$. Thus $x \in f^{-1}(\mathrm{cl'}_i(B))$ because of $f(x) \in Y - (\mathrm{int'}_i(Y - B)) = \mathrm{cl'}_i(B)$. Finally, we obtain $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl'}_i(B))$.

Definition 3.9 *Let* $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be a bijective map. If* f *is bi-continuous and* f^{-1} *bi-continuous, then it is called bi-homeomorphism.*

In our further discussion, we shall abbreviate "lower (upper) semicontinuous" to l.(u.)s.c.

Definition 3.10 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $(\mathbb{R}, \omega)$ *be a usual topological space. A function* $f : X \to \mathbb{R}$ *is called*

* cl_i*−l.s.c. if and only if for any* $a \in \mathbb{R}$, *the subset* $f^{-1}((a, \infty))$ *is open in the isotonic space* (X, cl_i),

* cl_i*−u.s.c. if and only if for any* $a \in \mathbb{R}$ *the subset* $f^{-1}((-\infty, a))$ *is open in the isotonic space* (X, cl_i).

Proposition 3.11 *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, *be a bi-isotonic space and* $(\mathbb{R}, \omega)$ *be a usual topological space. If a function* $f : X \to \mathbb{R}$ *is*

* cl_i-*l.s.c., then* $\mathrm{cl}_i\left(f^{-1}((-\infty,a))\right)\subseteq f^{-1}((-\infty,a])$,

* cl_i-*u.s.c., then* $\mathrm{cl}_i\left(f^{-1}((a,\infty))\right)\subseteq f^{-1}([a,\infty))$ *for any* $a\in\mathbb{R}$.

*Proof.*Let $f:X\to\mathbb{R}$ be cl_i-l.s.c., then for $a\in\mathbb{R}$, the subset $f^{-1}((a,\infty))$ is open in the isotonic space (X,cl_i), then $f^{-1}((a,\infty))=\mathrm{int}_i\left(f^{-1}((a,\infty))\right)$. On the other hand $f^{-1}((a,\infty))\subseteq f^{-1}([a,\infty))$ since $(a,\infty)\subseteq[a,\infty)$. It is known that f is isotonic which means that $\mathrm{int}_i\left(f^{-1}((a,\infty))\right)\subseteq\mathrm{int}_i\left(f^{-1}([a,\infty))\right)$. From these relations, we get$f^{-1}((a,\infty))\subseteq\mathrm{int}_i\left(f^{-1}([a,\infty))\right)$ Hence, we obtain $X-f^{-1}((a,\infty))\supseteq X-\mathrm{int}_i\left(f^{-1}([a,\infty))\right)$. As a consequence

$$f^{-1}(\mathbb{R}-(a,\infty))\supseteq\mathrm{cl}_i\left(f^{-1}(\mathbb{R}-[a,\infty))\right)$$

i.e., $\mathrm{cl}_i\left(f^{-1}((-\infty,a))\right)\subseteq f^{-1}((-\infty,a])$ is satisfied. The other case can be seen in a similar manner.

Definition 3.12 *Let* (X,cl_1,cl_2) *be a bi-isotonic space and* $(\mathbb{R},\omega)$ *be a usual topological space. A function* $f:(X,cl_1,cl_2)\to(\mathbb{R},\omega)$ *is called* cl_icl_j-*l.u.s.c.if* f *is* cl_i-*l.s.c. and* cl_j-*u.s.c.*

Proposition 3.13 *Let* (X,cl_1,cl_2) *be a bi-isotonic space,* $(\mathbb{R},\omega)$ *be a usual topological space and* $(\mathbb{R},\omega_1,\omega_2)$ *be a bitopological space where* $\omega_1=\{\mathbb{R},\varnothing\}\cup\{(a,\infty):a\in\mathbb{R}\}$ *is the right ray topology and* $\omega_2=\{\mathbb{R},\varnothing\}\cup\{(-\infty,a):a\in\mathbb{R}\}$ *is the left ray topology on* $\mathbb{R}$. *A function* $f:(X,cl_1,cl_2)\to(\mathbb{R},\omega_1,\omega_2)$ *is bi-continuous if and only if* $f:(X,cl_1,cl_2)\to(\mathbb{R},\omega)$ *is* cl_icl_j-*l.u.s.c. for each* $i,j\in\{1,2\}$ *such that* $i\neq j$.

*Proof.*We shall consider only the case with $i=1$ and $j=2$ since the other case can be proved in a similar manner.

($\Rightarrow$:) Assume that $f:(X,\mathrm{cl}_1,\mathrm{cl}_2)\to(\mathbb{R},\omega_1,\omega_2)$ is a bi-continuous function then there is the relation $\mathrm{cl}_i\left(f^{-1}(B)\right)\subseteq f^{-1}(\mathrm{cl}_{\omega_i}(B))$ for any $B\in P(\mathbb{R})$ and $i\in\{1,2\}$. If we assume $B=(-\infty,a)$ for any $a\in\mathbb{R}$ we see $\mathrm{cl}_{\omega_1}(B)=(-\infty,a]$ with respect to right ray topology ω_1. Hence we get $\mathrm{cl}_1\left(f^{-1}((-\infty,a))\right)\subseteq f^{-1}((-\infty,a])$ which means that $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega)$ cl_1- l.s.c. Similarly, if suppose $B=(a,\infty)$ for any $a\in\mathbb{R}$, then the closure of B with respect to right ray topology ω_2 becomes $\mathrm{cl}_{\omega_2}(B)=[a,\infty)$. So $\mathrm{cl}_2\left(f^{-1}((a,\infty))\right)\subseteq f^{-1}([a,\infty))$ is obtained. This proves that the function $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega)$ is cl_2-u.s.c. As a consequence $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega)$ is $\mathrm{cl}_1\mathrm{cl}_2-$l.u.s.c.

($\Leftarrow$:) Let $f:(X,\mathrm{cl}_1,\mathrm{cl}_2)\to(\mathbb{R},\omega)$ be a$\mathrm{cl}_1\mathrm{cl}_2-$l.u.s.c. function. This means that it is cl_1-l.s.c. and cl_2- u.s.c. i.e., $\mathrm{cl}_1\left(f^{-1}((-\infty,a))\right)\subseteq f^{-1}((-\infty,a])$ and $\mathrm{cl}_2\left(f^{-1}((a,\infty))\right)\subseteq f^{-1}([a,\infty))$ is satisfied for all $a\in\mathbb{R}$. If we take any ω_1- open set B_1 and ω_2- open set B_2 , then $B_1=(a,\infty)$ and $B_2=(-\infty,a)$ for any, $a\in\mathbb{R}$ we get $\mathrm{cl}_1\left(f^{-1}(B_2)\right)\subseteq f^{-1}(\mathrm{cl}_{\omega_1}(B_2))$ and. It is easily seen that $f:(X,\mathrm{cl}_i)\to(\mathbb{R},\omega_i)$ is $i-$continuous for each $i\in\{1,2\}$, that is, the function $f:(X,\tau_1,\tau_2)\to(\mathbb{R},\omega_1,\omega_2)$ is bicontinuous.

4 Separation Axioms in Bi-Isotonic Spaces

Definition 4.1 *A space* (X,cl) *is called* T_0-*space if there* $N_x\in\nu(x)$ *such that* $y\notin N_x$ *or there is* $N_y\in\nu(y)$ *such that* $x\notin N_y$ *for all distinct points* $x,y\in X$ *[27].*

Proposition 4.2 *An isotonic space* (X,cl) *is a* T_0-*space if and only if* $y\notin\mathrm{cl}(\{x\})$ *or* $x\notin\mathrm{cl}(\{y\})$ *for all distinct points* $x,y\in X$ *[27].*

Definition 4.3 *A generalized bi-closure* $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ *is called pairwise* T_0-*space if there is* $N_x\in\nu_1(x)$ *such that* $y\notin N_x$ *or there is* $N_y\in\nu_2(y)$ *such that* $x\notin N_y$ *for all distinct points* $x,y\in X$.

Proposition 4.4 *A bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise* T_0-*space if and only if* $y \notin \mathrm{cl}_1(\{x\})$ *or* $x \notin \mathrm{cl}_2(\{y\})$ *for all distinct points* $x, y \in X$.

*Proof.*Proposition 4.2 and Definition 4.3 require that there is require that there is $N_x \in \nu_1(x)$ such that $y \notin N_x$ iff $y \notin \mathrm{cl}_1(\{x\})$ for all distinct points $x, y \in X$. Similarly $N_y \in \nu_2(y)$ such that $x \notin N_y$ iff $x \notin N_y$. These complete the proof.

Definition 4.5 *A generalized bi-closure space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called*

* *pairwise* S_T_1-*space if there is* $N_x \in \nu_1(x)$ *such that* $y \notin N_x$ *and there is* $N_y \in \nu_2(y)$ *such that* $x \notin N_y$ *for all distinct points* $x, y \in X$,
* *pairwise* R_T_1-*space if* (X, cl_1) *and* (X, cl_2) *are both* T_1-*spaces.*

Proposition 4.6 *An isotonic space* (X, cl) *is* T_1-*space iff* $\mathrm{cl}(\{x\}) \subseteq \{x\}$ *for all* $x \in X$ *[27].*

Proposition 4.7 *If a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise* R_T_1-*space then it is pairwise* S_T_1-*space.*

*Proof.*Let bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise R_T_1-space. Then from Proposition 4.7 $\mathrm{cl}_1(\{x\}) \subseteq \{x\}$ and $\mathrm{cl}_2(\{y\}) \subseteq \{y\}$ for $x \neq y$, $x, y \in X$ since the isotonic spaces (X, cl_1) and (X, cl_2) are T_1- spaces. Hence $X - \{x\} \subseteq \mathrm{int}_1(X - \{x\})$ and $X - \{y\} \subseteq \mathrm{int}_2(X - \{y\})$. It is easily seen $y \in X - \{x\} \subseteq \mathrm{int}_1(X - \{x\})$ from $y \in X - \{x\}$ and $x \notin X - \{x\}$ because $x \neq y$. So we find $X - \{x\} \in \nu_1(y)$. Similarly $x \in \mathrm{int}_2(X - \{y\})$ which means that $X - \{y\} \in \nu_2(x)$. As a consequence, the bi-isotonic space X is pairwise S_T_1-space.

Definition 4.8 *A generalized bi-closure space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is called pairwise Hausdorff space, if there are* $U \in \nu_1(x)$ *and* $V \in \nu_2(y)$ *such that* $U \cap V = \emptyset$ *for all distinct points* $x, y \in X$.

Proposition 4.9 *A bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is pairwise Hausdorff space if and only if there is* $U \in \nu_2(x)$ *such that* $y \notin \mathrm{cl}_1(U)$ *or there is* $V \in \nu_1(y)$ *such that* $x \notin \mathrm{cl}_2(V)$ *for all distinct points* $x, y \in X$.

Proof.

($\Rightarrow$:) Let bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be pairwise Hausdorff. Then there are $U \in \nu_1(x)$ and $U \in \nu_2(x)$ such that $U \cap V = \emptyset$ Hence $y \in \mathrm{int}_1(V)$, i.e., $y \notin X - \mathrm{int}_1(V) = \mathrm{cl}_1(X - V)$ since $V \in \nu_1(y)$. Also $\mathrm{cl}_1(U) \subseteq \mathrm{cl}_1(X - V)$ because cl_1 is the isotonic operator and $U \subseteq X - V$. So there is $U \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(U)$. Similarly, there is $V \in \nu_1(y)$ such that $x \notin \mathrm{cl}_2(V)$.

($\Leftarrow$:) Suppose that there is $U \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(U)$ for all distinct points $x, y \in X$. Then $y \in X - \mathrm{cl}_1(U) = \mathrm{int}_1(X - U)$ and this means that $X - U \in \nu_1(y)$. If we call $X - U = V$ we find $V \in \nu_1(y)$ and $U \in \nu_2(x)$ satisfying $U \cap V = \emptyset$. As a consequence X becomes a pairwise Hausdorff space.

Proposition 4.10 *If bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise Hausdorff space then* (X, cl_1) *and* (X, cl_2) *are* T_1-*spaces.*

*Proof.*Let bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be pairwise Hausdorff. $x \notin \{y\}$ for all $x, y \in X$ such that $x \neq y$. Moreover $x \in \mathrm{int}_1(U)$ since there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $U \cap V = \emptyset$. Form here $x \notin X - \mathrm{int}_1(U) = \mathrm{cl}_1(X - U)$. Also $\mathrm{cl}_1(V) \subseteq \mathrm{cl}_1(X - U)$ is satisfied since $V \subseteq X - U$. Accordingly we obtain $x \notin \mathrm{cl}_1(V)$. On the other hand $\mathrm{cl}_1\{y\} \subseteq \mathrm{cl}_1(V)$, because of $\{y\} \subseteq V$ and cl_1 is an isotonic operator. In conclusion we get

$x \notin \mathrm{cl}_1\{y\}$, i.e. $\mathrm{cl}_1\{y\} \subseteq \{y\}$ which means that (X, cl_1) is a T_1-space. Similarly it can be proved that (X, cl_2) is a T_1-space, too.

If we consider this last proposition associated with Definition 4.5 and Proposition 4.10, we can give the following result.

Corollary 4.11 *Each pairwise Hausdorff bi-isotonic space is a pairwise R_T_1-space, thereby a S_T_1-space.*

Definition 4.12 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space if there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V) = \emptyset$ for all distinct points $x, y \in X$.*

Definition 4.13 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a generalized bi-closure space. It is said that (X, cl_1) is regular with respect to (X, cl_2) if there are $U \in \nu_1(x)$ and $V \in \nu_2(F)$ such that $U \cap V = \emptyset$ for all $x \in X$ and $F \subseteq X$ where $x \notin \mathrm{cl}_1(F)$.*

If (X, cl_1) is regular with respect to (X, cl_2) and (X, cl_2) is regular with respect to (X, cl_1) then $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called pairwise regular.

Proposition 4.14 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space. The isotonic space (X, cl_1) is regular with respect to the isotonic space (X, cl_2) if and only if there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N$ for all neighborhood $N \in \nu_1(x)$ of each $x \in X$.*

Proof.

($\Rightarrow$:) Let (X, cl_1) be regular with respect to the isotonic space (X, cl_2), then there are $U \in \nu_1(x)$ and $V \in \nu_2(F)$ such that $U \cap V = \emptyset$ for all $x \in X$ and $F \subseteq X$ where $x \notin \mathrm{cl}_1(F)$. Consider any $x \in X$ and $N \in \nu_1(x)$. In that case $x \notin X - \mathrm{int}_1(N) = \mathrm{cl}_1(X - N)$ since $x \in \mathrm{int}_1(N)$. So there is a $V \in \nu_2(X - N)$ such that $U \cap V = \emptyset$. Moreover, $X - \mathrm{cl}_2(U) \supseteq X - \mathrm{cl}_2(X - V) = \mathrm{int}_2(V)$ because of $\mathrm{cl}_2(U) \subseteq \mathrm{cl}_2(X - V)$. We get $X - N \subseteq \mathrm{int}_2(V) \subseteq X - \mathrm{cl}_2(U)$ since $V \in \nu_2(X - N)$ and $F \in \mathrm{int}_2(V)$. Finally there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N$.

($\Leftarrow$:) Assume that there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N$ for all neighborhood $N \in \nu_1(x)$ and consider a subset $F \subseteq X$ such that $x \notin \mathrm{cl}_1(F)$ for any $x \in X$. At that time $x \in X - \mathrm{cl}_1(F) = \mathrm{int}_1(X - F)$. Hence $X - F \in \nu_1(x)$. Under the assumption there is a $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq X - F$. Then $F \subseteq X - \mathrm{cl}_2(U)$. If we call $X - \mathrm{cl}_2(U) = V$, then there are $U \in \nu_1(x)$ and $V \in \nu_2(F)$ such that $U \cap V = \emptyset$ and this completes the proof.

The following result can be given from Definition 4.13 and Proposition 4.14.

Corollary 4.15 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise regular if and only if there is $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N'$ for all $N' \in \nu_1(x)$ and there is $V \in \nu_2(x)$ such that $\mathrm{cl}_1(V) \subseteq N''$ for all $N'' \in \nu_2(x)$.*

Definition 4.16 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called pairwise T_3-space if it is both pairwise regular and pairwise R_T_1-space.*

Proposition 4.17 *If a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise T_3-space then it is pairwise Hausdorff space.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise T_3-space. Then $\mathrm{cl}_2(\{y\}) \subseteq \{y\}$ is satisfied for all $y \in X$ since it is pairwise R_T_1-space. From here $X - \{y\} \in \nu_2(x)$ for any $x \in X - \{y\}$ since $X - \{y\} \subseteq \mathrm{int}_2(X - \{y\})$.

On the other hand there is $\mathrm{cl}_1(V) \subseteq X - \{y\}$ such that $V \in \nu_2(x)$ for $X - \{y\} \in \nu_2(x)$ since $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is pairwise regular. So we find a $V \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(V)$ and say X is pairwise Hausdorff space by Proposition 4.9.

Definition 4.18 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $A, B \subset X$. If there is a $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. function $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $f(A) = 0$ and $f(B) = 1$, then A is called $(i, j)-$completely separated from B.*

Definition 4.19 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called*

* *$(i, j)-$completely regular if every closed F is $(i, j)-$completely separated from each point $x \notin F$.*
* *pairwise completely regular if it is both $(1, 2)-$completely regular and $(2, 1)-$completely regular.*
* *pairwise $T_{3\frac{1}{2}}-$space if it is both pairwise completely regular and pairwise R_T_1-space.*

Proposition 4.20 *If a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is pairwise completely regular then it is pairwise regular.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise completely regular bi-isotonic space. Consider any closed subset F and any point x such that $x \notin F$ There is a $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. function $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $f(x) = 0$ and $f(F) = 1$. This means that the function $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega_1, \omega_2)$ is bi-continuous. On the other hand there are $\omega-$open neighborhoods U and V of the points 0 and 1, respectively, such that $U \cap V = \emptyset$ since the usual topological space $(\mathbb{R}, \omega)$ is Hausdorff. In that case $\mathrm{cl}_i\left(f^{-1}(U)\right) \subseteq f^{-1}(\mathrm{cl}_{\omega_i}(U))$, that is, $f^{-1}(0) = x \in f^{-1}(\mathrm{int}_{\omega_i}(U)) \subseteq \mathrm{int}_i\left(f^{-1}(U)\right)$. From here we can say that $f^{-1}(U) \in \nu_i(x)$. Analogously $f^{-1}(V) \in N_j\left(f^{-1}(1)\right) = N_j(F)$. As a consequence we obtain the subsets $f^{-1}(U) \in \nu_i(x)$ and $f^{-1}(V) \in N_j(F)$ such that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and this is sufficient to say $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise regular bi-isotonic space.

If we consider the last proposition and the definitions of pairwise $T_{3\frac{1}{2}}-$space and pairwise T_3-space, then we can give the following corollary.

Corollary 4.21 *If a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise $T_{3\frac{1}{2}}-$space then it is T_3-space.*

Definition 4.22 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space.*

(TN) *X is called pairwise t-normal, if there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \emptyset$ for all disjoint closed subsets $F, K \subseteq X$.*

(QN) *X is called pairwise quasi-normal, if there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \emptyset$ for all subsets $F, K \subseteq X$ such that $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \emptyset$.*

(N) *X is called pairwise normal, if there are $U \in \nu_1(\mathrm{cl}_1(F))$ and $V \in \nu_2(\mathrm{cl}_2(K))$ such that $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \emptyset$.*

It is easy to prove following proposition if we consider Definition 4.22 associated with the definition of closed sets in bi-isotonic spaces.

Proposition 4.23 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space.*

* *(N)⇒(TN).*
* *(QN)⇒(TN).*

Definition 4.24 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called T_4-space if it is both pairwise quasi-normal and pairwise R_T_1-space.*

So the following proposition can be seen easily.

Proposition 4.25 *If $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise T_4-bi-isotonic space then it is pairwise $T_{3\frac{1}{2}}-$space.*

Definition 4.26 *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space. The subsets $A, B \subseteq X$ are said to be semi- disjoint if $\mathrm{cl}_1(A) \cap B = A \cap \mathrm{cl}_2(B) = \emptyset$.*

Definition 4.27 *A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called*

* *pairwise completely normal if there are $U \in \nu_1(A)$ and $V \in \nu_2(B)$ such that $U \cap V = \emptyset$ for all semi-disjoint subsets $A, B \subseteq X$,*
* *T_5-space if it is both pairwise completely normal and R_T_1-space.*

Proposition 4.28 *The properties of pairwise T_0, R_T_1, S_T_1, Hausdorff, $T_{2\frac{1}{2}}$, regular, T_3, completely regular, $T_{3\frac{1}{2}}$, t-normal, quasi-normal, normal, T_4, completely normal and T_5-spaces in bi-isotonic spaces are topological properties.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ be two bi-isotonic spaces and $f : X \to Y$ be a bi-homeomorphism.

* Let X be a pairwise T_0-space. Consider any distinct points $x', y' \in Y$, then $f^{-1}(x') \neq f^{-1}(y')$. From here $f^{-1}(y') \notin \mathrm{cl}_1\left(\left\{f^{-1}(x')\right\}\right)$ $f^{-1}(x') \notin \mathrm{cl}_2\left(\left\{f^{-1}(y')\right\}\right)$. This gives us $y' = f\left(f^{-1}(y')\right) \notin f\left(\mathrm{cl}_1\left(\left\{f^{-1}(x')\right\}\right)\right) \subseteq \mathrm{cl'}_1\left(f\left(\left\{f^{-1}(x')\right\}\right)\right) = \mathrm{cl'}_1(\{x'\})$ or $x' = f\left(f^{-1}(x')\right) \notin f\left(\mathrm{cl}_2\left(\left\{f^{-1}(y')\right\}\right)\right) \subseteq \mathrm{cl'}_2\left(f\left(\left\{f^{-1}(y')\right\}\right)\right) = \mathrm{cl'}_2(\{y'\})$. That is $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise T_0-space.
* Let X be a pairwise R_T_1-space. Consider any $x' \in Y$, then there is a point $f^{-1}(x') = x \in X$. $\mathrm{cl}_i(\{x\}) \subseteq \{x\}$ is satisfied in (X, cl_i) for each $i \in \{1,2\}$ since the isotonic spaces (X, cl_1) and (X, cl_2) are both T_1-spaces. Moreover $f(\mathrm{cl}_i(A)) = \mathrm{cl'}_i(f(A))$ for all $A \in P(X)$ because of the functions $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ are $i-$homeomorphisms for each her $i \in \{1,2\}$. Thus,
$$\mathrm{cl'}_1(x') = \mathrm{cl'}_1(f(\{x\})) = f(\mathrm{cl}_1(\{x\})) \subseteq f(x) = \{x'\}$$
and
$$\mathrm{cl'}_2(x') = \mathrm{cl'}_2(f(\{x\})) = f(\mathrm{cl}_2(\{x\})) \subseteq f(x) = \{x'\}$$
is satisfied. This means that $(Y, \mathrm{cl'}_1)$ and $(Y, \mathrm{cl'}_2)$ are T_1-spaces. Consequently, the bi-isotonic space $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise R_T_1-space.
* Let X be a pairwise S_T_1-space. Consider any distinct points $x', y' \in Y$, then there are points $f^{-1}(x') = x$ and $f^{-1}(y') = y$ in X such that $x \neq y$. Hence there are $N_x \in \nu_1(x)$ and $N_y \in \nu_2(y)$ such that $x \notin N_y$ and $y \notin N_x$. If $A \in \nu_i(x)$ then $f(A) \in \nu'_i(f(x))$ for all $\forall A \in P(X)$ since the functions $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ are $i-$homeomorphisms for each $i \in \{1,2\}$. In this way, there are $f(N_x) \in \nu'_1(x')$ and $f(N_y) \in \nu'_2(y')$ such that $x' \notin f(N_y)$ and $y' \notin f(N_x)$. This completes the proof.
* Let X be a pairwise Hausdorff space. Consider any distinct points $x', y' \in Y$, then there are points $f^{-1}(x') = x$ and $f^{-1}(y') = y$ in X such that $x \neq y$. So, there is $U \in \nu_2(x)$ such that $y \notin \mathrm{cl}_1(U)$ or there is $V \in \nu_1(y)$ such that $x \notin \mathrm{cl}_2(V)$. On the other hand $f(\mathrm{cl}_i(A)) \subseteq \mathrm{cl'}_i(f(A))$ for all $A \in P(X)$ and $i \in \{1,2\}$. In this case, there is $f(U) \in \nu'_2(x')$ such that $y' \notin f(\mathrm{cl}_1(U)) \subseteq \mathrm{cl'}_1(f(U))$ or there is $f(V) \in \nu'_1(y')$ such that $x' \notin f(\mathrm{cl}_2(V)) \subseteq \mathrm{cl'}_2(f(V))$. This means that the bi-isotonic space $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise Hausdorff space.

* Let X be a pairwise $T_{2\frac{1}{2}}-$space and $x', y' \in Y$ such that $x' \neq y'$, then there are points $f^{-1}(x') = x$ and $f^{-1}(y') = y$ in X such that $x \neq y$. So, there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V) = \emptyset$. Also

$$f(\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V)) = f(\mathrm{cl}_1(U)) \cap f(\mathrm{cl}_2(V))$$

and

$$f(\mathrm{cl}_1(U)) \cap f(\mathrm{cl}_2(V)) = \mathrm{cl'}_1(f(U)) \cap \mathrm{cl'}_2(f(V))$$

are satisfied since the functions $f : (X, \mathrm{cl}_i) \to (Y, \mathrm{cl'}_i)$ are $i-$homeomorphisms for all $i \in \{1,2\}$. Moreover, $f(A) \in \nu'_i(f(x))$, provided that $A \in \nu_i(x)$ for $\forall A \in P(X)$. Hence, there are $f(U) \in \nu'_1(x')$ and $f(V) \in \nu'_2(y')$ such that $\mathrm{cl'}_1(f(U)) \cap \mathrm{cl'}_2(f(V)) = \emptyset$. From here $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space.

* Let X be a pairwise regular. Consider any $x' \in Y$, then $f^{-1}(x') = x \in X$ and there is $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq N'$ for all $N' \in \nu_1(x)$ and there is $V \in \nu_2(x)$ such that $\mathrm{cl}_1(V) \subseteq N''$ for all $N'' \in \nu_2(x)$.

* Let X be a pairwise T_3-space. Then it is pairwise regular and pairwise R_T_1-space. The aforementioned proposition requires that $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise regular and pairwiseR_T_1-space. Thus, Y is a pairwise T_3-space, too.

* Let X be a pairwise completely regular space. Consider any $\mathrm{cl'}_1\mathrm{cl'}_2-$closed subset $F' \subset Y$and any point $x' \in Y$ such that $x' \notin F'$. Then there is $\mathrm{cl}_1\mathrm{cl}_2-$closed subset $f^{-1}(F') = F \subset X$ and a point $x \in X$ such that $x \notin F$ since $f : X \to Y$ is a bi-homeomorphism. There is a $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. a function $g : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $g(x) = 0$ and $g(F) = 1$ because X is pairwise completely regular space. From here $g : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega_1, \omega_2)$ is a bi-continuous function.

$$\begin{tikzcd}
(X, \mathrm{cl}_1, \mathrm{cl}_2) \arrow[rr, "g"] \arrow[dr, "f"'] & & (\mathbb{R}, \omega_1, \omega_2) \\
& (Y, \mathrm{cl'}_1, \mathrm{cl'}_2) \arrow[ur, "g \circ f^{-1}"'] &
\end{tikzcd}$$

This diagram indicates that the function $g \circ f^{-1} : (Y, \mathrm{cl'}_1, \mathrm{cl'}_2) \to (\mathbb{R}, \omega_1, \omega_2)$ is bi-continuous and so $g \circ f^{-1} : (Y, \mathrm{cl'}_1, \mathrm{cl'}_2) \to (\mathbb{R}, \omega)$ is a $\mathrm{cl'}_i\mathrm{cl'}_j-$l.u.s.c. function. Also $g(x) = g\left(f^{-1}(x')\right) = g \circ f^{-1}(x') = 0$ and $g(F) = g\left(f^{-1}(F')\right) = g \circ f^{-1}(F') = 1$ are satisfied. That is $\mathrm{cl'}_1\mathrm{cl'}_2-$ closed subset$F'$of bi-isotonic space $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is $(1,2)$ and $(2,1)$ $-$completely separated from each point $x' \notin F'$ and this completes the proof.

* Let X be a pairwise $T_{3\frac{1}{2}}-$space. In this case, $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is both pairwise completely regular and pairwise R_T_1-space. So, $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise completely regular and pairwise R_T_1-space, too. Thus the bi-isotonic space Y is a pairwise $T_{3\frac{1}{2}}-$space.

* Let X be a pairwise t-normal bi-isotonic space. Let us take two separated closed subsets F' and K' in Y. Then $f^{-1}(F') = F$ and $f^{-1}(K') = K$ are disjoint closed subsets of X. There are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$ since X is a pairwise t-normal space. From Proposition 3.8, we see that there are $f(U) \in N'_1(F')$ and $f(V) \in N'_2(K')$ such that $f(U) \cap f(V) = \varnothing$. Consequently, Y is a pairwise t-normal bi-isotonic space.

* Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise quasi-normal bi-isotonic space. Consider two non-empty subsets F' and K' of Ysuch that $\mathrm{cl'}_1(F') \cap \mathrm{cl'}_2(K') = \varnothing$. It is easily seen

that$\mathrm{cl}_1\left(f^{-1}(F')\right)\cap \mathrm{cl}_2\left(f^{-1}(K')\right)=f^{-1}(\varnothing)=\varnothing$since$f$ is bi-homeomorphism. There are $U\in\nu_1\left(f^{-1}(F')\right)$ and $V\in\nu_2\left(f^{-1}(K')\right)$ such that $U\cap V=\varnothing$, since X is a pairwise quasi-normal space. In that way Y is a pairwise quasi-normal space from Proposition 3.8 .

* Let X be a pairwise normal space and consider the non-empty subsets F' and K' in Y such that $\mathrm{cl'}_1(F')\cap \mathrm{cl'}_2(K')=\varnothing$. Then there are the non-empty subsets F and K in X such that$\mathrm{cl}_1(F)\cap\mathrm{cl}_2(K)=\varnothing$ since f is a bi-homeomorphism satisfying $f(\mathrm{cl}_1(F))=\mathrm{cl'}_1(F')$ and $f(\mathrm{cl}_2(K))=\mathrm{cl'}_2(K')$. Also, there are $U\in\nu_1(\mathrm{cl}_1(F))$ and $V\in\nu_2(\mathrm{cl}_2(K))$ such that $U\cap V=\varnothing$ since X is a pairwise normal space. In these considerations, we find $f(U)\in N'_1(\mathrm{cl'}_1(F'))$ and $f(V)\in N'_2(\mathrm{cl'}_2(K))$ satisfying $f(U)\cap f(V)=\varnothing$. Finally, we see that the bi-isotonic space Y is a pairwise normal space.

* Let $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ be a pairwise T_4-space. This means that $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ is a pairwise normal and pairwise R_T_1-space. If $f:X\to Y$ is a bi-homeomorphism then $(Y,\mathrm{cl'}_1,\mathrm{cl'}_2)$ is pairwise normal and pairwise R_T_1-space. So, the bi-isotonic space Y is a pairwise T_4-space, too.

* Let X be a pairwise completely normal space and $A',B'\subset Y$ be separated sets, i.e., $\mathrm{cl'}_1(A')\cap B'=A'\cap\mathrm{cl}_2(B')=\varnothing$. Thus, $f^{-1}(\mathrm{cl'}_1(A'))\cap f^{-1}(B')=f^{-1}(B')\cap f^{-1}(\mathrm{cl'}_2(A'))=\varnothing$ is satisfied and there are two sets $f^{-1}(A')=A$ and $f^{-1}(B')=B$ in the bi-isotonic space X such that $\mathrm{cl}_1(A)\cap B=A\cap\mathrm{cl}_2(B)=\varnothing$ since the function f is bi-homeomorphism. From the hypothesis, we see $U\in\nu_1(A)$ and $V\in\nu_2(B)$ such that $U\cap V=\varnothing$. Then, we find the sets $f(U)\in N'_1(A')$ and$f(V)\in N'_2(B')$ satisfying $f(U)\cap f(V)=\varnothing$. As a consequence, the bi-isotonic space $(Y,\mathrm{cl'}_1,\mathrm{cl'}_2)$ is pairwise completely normal.

* Let X be a pairwise T_5-space. Then $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ is a pairwise completely normal and pairwise R_T_1-space. $(Y,\mathrm{cl'}_1,\mathrm{cl'}_2)$ pairwise completely normal and pairwise R_T_1-space since $f:X\to Y$ is a bi-homeomorphism. Finally, Y is a pairwise T_5-space, too.

Proposition 4.29 *In a bi-isotonic space, the properties of being a pairwise T_0, pairwise R_T_1, pairwise S_T_1, pairwise Hausdorff, pairwise $T_{2\frac{1}{2}}$, pairwise regular, pairwise T_3, pairwise completely regular, pairwise completely normal and pairwise T_5-space are hereditary properties.*

*Proof.*Let $\left(Y,\mathrm{cl}_1^Y,\mathrm{cl}_2^Y\right)$ and $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ be bi-isotonic spaces.

* Assume that the bi-isotonic space $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ is a pairwise T_0-space and $x,y\in Y$ such that $x\neq y$. Then $y\notin\mathrm{cl}_1(\{x\})$ or $x\notin\mathrm{cl}_2(\{y\})$ for $x,y\in X$. From Proposition 2.8, we can say $y\notin\mathrm{cl}_1(\{x\})\cap Y=\mathrm{cl}_1^Y(\{x\})$ or $x\notin\mathrm{cl}_2(\{y\})\cap Y=\mathrm{cl}_2^Y(\{x\})$. Thus, the subspace $(Y,\mathrm{cl'}_1,\mathrm{cl'}_2)$ is obtained as a pairwise T_0-space.

* Assume that the bi-isotonic space $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ is a pairwise R_T_1-space. In this case (X,cl_1) and (X,cl_2) spaces are T_1-spaces. Hence $\mathrm{cl}_1(\{x\})\subseteq\{x\}$ and $\mathrm{cl}_2(\{x\})\subseteq\{x\}$ for every $x\in Y\subseteq X$. It is found that $\mathrm{cl}_1^Y(\{x\})=\mathrm{cl}_1(\{x\})\cap Y\subseteq\{x\}\cap Y=\{x\}$ and $\mathrm{cl}_2^Y(\{x\})=\mathrm{cl}_2(\{x\})\cap Y\subseteq\{x\}\cap Y=\{x\}$ in $\left(Y,\mathrm{cl}_1^Y\right)$ and $\left(Y,\mathrm{cl}_2^Y\right)$, respectively. Then $\left(Y,\mathrm{cl}_1^Y\right)$ and $\left(Y,\mathrm{cl}_2^Y\right)$ spaces are T_1-spaces. Eventually, it is obtained that $(Y,\mathrm{cl'}_1,\mathrm{cl'}_2)$ is a pairwise R_T_1-space

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise S_T_1-space and $x, y \in Y$ such that $x \neq y$. Then there are $N_x \in \nu_1(x)$ and $N_y \in \nu_2(y)$ such that $y \notin N_x$ and $x \notin N_y$, respectively, for the distinct points $x, y \in X$. There are $N_x^Y \in \nu_1^Y(x)$ and $N_y^Y \in \nu_2^Y(y)$ such that $y \notin N_x \cap Y = N_x^Y$ and $x \notin N_y \cap Y = N_y^Y$ since Y is a subspace. Thus, $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise S_T_1-space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise Hausdorff space and $x, y \in Y$ such that $x \neq y$. By the hypothesis there is $U \in \nu_2(x)$ or $V \in \nu_1(y)$ such that $y \notin \mathrm{cl}_1(U)$ or $x \notin \mathrm{cl}_2(V)$, respectively, as $x \neq y$ for every $x, y \in X$. From Definition 2.5., it is seen that $U \in \nu_2^Y(x)$ or $V \in \nu_1^Y(x)$ since $x \notin \mathrm{cl}_2(V) \cap Y = \mathrm{cl}_2^Y(V)$ or $y \notin \mathrm{cl}_1(U) \cap Y = \mathrm{cl}_1^Y(U)$, respectively. So $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ becomes a pairwise Hausdorff space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space and $x, y \in Y$ for $x \neq y$. By the hypothesis, there are $U \in \nu_1(x)$ and $V \in \nu_2(y)$ such that $\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V) = \varnothing$. Then, there are $U \in \nu_2^Y(x)$ and $V \in \nu_2^Y(y)$ such that $\mathrm{cl}_1^Y(U) \cap \mathrm{cl}_2^Y(V) = (\mathrm{cl}_1(U) \cap \mathrm{cl}_2(V)) \cap Y = \varnothing$. As a result $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise $T_{2\frac{1}{2}}-$space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise regular space and $x, y \in Y$ for $x \neq y$. Then, there is a set $U \in \nu_1(x)$ such that $\mathrm{cl}_2(U) \subseteq \nu_1$ for every neighborhood $\nu_1 \in \nu_1(x)$ and there is a set $V \in \nu_2(y)$ such that $\mathrm{cl}_1(V) \subseteq \nu_2$ for every neighborhood $\nu_2 \in \nu_2(x)$. From here, there are $U \in \nu_1^Y(x)$ and $V \in \nu_2^Y(x)$ such that $\mathrm{cl}_2^Y(U) = \mathrm{cl}_2(U) \cap Y \subseteq \nu_1 \cap Y = \nu_1^Y$ and $\mathrm{cl}_1^Y(V) = \mathrm{cl}_1(V) \cap Y \subseteq \nu_2 \cap Y = \nu_2^Y$ for the neighborhoods $\nu_1 \cap Y = \nu_1^Y \in \nu_1^Y(x)$ and $\nu_2 \cap Y = \nu_2^Y \in \nu_2^Y(x)$, respectively. It is seen that $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise regular.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a T_3-space. Then X is both pairwise regular and pairwise R_T_1-space and so, $Y \subset X$ subspace is both pairwise regular and pairwise R_T_1-space. This means that the bi-isotonic subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a T_3-space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ a pairwise completely regular. Let us take any $\mathrm{cl}_1^Y \mathrm{cl}_2^Y-$closed set F and any point $x \in Y$ such that $x \notin F$. Then $x \notin \mathrm{cl}_1\mathrm{cl}_2(F)$ for the closed subset $\mathrm{cl}_1\mathrm{cl}_2(F)$ in X. By the hypothesis, there is a function $\mathrm{cl}_i\mathrm{cl}_j-$l.u.s.c. $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (\mathbb{R}, \omega)$ such that $f(x) = 0$ and $f(\mathrm{cl}_1\mathrm{cl}_2(F)) = 1$. If we denote the restriction function of f to Y with $f|_Y = g$, it is provided $g(x) = 0$ and $g(F) = 1$ since $g : \left(X, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right) \to (\mathbb{R}, \omega)$ is $\mathrm{cl}_i^Y \mathrm{cl}_j^Y-$l.u.s.c. Thus, the bi-isotonic subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is pairwise completely regular space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a $T_{3\frac{1}{2}}-$space. Then X is both pairwise completely regular and pairwise R_T_1-space. Thus, $Y \subset X$ subspace is both pairwise completely regular and pairwise R_T_1-space. So $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ bi-isotonic subspace is $T_{3\frac{1}{2}}-$space, too.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise completely normal. Let $A, B \subseteq Y$ be separated sets. Namely $\mathrm{cl}_1^Y(A) \cap B = A \cap \mathrm{cl}_2^Y(B) = \varnothing$. Then $\mathrm{cl}_1(A) \cap B = A \cap \mathrm{cl}_2(B) = \varnothing$. By the hypothesis, there are the neighborhoods $U \in \nu_1(A)$ and $V \in \nu_2(B)$ such that $U \cap V = \varnothing$. So, it is obtained that there are $U^Y \in \nu_1^Y(A)$ and $V^Y \in \nu_2^Y(B)$such that $U^Y \cap V^Y = \varnothing$. As a result, the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is found as pairwise completely normal space.

* Assume that the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a T_5-space. Then X is pairwise completely normal and pairwise R_T_1-space. So, the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is T_5-space since it is pairwise completely normal and pairwise R_T_1-space.

Proposition 4.30 *Let (X, cl_1, cl_2) be a bi-isotonic space and $Y \subseteq X$ be a closed subset. If (X, cl_1, cl_2) is a pairwise t-normal, pairwise quasi-normal, pairwise normal space and pairwise T_4-space, then $\left(Y, cl_1^Y, cl_2^Y\right)$ is a pairwise t-normal, pairwise quasi-normal, pairwise normal and pairwise T_4-space, respectively.*

*Proof.*Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and subset $Y \subseteq X$ be closed.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise t-normal and $F, K \subseteq Y$ be non-empty discrete closed subsets. Then the subsets F, K in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ are closed from the Proposition 2.4. By the hypothesis, there are the neighborhoods $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$. So there are the sets $U^Y \in \nu_1^Y(F)$, $V^Y \in \nu_2^Y(K)$ such that $U^Y \cap V^Y = (U \cap Y) \cap (V \cap Y) = \varnothing$. Namely, the bi-isotonic subspace $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ is a pairwise t-normal space.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise quasi-normal and $F, K \subseteq Y$ be non-empty discrete subsets such that $\mathrm{cl}_1^Y(F) \cap \mathrm{cl}_2^Y(K) = \varnothing$. Then there are non-empty discrete subsets $F, K \subseteq X$ satisfying $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \varnothing$. By the hypothesis, there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$. As a consequence, it is easy to find that there are $U^Y \in \nu_1^Y(F)$ and $V^Y \in \nu_2^Y(K)$ such that $U^Y \cap V^Y = \varnothing$, which means that the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is a pairwise quasi-normal space.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise quasi-normal and $F, K \subseteq Y$ be non-empty discrete subsets such that $\mathrm{cl}_1^Y(F) \cap \mathrm{cl}_2^Y(K) = \varnothing$. Then there are non-empty discrete subsets $F, K \subseteq X$ satisfying $\mathrm{cl}_1(F) \cap \mathrm{cl}_2(K) = \varnothing$. By the hypothesis, there are $U \in \nu_1(F)$ and $V \in \nu_2(K)$ such that $U \cap V = \varnothing$. As a consequence, it is easy to find that there are $U^Y \in \nu_1^Y(F)$ and $V^Y \in \nu_2^Y(K)$ such that $U^Y \cap V^Y = \varnothing$, which means that the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is a pairwise quasi-normal space.

* Let the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a T_4-space. Then it is pairwise quasi-normal and pairwise R_T_1-space. So the bi-isotonic subspace $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is pairwise quasi-normal and pairwise R_T_1-space provided that Y is closed subset of X . This completes the proof.

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Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

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1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
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Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80, Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50, Euro47	US$30, Euro28
Life member*	Calculated as below*	US$750, Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73 − age) × ￥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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