Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 82, No. 3 Whole Number 293

December 2019

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

The International Society for Mathematical Sciences (ISMS) is an international society consisting of mathematical scientists throughout the world.

The main activities of the ISMS are to publish (1) the (**print and online**) journal **Scientiae Mathematicae Japonicae (SCMJ)** and (2) Notices from the ISMS and to hold **assembly meeetings** in Japan and **international internet meetings** (distance symposium) of mathematical sciences (IVMS) accessible from all over the world.

SCMJ is the 21^{st} Century New Unified Series of **Mathematica Japonica (MJ)** and **Scientiae Mathematicae (SCM). MJ** was first published in 1948 and was one of the oldest mathematical journals in Japan. **SCM** was an online and print journal started in 1998 in celebration of the semi-centurial anniversary and received 26000 visits per month from 50 countries in the world. **SCMJ** contains original papers in mathematical sciences submitted from all over the world and receives **38000** visits per month now. Not only papers in pure and applied mathematics but those devoted to mathematical statistics, operations research, informatics, computer science, biomathematics, mathematical economics and other mathematical sciences are also welcome. The journal is published in January, March, May, July, September, and November in each calendar year.

The ISMS has enhanced the journal, begining from July 1995, by including excellent Research-Expository papers in the section "**International Plaza** for Mathematical Sciences " as well as original research papers. The section provides papers dealing with broad overviews of contemporary mathmatical sciences, written by experts mainly at our invitation. Papers shedding lights on open problems or new directions or new breakthroughs for future research are especially welcome.

As is shown in the **Editorial Board of SCMJ**, we have invited **many distinguished professors** of 20 countries as editors, who will receive and referee the papers of their special fields with their **high standard**.

Beginning from 2007, we make the online version of SCMJ more readable and convenient to the readers by adding the specialized contents. By this, the readers can access to the online version, in which the papers appear in the order of acceptance, from (i) the contents of the printed version, and (ii) the specialized contents of a volume. From 2007, the subscription fee of the printed version plus the online version of SCMJ becomes lower and the same of the printed version only. Therefore, the subscribers of the printed version can read the online version without no additional cost.

For benefit of the ISMS members, we publish "Notices from the ISMS" 6 times a year. We are enhancing it by adding interesting articles, including book reviewing, written by eminent professors.

The ISMS has set up a **videoconferencing system** (**IVMS**) which can connect up to twenty sites of a reserch group in the same or **different countries in the world**. Using this system, speakers of the session can write on a white board or an OHP sheet or use PowerPoint. On the other hand participants can ask questions or make comments from any connected site in the world. All these are performed similarly to the traditional meetings.

To connect with our system, you can use your own videoconferencing system only if it satisfies the International Telecommunication Union-Technical Committee Standards (ITU-T Standard).

Copyright Transfer Agreement

A copyright transfer agreement is required before a paper is published in this journal. By submitting a paper to this journal, authors are regarded to certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings or similar publication.

For more information, please visit http://www.jams.or.jp.

NORM INEQUALITIES RELATED TO THE MATRIX GEOMETRIC MEAN OF NEGATIVE POWER

Mohsen Kian* and Yuki Seo**

Received February 12, 2018; revised March 27, 2018

Abstract. In this paper, we show norm inequalities related to the matrix geometric mean of negative power for positive definite matrices: For positive definite matrices A and B,

$$\left|\!\left|\!\left| e^{(1-\beta)\log A+\beta\log B} \right|\!\right|\!\right| \le |\!|\!| A \natural_\beta B |\!|\!| \le \left|\!\left|\!\left| A^{1-\beta}B^{\beta} \right|\!\right|\!\right|$$

for every unitarily invariant norm and $-1 \le \beta \le -\frac{1}{2}$, where the β-quasi geometric mean $A \natural_\beta B$ is defined by $A \natural_\beta B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\beta}A^{\frac{1}{2}}$. For our purposes, we show the Ando-Hiai log-majorization of negative power.

1 Introduction. Let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices and denote the matrix absolute value of any $A \in \mathbb{M}_n$ by $|A| = (A^*A)^{\frac{1}{2}}$. For $A \in \mathbb{M}_n$, we write $A \ge 0$ if A is positive semidefinite and $A > 0$ if A is positive definite, that is, A is positive and invertible. For two Hermitian matrices A and B, we write $A \ge B$ if $A - B \ge 0$, and it is called the Löwner ordering. A norm $|\!|\!|\cdot|\!|\!|$ on $\mathbb{M}_n$ is said to be unitarily invariant if $|\!|\!|UXV|\!|\!| = |\!|\!|X|\!|\!|$ for all $X \in \mathbb{M}_n$ and unitary U, V.

Let A and B be two positive definite matrices. The arithmetic-geometric mean inequality says that

$$A \sharp_\alpha B \le (1-\alpha)A + \alpha B \qquad \text{for all } \alpha \in [0,1], \tag{1.1}$$

where the α-geometric mean $A \sharp_\alpha B$ is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}} \qquad \text{for all } \alpha \in [0,1],$$

also see [11]. As another matrix geometric mean, we recall that the chaotic geometric mean $A \diamondsuit_\alpha B$ is defined by

$$A \diamondsuit_\alpha B = e^{(1-\alpha)\log A+\alpha\log B} \qquad \text{for all } \alpha \in \mathbb{R},$$

also see [5, Section 3.5]. If A and B commute, then $A \diamondsuit_\alpha B = A \sharp_\alpha B = A^{1-\alpha}B^{\alpha}$ for $\alpha \in [0,1]$. In [4], Bhatia and Grover showed precise norm estimations of the arithmetic-geometric mean inequality (1.1) as follows: For each $\alpha \in [0,1]$ and any unitarily invariant norm $|\!|\!|\cdot|\!|\!|$

$$\begin{aligned} |\!|\!|A \sharp_\alpha B|\!|\!| &\le |\!|\!|A \diamondsuit_\alpha B|\!|\!| \le \left|\!\left|\!\left|B^{\frac{\alpha}{2}}A^{1-\alpha}B^{\frac{\alpha}{2}}\right|\!\right|\!\right| \\ &\le \left|\!\left|\!\left|\frac{1}{2}\left(A^{1-\alpha}B^{\alpha} + B^{\alpha}A^{1-\alpha}\right)\right|\!\right|\!\right| \le \left|\!\left|\!\left|A^{1-\alpha}B^{\alpha}\right|\!\right|\!\right| \le |\!|\!|(1-\alpha)A + \alpha B|\!|\!| \end{aligned}$$

2000 *Mathematics Subject Classification.* Primary 15A45; Secondary 47A64.
Key words and phrases. Ando-Hiai inequality, matrix geometric mean, unitarily invariant norm, positive definite matrix.

and

$$\begin{aligned}\||A \sharp_\alpha B\|| &\le \||A \diamondsuit_\alpha B\|| \\ &\le \left\|\left|\left(B^{\frac{\alpha p}{2}}A^{(1-\alpha)p}B^{\frac{\alpha p}{2}}\right)^{\frac{1}{p}}\right|\right\| \le \left\|\left|((1-\alpha)A^p+\alpha B^p)^{\frac{1}{p}}\right|\right\| \quad \text{for all } p>0.\end{aligned}$$

For convenience in symbolic expression, we define $A \natural_\beta B$ for $\beta \in [-1,0)$ and positive definite matrices A, B as follows:

$$A \natural_\beta B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\beta A^{\frac{1}{2}} \quad \text{for all } \beta \in [-1,0), \tag{1.2}$$

whose formula is the same as $\sharp_\alpha$. Though $A \natural_\beta B$ for $\beta \in [-1,0)$ are not matrix means in the sense of Kubo-Ando theory [11], it is known in [7] that $A \natural_\beta B$ have matrix mean like properties for any positive definite matrices A and B. Thus we call (1.2) the β-quasi geometric mean for $\beta \in [-1,0)$. For more detail, see [7].

On the other hand, the following reverse arithmetic-geometric mean inequality holds:

$$(1-\beta)A + \beta B \le A \natural_\beta B \qquad \text{for all } \beta \in [-1,0),$$

also see [8]. Though we have no relation among $A \natural_\beta B$, $A \diamondsuit_\beta B$ and $A^{1-\beta}B^\beta$ for $\beta \in [-1,0)$ under the Löwner ordering, it follows from a proof similar to Bhatia-Grover's one in [4] that for each $\beta \in \mathbb{R}$ and any unitarily invariant norm $\||\cdot\||$

$$\||A \diamondsuit_\beta B\|| \le \left\|\left|B^{\frac{\beta}{2}}A^{1-\beta}B^{\frac{\beta}{2}}\right|\right\| \le \left\|\left|\frac{1}{2}\left(A^{1-\beta}B^\beta + B^\beta A^{1-\beta}\right)\right|\right\| \le \left\|\left|A^{1-\beta}B^\beta\right|\right\|. \tag{1.3}$$

Also, by the Lie-Trotter formula $\lim_{t\to 0}\left(e^{\frac{t}{2}B}e^{tA}e^{\frac{t}{2}B}\right)^{\frac{1}{t}} = e^{A+B}$ and the Araki-Cordes inequality $\||B^tA^tB^t\|| \le \||(BAB)^t\||$ for all $t \in [0,1]$, also see [3, Exercise IX.1.5,Theorem IX.2.10], it follows that for each $\beta \in [-1,0)$

$$\||A \diamondsuit_\beta B\|| \le \left\|\left|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right|\right\| \tag{1.4}$$

holds for all $q > 0$ and $\left\|\left|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right|\right\|$ decreases to $\||A \diamondsuit_\beta B\||$ as $q \downarrow 0$. It is natural to ask what is the estimate of the β-quasi geometric mean in the norm inequalities (1.3) and (1.4) for $\beta \in [-1,0)$.

In this paper, we show norm inequalities related to the β-quasi geometric mean of negative power, the chaotic geometric mean $A \diamondsuit_\beta B$ and $A^{1-\beta}B^\beta$ for positive definite matrices A, B. Moreover, we show precise norm estimations of the reverse arithmetic-geometric mean inequality under the assumption $A \ge B$. For our purposes, we need the Ando-Hiai log-majorization of negative power.

2 Preliminaries. In this section, we have some preliminary results on the log majorization of matrices. For Hermitian matrices H, K the weak majorization $H \prec_w K$ means that

$$\sum_{i=1}^k \lambda_i(H) \le \sum_{i=1}^k \lambda_i(K) \qquad \text{for } k = 1,2,\ldots,n,$$

where $\lambda_1(H) \ge \cdots \ge \lambda_n(H)$ and $\lambda_1(K) \ge \cdots \ge \lambda_n(K)$ are the eigenvalues of H and K respectively. Further, the majorization $H \prec K$ means that $H \prec_w K$ and the equality holds for $k = n$ in the above, i.e., $\mathrm{Tr}H = \mathrm{Tr}K$. For $A, B \geq 0$ let us write $A \prec_{w(\log)} B$ and refer to the weak log majorization if

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B) \qquad \text{for } k = 1, 2, \ldots, n.$$

Further the log majorization $A \prec_{(\log)} B$ means that $A \prec_{w(\log)} B$ and the equality holds for $k = n$ in the above, i.e.,

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B) \quad \text{i.e.,} \quad \det A = \det B.$$

Note that when $A, B > 0$ the log majorization $A \prec_{(\log)} B$ is equivalent to $\log A \prec \log B$. It is known that for positive semidefinite $A, B \geq 0$,

$$A \prec_{w(\log)} B \Longrightarrow A \prec_w B \Longrightarrow \|A\| \leq \|B\|$$

for any unitarily invariant norm. See [1, 12] for theory of majorization for matrices.

For each matrix X and $k = 1, 2, \ldots, n$, let $C_k(X)$ denote the k-fold antisymmetric tensor power of X. See [12] for details. Then (1)-(3) below are basic facts, (4) is easily seen from (2) and (3), and (5) follows from the Binet-Cauchy theorem.

Lemma 2.1. (1) $C_k(X^*) = C_k(X)^*$.

(2) *$C_k(XY) = C_k(X)C_k(Y)$ for every pair of matrices X, Y.*

(3) *$C_k(X^{-1}) = C_k(X)^{-1}$ for nonsingular X.*

(4) *$C_k(A^p) = C_k(A)^p$ for every positive definite $A > 0$ and all $p \in \mathbb{R} \setminus \{0\}$.*

(5) *For every $A > 0$, $\prod_{i=1}^{k} \lambda_i(A) = \lambda_1(C_k(A))$ for $k = 1, 2, \ldots, n$ and consequently, for $A, B > 0$, $\lambda_1(C_k(A)) \leq \lambda_1(C_k(B))$ for all $k = 1, \ldots, n$ if and only if $A \prec_{w(\log)} B$.*

3 Ando-Hiai Log-Majorization of negative power.

For $0 \leq \alpha \leq 1$, the matrix α-geometric mean is the matrix mean corresponding to the matrix monotone function t^α. Note that $A \sharp_\alpha B = B \sharp_{1-\alpha} A$ and if $AB = BA$ then $A \sharp_\alpha B = A^{1-\alpha}B^\alpha$, and $(A, B) \mapsto A \sharp_\alpha B$ is jointly monotone, also see [5, Lemma 3.2].

On the other hand, the β-quasi geometric mean for $\beta \in [-1, 0)$ has the following properties in [7]; for any positive definite matrices A, B and C

(i) **consistency with scalars:** If A and B commute, then $A \natural_\beta B = A^{1-\beta}B^\beta$.

(ii) **homogeneity:** $(\alpha A) \natural_\beta (\alpha B) = \alpha(A \natural_\beta B)$ for all $\alpha > 0$.

(iii) **right reverse monotonicity:** $B \leq C$ implies $A \natural_\beta B \geq A \natural_\beta C$.

We recall the log-majorization theorem due to Ando-Hiai [2]: For each $\alpha \in [0, 1]$

$$A^r \sharp_\alpha B^r \prec_{(\log)} (A \sharp_\alpha B)^r \qquad \text{for } r \geq 1,$$

or equivalently

$$(A^p \sharp_\alpha B^p)^{\frac{1}{p}} \prec_{(\log)} (A^q \sharp_\alpha B^q)^{\frac{1}{q}} \qquad \text{for } 0 < q < p.$$

To show the main theorem related to the β-quasi geometric mean for $\beta \in [-1, 0)$, we need the following Ando-Hiai log-majorization of negative power $\beta \in [-1, 0)$:

Theorem 3.1. *For every positive definite matrices $A, B > 0$ and $\beta \in [-1, 0)$,*

$$A^r \natural_\beta B^r \prec_{(\log)} (A \natural_\beta B)^r \qquad \text{for all } 0 < r \le 1 \tag{3.1}$$

or equivalently

$$(A \natural_\beta B)^r \prec_{(\log)} (A^r \natural_\beta B^r) \qquad \text{for all } r \ge 1, \tag{3.2}$$

$$(A^q \natural_\beta B^q)^{\frac{1}{q}} \prec_{(\log)} (A^p \natural_\beta B^p)^{\frac{1}{p}} \qquad \text{for all } 0 < q \le p. \tag{3.3}$$

Proof. The equivalence of (3.1)-(3.3) is immediate. It is easy to see by Lemma 2.1 that for $k = 1, \dots, n$

$$C_k(A^r \natural_\beta B^r) = C_k(A)^r \natural_\beta C_k(B)^r$$

and

$$C_k((A \natural_\beta B)^r) = (C_k(A) \natural_\beta C_k(B))^r.$$

Also,

$$\det(A^r \natural_\beta B^r) = (\det A)^{r(1-\beta)}(\det B)^{r\beta} = \det(A \natural_\beta B)^r.$$

Hence, in order to prove (3.1), it suffices to show that

$$\lambda_1(A^r \natural_\beta B^r) \le \lambda_1(A \natural_\beta B)^r \qquad \text{for all } 0 < r \le 1. \tag{3.4}$$

For this purpose we may prove that $A \natural_\beta B \le I$ implies $A^r \natural_\beta B^r \le I$, because both sides of (3.4) have the same order of homogeneity for A, B, so that we can multiply A, B by a positive constant.

First let us assume $\frac{1}{2} \le r \le 1$ and write $r = 1 - \varepsilon$ with $0 \le \varepsilon \le \frac{1}{2}$. Let $C = A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}$. Then $B^{-1} = A^{-\frac{1}{2}} C A^{-\frac{1}{2}}$ and $A \natural_\beta B = A^{\frac{1}{2}} C^{-\beta} A^{\frac{1}{2}}$. If $A \natural_\beta B \le I$, then $C^{-\beta} \le A^{-1}$ so that $A \le C^\beta$ and $A^\varepsilon \le C^{\beta\varepsilon}$ for $0 \le \varepsilon \le \frac{1}{2}$ by Löwner-Heinz inequality. Since $-\beta \in (0, 1]$ and $1 - \varepsilon \in [\frac{1}{2}, 1]$, we now get

$$\begin{aligned}
A^r \natural_\beta B^r &= A^{\frac{1-\varepsilon}{2}} (A^{\frac{\varepsilon-1}{2}} B^{1-\varepsilon} A^{\frac{\varepsilon-1}{2}})^\beta A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1-\varepsilon}{2}} (A^{\frac{1-\varepsilon}{2}} (B^{-1})^{1-\varepsilon} A^{\frac{1-\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1-\varepsilon}{2}} (A^{\frac{1-\varepsilon}{2}} (A^{-\frac{1}{2}} C A^{-\frac{1}{2}})^{1-\varepsilon} A^{\frac{1-\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1-\varepsilon}{2}} (A^{-\frac{\varepsilon}{2}} [A \sharp_{1-\varepsilon} C] A^{-\frac{\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1}{2}-\varepsilon} [A^\varepsilon \sharp_{-\beta} (A \sharp_{1-\varepsilon} C)] A^{\frac{1}{2}-\varepsilon} \\
&\le A^{\frac{1}{2}-\varepsilon} [C^{\beta\varepsilon} \sharp_{-\beta} (C^\beta \sharp_{1-\varepsilon} C)] A^{\frac{1}{2}-\varepsilon},
\end{aligned}$$

using the joint monotonicity of matrix geometric means. Since a direct computation yields

$$C^{\beta\varepsilon} \sharp_{-\beta} (C^\beta \sharp_{1-\varepsilon} C) = C^{\beta(2\varepsilon-1)}$$

and by Löwner-Heinz inequality and $0 \le 1 - 2\varepsilon \le 1$, $C^{-\alpha} \le A^{-1}$ implies $C^{-\beta(1-2\varepsilon)} \le A^{-(1-2\varepsilon)}$ and thus we get

$$A^r \natural_\beta B^r \le A^{\frac{1}{2}-\varepsilon} C^{\beta(2\varepsilon-1)} A^{\frac{1}{2}-\varepsilon} \le A^{\frac{1}{2}-\varepsilon} A^{-1+2\varepsilon} A^{\frac{1}{2}-\varepsilon} = I.$$

Therefore (3.4) is proved in the case of $\frac{1}{2} \le r \le 1$.

When $0 < r < \frac{1}{2}$, writing $r = 2^{-k}(1-\varepsilon)$ with $k \in \mathbb{N}$ and $0 \le \varepsilon \le \frac{1}{2}$, and repeating the argument above we have

$$\begin{aligned}\lambda_1(A^r \natural_\beta B^r) &\le \lambda_1(A^{2^{-(k-1)}(1-\varepsilon)} \natural_\beta B^{2^{-(k-1)}(1-\varepsilon)})^{\frac{1}{2}} \\ &\vdots \\ &\le \lambda_1(A^{1-\varepsilon} \natural_\beta B^{1-\varepsilon})^{2^{-k}} \\ &\le \lambda_1(A \natural_\beta B)^r\end{aligned}$$

and so the proof is complete. □

By Theorem 3.1, we have the following results:

Theorem 3.2. *Let A and B be positive definite matrices and $\|\!|\cdot|\!\|$ any unitarily invariant norm, and $\beta \in [-1, 0)$. If f is a continuous non-decreasing function on $[0, \infty)$ such that $f(0) \ge 0$ and $f(e^t)$ is convex, then*

$$\|\!|f(A^r \natural_\beta B^r)|\!\| \le \|\!|f((A \natural_\beta B)^r)|\!\| \qquad \text{for all } 0 < r \le 1.$$

In particular,

$$\|\!|A^r \natural_\beta B^r|\!\| \le \|\!|(A \natural_\beta B)^r|\!\| \qquad \text{for all } 0 < r \le 1$$

or equivalently

$$\|\!|(A \natural_\beta B)^r|\!\| \le \|\!|(A^r \natural_\beta B^r)|\!\| \qquad \text{for all } r \ge 1,$$

$$\left\|\!\left|(A^q \natural_\beta B^q)^{\frac{1}{q}}\right|\!\right\| \le \left\|\!\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\!\right\| \qquad \text{for all } 0 < q \le p.$$

Proof. By [9, Proposition 4.4.13], if $A \prec_{w(\log)} B$ for positive definite matrices A and B and f is a continuous non-decreasing function on $[0, \infty)$ such that $f(0) \ge 0$ and $f(e^t)$ is convex, then $f(A) \prec_w f(B)$ and so $\|\!|f(A)|\!\| \le \|\!|f(B)|\!\|$. Hence Theorem 3.2 follows from Theorem 3.1. □

Corollary 3.3. *For every positive definite matrices $A, B > 0$ and $\beta \in [-1, 0)$,*

$$A \natural_\beta B \le I \quad \text{implies} \quad A^r \natural_\beta B^r \le I \quad \text{for all } 0 < r \le 1.$$

4 Norm inequalities for quasi geometric mean.

In this section, we show the main norm inequalities related to the quasi geometric mean for positive definite matrices. By [5, Lemma 5.5], we have the following quasi-geometric mean version of the Lie-Trotter formula: If A and B are positive definite matrices, then for each $\beta \in [-1, 0)$

$$A \diamondsuit_\beta B = \lim_{p \to 0} (A^p \natural_\beta B^p)^{\frac{1}{p}} \tag{4.1}$$

and so for each $\beta \in [-1, 0)$ $\left\|\!\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\!\right\|$ decreases to $\|\!|A \diamondsuit_\beta B|\!\|$ as $p \downarrow 0$. Hence we have the following norm inequality for the quasi geometric mean of negative power:

Theorem 4.1. *Let A and B be positive definite matrices. Then for every unitarily invariant norm*

$$\|\!|A \diamondsuit_\beta B|\!\| \le \|\!|A \natural_\beta B|\!\| \qquad \text{for all } \beta \in [-1, 0).$$

Proof. By Theorem 3.2, it follows that

$$\left|\left|\left|(A^q \natural_\beta B^q)^{\frac{1}{q}}\right|\right|\right| \le \left|\left|\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\right|\right| \qquad \text{for all } 0<q<p$$

and as $q \to 0$ and $p=1$ we have the desired inequality by (4.1). $\square$

Theorem 4.2. *Let A and B be positive definite matrices. Then for every unitarily invariant norm*

$$|||A \natural_\beta B||| \le |||A^{1-\beta}B^\beta||| \qquad \textit{for all } \beta \in [-1,-\tfrac{1}{2}]. \tag{4.2}$$

Proof. For the matrix norm $\|\cdot\|$, by the Araki-Cordes inequality $\|B^tA^tB^t\| \le \|(BAB)^t\|$ for all $t\in[0,1]$, we have for $-1 \le \beta \le -\frac{1}{2}$

$$\begin{aligned}
\|A \natural_\beta B\| &= \left\|A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\beta A^{\frac{1}{2}}\right\| \\
&= \left\|A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-\beta} A^{\frac{1}{2}}\right\| \\
&\le \left\|A^{-\frac{1}{2\beta}}A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}A^{-\frac{1}{2\beta}}\right\|^{-\beta} \qquad \text{by } \tfrac{1}{2} \le -\beta \le 1 \\
&= \left\|A^{\frac{\beta-1}{2\beta}}B^{-1}A^{\frac{\beta-1}{2\beta}}\right\|^{-\beta} \\
&\le \left\|A^{1-\beta}B^{2\beta}A^{1-\beta}\right\|^{\frac{1}{2}} \qquad \text{for } \tfrac{1}{2} \le -\tfrac{1}{2\beta} \le 1 \\
&= \left\|(A^{1-\beta}B^{2\beta}A^{1-\beta})^{\frac{1}{2}}\right\|
\end{aligned}$$

and this implies

$$\lambda_1(A \natural_\beta B) \le \lambda_1((A^{1-\beta}B^{2\beta}A^{1-\beta})^{\frac{1}{2}}) = \lambda_1(|B^\beta A^{1-\beta}|).$$

Replacing A and B by (5) of Lemma 2.1, we obtain

$$\prod_{i=1}^k \lambda_i(A \natural_\beta B) \le \prod_{i=1}^k \lambda_i(|B^\beta A^{1-\beta}|) \qquad \text{for } k=1,\dots,n.$$

Hence we have the weak log majorization $A \natural_\beta B \prec_{w(\log)} \left|B^\beta A^{1-\beta}\right|$ and this implies

$$|||A \natural_\beta B||| \le |||\,|B^\beta A^{1-\beta}|\,||| = |||B^\beta A^{1-\beta}||| = |||A^{1-\beta}B^\beta|||$$

for every unitarily invariant norm and so we have the desired inequality (4.2). $\square$

Remark 4.3. In Theorem 4.2, the inequality $|||A \natural_\beta B||| \le |||A^{1-\beta}B^\beta|||$ does not always hold for $-1/2<\beta<0$. In fact, if we put $\beta=-\frac{1}{3}$, $A=\begin{pmatrix}2&1\\1&2\end{pmatrix}$ and $B=\begin{pmatrix}2&1\\1&1\end{pmatrix}$, then we have the matrix norm $\left\|A \natural_{-\frac{1}{3}} B\right\| = 3.385$ and $\left\|A^{\frac{4}{3}}B^{-\frac{1}{3}}\right\| = 3.375$, and so $\|A \natural_\beta B\| > \left\|A^{1-\beta}B^\beta\right\|$.

Theorem 4.4. *Let A and B be positive definite matrices. Then for every unitarily invariant norm*

$$|||A \diamondsuit_\beta B||| \le \left|\left|\left|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right|\right|\right| \le |||A \natural_\beta B||| \quad \textit{for } 0<q\le \tfrac{1}{2} \textit{ and } \beta\in[-1,0) \tag{4.3}$$

and

$$|||A \natural_\beta B||| \le |||A^{1-\beta}B^\beta||| \le \left|\left|\left|\left(B^{\frac{\beta p}{2}}A^{(1-\beta)p}B^{\frac{\beta p}{2}}\right)^{\frac{1}{p}}\right|\right|\right| \quad \textit{for } p\ge 2 \textit{ and } \beta\in[-1,-\tfrac{1}{2}]. \tag{4.4}$$

Proof. Since the first inequality in (4.3) follows from the Lie-Trotter formula (4.1), we show the second inequality in (4.3). By Theorem 3.2, we have $\left\|\left\|(A^r \natural_\beta B^r)^{\frac{1}{r}}\right\|\right\| \leq \|\|A \natural_\beta B\|\|$ for all $0 < r \leq 1$. For the matrix norm $\|\|\cdot\|\|$, by the Araki-Cordes inequality $\|\|(BAB)^t\|\| \leq \|\|B^t A^t B^t\|\|$ for all $t \geq 1$, we have for $0 < r \leq 1$

$$\begin{aligned}
\|\|A \natural_\beta B\|\| &\geq \|\|A^r \natural_\beta B^r\|\|^{\frac{1}{r}} \\
&= \|\|B^r \natural_{1-\beta} A^r\|\|^{\frac{1}{r}} \\
&= \left\|\left\|B^{\frac{r}{2}}(B^{-\frac{r}{2}}A^r B^{-\frac{r}{2}})^{1-\beta}B^{\frac{r}{2}}\right\|\right\|^{\frac{1}{r}} \\
&\geq \left\|\left\|B^{\frac{\beta r}{2(1-\beta)}}A^r B^{\frac{\beta r}{2(1-\beta)}}\right\|\right\|^{\frac{1-\beta}{r}} \qquad \text{by } 0 < \tfrac{1}{1-\beta} < 1 \\
&\geq \left\|\left\|B^{\frac{\beta r}{4}}A^{\frac{(1-\beta)r}{2}}B^{\frac{\beta r}{4}}\right\|\right\|^{\frac{2}{r}} \qquad \text{by } \tfrac{1}{2} < \tfrac{1-\beta}{2} \leq 1.
\end{aligned}$$

If we put $q = \frac{r}{2}$, then we have $\left\|\left\|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right\|\right\| \leq \|\|A \natural_\beta B\|\|$ for $0 < q \leq \frac{1}{2}$ and this implies

$$\lambda_1(\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}) \leq \lambda_1(A \natural_\beta B).$$

Replacing A and B by (5) of Lemma 2.1, we obtain

$$\prod_{i=1}^{k} \lambda_1(\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}) \leq \prod_{i=1}^{k} \lambda_1(A \natural_\beta B) \qquad \text{for } k = 1, \ldots, n,$$

which gives the second inequality in (4.3).

Next, for $s \geq 1$, it follows from Theorem 3.2 that $\|\|A \natural_\beta B\|\| \leq \left\|\left\|(A^s \natural_\beta B^s)^{\frac{1}{s}}\right\|\right\|$. For the matrix norm $\|\|\cdot\|\|$, we have

$$\begin{aligned}
\|\|A \natural_\beta B\|\| &\leq \|\|A^s \natural_\beta B^s\|\|^{\frac{1}{s}} \\
&= \left\|\left\|A^{\frac{s}{2}}(A^{\frac{s}{2}}B^{-s}A^{\frac{s}{2}})^{-\beta}A^{\frac{s}{2}}\right\|\right\|^{\frac{1}{s}} \\
&\leq \left\|\left\|A^{\frac{-(1-\beta)s}{2\beta}}B^{-s}A^{\frac{-(1-\beta)s}{2\beta}}\right\|\right\|^{-\frac{\beta}{s}} \qquad \text{by } \tfrac{1}{2} \leq -\beta \leq 1 \\
&\leq \left\|\left\|A^{(1-\beta)s}B^{2\beta s}A^{(1-\beta)s}\right\|\right\|^{\frac{1}{2s}} \qquad \text{by } \tfrac{1}{2} \leq -\tfrac{1}{2\beta} \leq 1.
\end{aligned}$$

If we put $p = 2s$, then we have

$$\begin{aligned}
\|\|A \natural_\beta B\|\| &\leq \left\|\left\|A^{\frac{(1-\beta)p}{2}}B^{\beta p}A^{\frac{(1-\beta)p}{2}}\right\|\right\|^{\frac{1}{p}} \\
&= \operatorname{spr}(A^{\frac{(1-\beta)p}{2}}B^{\beta p}A^{\frac{(1-\beta)p}{2}})^{\frac{1}{p}} \\
&= \operatorname{spr}(B^{\frac{\beta p}{2}}A^{(1-\beta)p}B^{\frac{\beta p}{2}})^{\frac{1}{p}} \\
&= \left\|\left\|B^{\frac{\beta p}{2}}A^{(1-\beta)p}B^{\frac{\beta p}{2}}\right\|\right\|^{\frac{1}{p}} \\
&= \left\|\left\|\left(B^{\frac{\beta p}{2}}A^{(1-\beta)p}B^{\frac{\beta p}{2}}\right)^{\frac{1}{p}}\right\|\right\|
\end{aligned}$$

for $p \geq 2$, where $\operatorname{spr}(X)$ is the spectral radius of X. By the argument similar to above, we have the inequality (4.4). $\square$

Let A and B be positive definite matrices in $\mathbb{M}_n$ and $\beta \in [-1, 0)$. Since there is the case that $(1-\beta)A+\beta B$ is not positive semidefinite, we have no relation between $|||(1-\beta)A+\beta B|||$ and $|||A \natural_\beta B|||$ though $(1-\beta)A+\beta B \le A \natural_\beta B$. Suppose that $A \ge B$. Then $(1-\beta)A^p + \beta B^p$ is positive definite for all $p \in (0, 1]$. In particular $0 < (1-\beta)A + \beta B \le A \natural_\beta B$ and so $|||(1-\beta)A+\beta B||| \le |||A \natural_\beta B|||$ for every unitarily invariant norm. Thus under the assumption $A \ge B$, we consider the refinement of this norm inequality. For this, we need the following result due to J. I. Fujii [6]: A real valued continuous function f on an interval J is matrix concave if and only if

$$f((1-\beta)H+\beta K) \le (1-\beta)f(H)+\beta f(K) \tag{4.5}$$

for all Hermitian matrices H and K with $\sigma(H), \sigma(K)$ and $\sigma((1-\beta)H+\beta K) \subset J$ and $\beta \in [-1, 0)$.

Let $0 < q < p \le 1$. Then the function $f(t) = t^{\frac{q}{p}}$ on $[0, \infty)$ is matrix concave and by (4.5)

$$((1-\beta)A^p+\beta B^p)^{\frac{q}{p}} \le (1-\beta)A^q+\beta B^q. \tag{4.6}$$

Note that $(1-\beta)A^p+\beta B^p > 0$ for all $p \in (0, 1]$ since $A \ge B$. This implies that

$$\lambda_i\left((1-\beta)A^p+\beta B^p\right)^{\frac{q}{p}} \le \lambda_i\left((1-\beta)A^q+\beta B^q\right) \quad \text{for all } i = 1, \dots, n.$$

Taking q-th roots of both sides, we obtain

$$\lambda_i\left((1-\beta)A^p+\beta B^p\right)^{\frac{1}{p}} \le \lambda_i\left((1-\beta)A^q+\beta B^q\right)^{\frac{1}{q}} \quad \text{for all } i = 1, \dots, n$$

and so $\left|\left|\left|((1-\beta)A^p+\beta B^p)^{\frac{1}{p}}\right|\right|\right|$ is a decreasing function of p.

On the other hand, taking the logarithm of both sides in (4.6) and by (4.5), we obtain

$$\begin{aligned}\log\left((1-\beta)A^p+\beta B^p\right)^{\frac{1}{p}} &\le \frac{1}{q}\log\left((1-\beta)A^q+\beta B^q\right)\\ &\le (1-\beta)\log A+\beta\log B\end{aligned}$$

and this implies

$$\lambda_i\left(\log((1-\beta)A^p+\beta B^p)^{\frac{1}{p}}\right) \le \lambda_i\left((1-\beta)\log A+\beta\log B\right) \quad \text{for all } i = 1, \dots, n.$$

Taking the exponent of both sides, we obtain

$$\lambda_i\left((1-\beta)A^p+\beta B^p\right)^{\frac{1}{p}} \le \lambda_i\left(e^{(1-\beta)\log A+\beta\log B}\right) \quad \text{for all } i = 1, \dots, n$$

and so

$$\left|\left|\left|((1-\beta)A^p+\beta B^p)^{\frac{1}{p}}\right|\right|\right| \le \left|\left|\left|e^{(1-\beta)\log A+\beta\log B}\right|\right|\right|$$

for all $p \in (0, 1]$. Summing up, we obtain the following result:

Theorem 4.5. *Let A and B be positive definite matrices in $\mathbb{M}_n$ such that $A \ge B$ and $\beta \in [-1, 0)$. Then for every unitarily invariant norm*

$$|||(1-\beta)A+\beta B||| \le \left|\left|\left|((1-\beta)A^p+\beta B^p)^{\frac{1}{p}}\right|\right|\right| \le |||A \diamondsuit_\beta B||| \le |||A \natural_\beta B|||$$

for all $p \in (0, 1]$.

Finally, as an application, we show a refinement of the generalized Golden-Thompson inequality in terms of the quasi geometric means. Let H and K be Hermitian matrices. The Golden-Thompson trace inequality is

$$\mathrm{Tr}[e^{H+K}] \leq \mathrm{Tr}[e^H e^K].$$

Hiai-Petz [10] proved the complemented Golden-Thompson inequality:

$$\left|\!\left|\!\left| e^H \sharp_\alpha e^K \right|\!\right|\!\right| \leq \left|\!\left|\!\left| e^{(1-\alpha)H+\alpha K} \right|\!\right|\!\right| \quad \text{for all } \alpha \in [0,1]$$

for every unitarily invariant norm. By Theorem 4.1 and Theorem 4.2, we have a refinement of the Golden-Thompson inequality in terms of the quasi geometric means:

$$\left|\!\left|\!\left| e^{(1-\beta)H+\beta K} \right|\!\right|\!\right| \leq \left|\!\left|\!\left| e^H \natural_\beta e^K \right|\!\right|\!\right| \leq \left|\!\left|\!\left| e^{(1-\beta)H} e^{\beta K} \right|\!\right|\!\right|$$

for all $\beta \in [-1, -\frac{1}{2}]$ and so

$$\mathrm{Tr}[e^{H+K}] \leq \mathrm{Tr}[e^{\frac{1}{1-\beta}H} \natural_\beta e^{\frac{1}{\beta}K}] \leq \mathrm{Tr}[e^H e^K].$$

In particular, if we put $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, then we have

$$\mathrm{Tr}[e^{2H} \sharp_{\frac{1}{2}} e^{2K}] \leq \mathrm{Tr}[e^{H+K}] \leq \mathrm{Tr}[e^{\frac{2}{3}H} \natural_{-\frac{1}{2}} e^{-2K}] \leq \mathrm{Tr}[e^H e^K].$$

Acknowledgement. The second author is partially supported by JSPS KAKENHI Grant Number JP 16K05253.

References

[1] T. Ando, *Majorization, doubly stochastic matrices, and comparison of eigenvalues*, Linear Algebra Appl., **118** (1989), 163–248.

[2] T. Ando and F. Hiai, *Log-majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197,198** (1994), 113–131.

[3] R. Bhatia, *Matrix Anaysis*, Springer, New York, 1997.

[4] R. Bhatia and P. Grover, *Norm inequalities related to the matrix geometric mean*, Linear Algebra Appl., **437** (2012), 726–733.

[5] M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 4, Element, Zagreb, 2012.

[6] J. I. Fujii, *An external version of the Jensen operator inequality*, Sci. Math. Japon., **73**, No.2&3 (2011), 125–128.

[7] J.I. Fujii and Y. Seo, *Tsallis relative operator entropy with negative paramaeters*, Adv. Oper. Theory, **1** (2016), No.2, 219–236.

[8] T. Furuta, *Invitation to linear operators*, Taylor&Francis, London, 2001.

[9] F. Hiai, *Matrix Analysis: Matrix Monotone Functions, Matrix Means, and Majorization*, Inter. Information Sci., **16**, No. 2 (2010), 139–248.

[10] F. Hiai and D. Petz, *The Golden-Thompson trace inequality is complemented*, Linear Algebra Appl., **181** (1993), 153–185.

[11] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246**(1980), 205–224.

[12] A.W. Marshall and I. Olkin, *Inequalities: Theory of majorization and its applications*, Mathematics in Science and Engineering, Vol. **143**, Academic Press, 1979.

Communicated by *Masatoshi Fujii*

* Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P. O. Box 1339, Bojnord 94531, Iran
E-mail address : kian@member.ams.org

** Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
E-mail address : yukis@cc.osaka-kyoiku.ac.jp

ON THE BANG-BANG PRINCIPLE FOR DIFFERENTIAL INCLUSIONS IN A REFLEXIVE SEPARABLE BNANACH SPACE

Hiroshi Tateishi

Received February 10, 2018 ; revised March 9, 2018

Abstract. In this paper, we consider the relation existing between the solutions of the following differential inclusions: (I) $\dot{x} \in \Gamma(t,x), x(0) = 0$ and (II) $\dot{x} \in \operatorname{ext}\Gamma(t,x), x(0) = 0$ defined on a reflexive separable Banach space. In particular, we establish the sufficient conditions which guarantee the set of solutions of (II) is dense in the set of solutions of (I) with respect to the (weak) uniformly continuous topology.

Let $(\mathfrak{X}, \|\cdot\|)$ be a real reflexive separable Banach space and T be a positive real number. Let $\Gamma : [0,T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ be a correspondence (=multi-valued function). We consider a relation existing between the sets of solutions of the following differential inclusions:

(I) $\dot{x} \in \Gamma(t,x), x(0) = 0$, and
(II) $\dot{x} \in \operatorname{ext}\Gamma(t,x), x(0) = 0$,

where $\operatorname{ext} A$ stands for the weak-closure of the extreme points of A. By a solution of (I) and (II), we mean an absolutely continuous function $x : [0,T] \to \mathfrak{X}$ that satisfies $\dot{x} \in \Gamma(t, x(t))$ a.e. in $t \in [0,T]$ and $x(0) = 0$ in the case of (I) and $\dot{x} \in \operatorname{ext}\Gamma(t,x(t))$ a.e. in $t \in [0,T]$ and $x(0) = 0$ in the case of (II). We denote by $\mathcal{R}$ and $\mathcal{R}_*$ the set solutions of (I) and (II) respectively. Tateishi [5, 6] established the existence of solutions of the differential inclusions (I) under the following assumptions:

(i) Γ is nonempty and weakly compact-valued, i.e., $\Gamma(t,x)$ is nonempty and weakly compact for each $(t,x) \in [0,T] \times \mathfrak{X}$,
(ii) for each fixed $t \in [0,T]$, the correspondence $t \twoheadrightarrow \Gamma(t,x)$ is continuous with respect to the weak topology for $\mathfrak{X}$,
(iii) for each fixed $x \in \mathfrak{X}$, the correspondence $t \twoheadrightarrow \Gamma(t,x)$ is measurable, and
(iv) there exists $M > 0$ such that $\sup\{\|y\| \mid y \in \Gamma(t,x), t \in [0,T], x \in \mathfrak{X}\} \leq M$.

Furthermore, Tateishi [6, 7] examined the relations existing between the solutions set of (I) and (III): $\dot{x} \in \overline{\mathrm{co}}\,\Gamma(t,x), x(0) = 0$. The aim of this paper is to establish the relation between the sets of solutions (I) and (II). Bressan [1, 2] established the existence of solutions of both of the problems and obtained the closure result $\mathcal{R} = \mathcal{R}_*$ in the case that $\mathfrak{X}$ is a finite dimensional space. In this paper, we generalize his theorem to infinite dimensional spaces.

Date: March 9, 2018.
2010 *Mathematics Subject Classification.* Primary 34A60, 49J30.
Key words and phrases. bang-bang principle, differential inclusion, reflexive separable Banach space, weak topology.

1. Preliminaries

In this section, we offer some notations and lemmata used in this paper.

Let $(\mathfrak{X}, \|\cdot\|)$ be a reflexive separable Banach space with $\mathfrak{X}^*$ its dual. We denote by $\mathfrak{X}^w$ the space $\mathfrak{X}$ endowed with the weak topology. Let $\mathfrak{S} = \{x \in \mathfrak{X} \mid \|x\| \leq \max(MT, M)\}$ where M and T are constants which appear in the Introduction section. The set $\mathfrak{S}$ endowed with the relative topology of $\mathfrak{X}^w$ is denoted by $\mathfrak{S}^w$. The following proposition is from Larman and Rogers [4, Theorem 2].

Proposition 1. *Let E be a Hausdorff locally convex topological vector space. Let X be a metrizable compact subset of E. Let V be the linear subspace generated by the set $\overline{\mathrm{co}}\, X$. Then it is possible to introduce a norm on V so that the relative topologies of $\overline{\mathrm{co}}\, X$, as a subset of E, and as a subset of the normed space V, coincide.*

$\mathfrak{X}^w$ is a Hausdorff locally convex topological vector space and $\mathfrak{S}^w$ is a metrizable and compact subset of $\mathfrak{X}^w$. Furthermore, the linear subspace generated by $\mathfrak{S}^w$ is the whole space $\mathfrak{X}$. Hence, we can, by the above proposition, introduce a norm $\|\cdot\|^w$ on $\mathfrak{X}^w$ so that the topology on $\mathfrak{S}^w$ and the relative topology as a subset of the normed vector space $(\mathfrak{X}, \|\cdot\|^w)$ coincide.

We denote by h the Hausdorff distance on $\mathfrak{S}^w$ induced by $\|\cdot\|^w$, that is, $h(A, A') = \max\{\sup_{x\in A'} d(x, A), \sup_{x\in A} d(x, A')\}$ for any closed subsets A, A' of $\mathfrak{S}^w$, where $d(x, A) = \inf\{\|x - y\|^w \mid y \in A\}$. For $A \subset \mathfrak{S}$ and $\alpha > 0$, we set $B[A, \alpha] = \{x \in \mathfrak{S} \mid d(x, A) < \alpha\}$. We denote by μ, the Lebesgue measure defined on the interval $[0, T]$.

Lemma 1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space and $\Gamma : \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ be convex, compact-valued and continuous. Then the map $\mathrm{ext}\,\Gamma : \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ is lower hemi-continuous.*

Proof. Let $x_0, y_0 \in \mathfrak{X}$ with $y_0 \in \mathrm{ext}\,\Gamma(x_0)$ and $\{x_n\}$ be a sequence which converges to x_0. We must show that for some subsequence $x_{n'}$ of x_n and some $y_{n'} \in \mathrm{ext}\,\Gamma(x_{n'})$, we have $y_{n'} \to y_0$. Since Γ is continuous, there exists a sequence $y_n \in \Gamma(x_n)$ such that $y_n \to y_0$. Since Γ is compact and convex-valued, the Krein-Milman theorem implies that $\Gamma(x_n) = \overline{\mathrm{co}}\,\mathrm{ext}\,\Gamma(x_n)$. Hence, for each $n \in \mathbb{N}$, there exists $\alpha_n^i \geq 0, \sum_i \alpha_n^i = 1$ $(i \in \mathbb{N})$, where only finitely many α_n^i are not equal to zero, and $z_n^i \in \mathrm{ext}\,\Gamma(x_n)$ such that $\|y_n - \sum_i \alpha_n^i z_n^i\| \leq 1/n$. Let $y_n^i \in \Gamma(x_0)$ be such that $\|z_n^i - y_n^i\| \leq h(\Gamma(x_n), \Gamma(x_0))$, where h is the Hausdorff metric defined by $\|\cdot\|$. Since $\Gamma(x_0)$ is compact, there exist, for each fixed i, converging subsequences $y_{n'}^i$ to y_0^i and $\alpha_{n'}^i$ to α_0^i. Then $\sum_i \alpha_0^i y_0^i = y_0$ and since y_0 is an extreme point of $\Gamma(x_0)$, we have each y_0^i is equal to y_0 for all i with $\alpha_0^i > 0$. Let i^* be such that $\alpha_0^{i^*} > 0$. Then $\limsup_{n'} \|z_{n'}^{i^*} - y_0\| \leq \limsup_{n'} \|z_{n'}^{i^*} - y_{n'}^{i^*}\| + \limsup_{n'} \|y_{n'}^{i^*} - y_0\| = 0$. Hence $y_{n'}$ also converges to y_0 and this completes the proof. □

Lemma 2. *Let $F : [0, T] \times \mathfrak{X}^w \twoheadrightarrow \mathfrak{X}^w$ be lower hemi-continuous and $V \subset \mathfrak{X}^w$ be open. Then the correspondence $H : [0, T] \times \mathfrak{X}^w \twoheadrightarrow \mathfrak{X}^w$ defined by $H(t, x) = \overline{F(t, x) \cap V}$ is lower hemi-continuous, where $\overline{A}$ stands for the closure with respect to the weak topology of $\mathfrak{X}$.*

Proof. Let K be a weakly closed subset of $\mathfrak{X}$. Then we have the following implications:

$$H(t, x) \subset K \Leftrightarrow F(t, x) \cap V \subset K \Leftrightarrow F(t, x) \subset K \cup V^c.$$

Since $K \cup V^c$ is weakly closed and F is lower hemi-continuous, the set $\{(t,x) \mid H(t,x) \subset K\} = \{(t,x) \mid F(t,x) \subset K \cup V^c\}$ is closed in $[0,T] \times \mathfrak{X}^w$. It follows that H is lower hemi-continuous. □

Lemma 3. *Let $I_0 \subset [0,T]$ be a measurable set with $\mu(I_0) = \sigma$ and let M and ϵ be given positive real numbers. Then the solution $\psi : [0,T] \to \mathbb{R}$ of the differential equation*

$$\dot{\psi}(t) = \psi(t) + 2M\chi_{I_0}(t) + 4\epsilon, \psi(0) = 0 \tag{1}$$

is positive, monotonically increasing and satisfies the following inequality:

$$\psi(T) \leq 2M\sigma e^T + 4\epsilon(e^T - 1).$$

Proof. It is easy to verify that the solution ψ is positive and monotonically increasing. By calculating the solution ψ of (1) directly, we obtain

$$\psi(T) = 2M \cdot \int_0^T \chi_{I_0}(s)e^{(T-s)}ds + 4\epsilon \int_0^T e^{(T-s)}ds \leq 2M\sigma e^T + 4\epsilon(e^T - 1).$$

□

2. Main Theorem

Theorem 1. *Let $\Gamma : [0,T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ be a correspondence which satisfies the conditions:*

(i) *Γ is convex and weakly compact-valued, that is $\Gamma(t,x)$ is convex and weakly compact for each $(t,x) \in [0,T] \times \mathfrak{X}$,*
(ii) *Γ is continuous, where $\mathfrak{X}$ is endowed with the weak topology.*
(iii) *$h(\Gamma(t,x), \Gamma(t,y)) \leq \|x-y\|^w$, and*
(iv) *there exists $M > 0$ such that $\sup\{\|y\| \mid y \in \Gamma(t,x), t \in [0,T], x \in \mathfrak{X}\} \leq M$.*

Then $\mathcal{R} = \overline{\mathcal{R}_*}$, that is the set of solutions of (II) *is dense in the set of solutions of* (I) *with respect to the* (*weak*) *uniform convergence topology.*

Proof. **Step 1.** Let v be a solution of (I) and let ϵ be a positive real number. Then there exists, an open subset I_0 of $[0,T]$ with $\mu(I_0) < \epsilon$ such that, for all $t \in I_1 = [0,T] \cap I_0^c$, $\dot{v}(t)$ exists and lies in $\Gamma(t, v(t))$, and the restriction $\dot{v}\mid_{I_1}$ of $\dot{v}$ to I_1 is continuous. We may also assume that $[0,\tau_0] \subset I_0$ for some $\tau_0 > 0$.

Step 2. Let $\mathfrak{M}$ be the set of $\{u,\tau\}$ of an absolutely continuous mapping u and a positive constant $0 \leq \tau \leq T$ such that u is defined on the closed interval $[0,\tau]$ and satisfies $\dot{u} \in \operatorname{ext}\Gamma(t,u(t))$ a.e. in $t \in [0,\tau], u(0) = 0$,

$$\|u(\tau) - v(\tau)\|^w \leq \psi(\tau), \text{ and} \tag{2}$$

$$\|u(t) - v(t)\|^w \leq \psi(t) + 2M\epsilon \text{ for all } t \in [0,\tau], \tag{3}$$

where ψ is a solution of (1).

Step 3. Since $[0,\tau_0] \subset I_0$, the pair $\{u,\tau_0\}$ for every solution u of (II) satisfies the above properties, thus the set $\mathfrak{M}$ is nonempty. Let us define a partial ordering $\precsim_{\mathfrak{M}}$ on $\mathfrak{M}$ by $(u_1,\tau_1) \precsim_{\mathfrak{M}} (u_2,\tau_2) \Leftrightarrow \tau_1 \leq \tau_2$ and u_2 is an extension of u_1. Then Zorn's lemma implies that there exists a maximal element (u^*,τ^*) of $\mathfrak{M}$.

Step 4. Since $\epsilon > 0$ is arbitrary, the equations (2) and (3) imply that the solution u^* of (II) can be arbitrarily near to the solution v with respect to the (weak) uniform convergence topology on $[0, \tau^*]$. In the following two steps, we show that τ^* obtained in Step 3 equals T. In this step, we consider the case $\tau^* \in I_0$. Then, since I_0 is open, there exists a positive number δ such that $[\tau^*, \tau^* + \delta] \subset I_0$. Then, we have an absolutely continuous function $w : [\tau^*, \tau^* + \delta] \to X$ satisfying $\dot{w}(t) \in \operatorname{ext}\Gamma(t, w(t))$ for $t \in [\tau^*, \tau^* + \delta]$, $w(\tau^*) = u^*(\tau^*)$. Let us define $w^* : [0, \tau^* + \delta] \to \mathfrak{X}$ by

$$w^*(t) = \begin{cases} u^*(t) & \text{for } t \in [0, \tau^*] \\ w(t) & \text{for } t \in [\tau^*, \tau^* + \delta]. \end{cases}$$

Then, for $t \in [\tau^*, \tau^* + \delta]$, we obtain the estimation:

$$\begin{aligned} &\|w^*(t) - v(t)\|^w \\ &\le \|w^*(\tau^*) - v(\tau^*)\|^w + \int_{\tau^*}^t \|\dot{w}^*(s) - \dot{v}(s)\|^w ds \\ &\le \|w^*(\tau^*) - v(\tau^*)\|^w + \int_{\tau^*}^t 2M\chi_{I_0}(s) ds \\ &\le \psi(\tau^*) + \int_{\tau^*}^t \dot{\psi}(s) ds = \psi(t), \end{aligned}$$

where the first inequality is an immediate consequence of the fundamental theorem of calculus, and the second follows from assumption (iv) and the third from (2) and the definition of ψ. Thus w^* belongs to $\mathfrak{M}$, which contradicts the maximality of u^*.

Step 5. In this step, we consider the case $\tau^* \in I_1$. By assumption (iii) and (2), we have $y^* \in \Gamma(\tau^*, u^*(\tau^*))$ such that $\|y^* - \dot{v}(\tau^*)\|^w \le \psi(\tau^*)$. Since $\dot{v}^* |_{I_1}$ is continuous by assumption and $\operatorname{ext}\Gamma : I_1 \times \mathfrak{X}^w \twoheadrightarrow \mathfrak{X}^w$ is lower hemi-continuous by Lemma 1, we have a positive constant $0 < \delta \le \epsilon$ such that $\|\dot{v}^*(t) - \dot{v}^*(\tau)\|^w < \epsilon$ for $t \in I_1 \cap [\tau^*, \tau^* + \delta]$, and

(4) $\quad \operatorname{ext}\Gamma(\tau^*, u^*(\tau^*)) \subset B[\operatorname{ext}\Gamma(t, x), \epsilon]$ for $t \in [\tau^*, \tau^* + \delta]$ and $x \in B[u^*(\tau^*), M\delta]$.

By the Krein-Milman theorem, we have: $y^* \in \Gamma(\tau^*, u^*(\tau^*)) = \overline{\text{co}}\operatorname{ext}\Gamma(\tau^*, u^*(\tau^*))$, where $\overline{\text{co}}$ stands for the closed convex hull of A. Thus we obtain, for any $\epsilon > 0$, finite points $y_1, y_2, \ldots, y_m$ in $\operatorname{ext}\Gamma(\tau^*, u^*(\tau^*))$ and nonnegative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $\|y^* - \sum_i \lambda_i y_i\|^w < \epsilon$. Then we obtain, by Lyapunov's convexity theorem, a set of m measurable partition $J_0, J_1, \ldots, J_m$ such that $t < s$ for $t \in J_i, s \in J_j$ with $i < j, \cup_i J_i = I_1 \cap [\tau^*, \tau^* + \delta]$, and $\mu(J_i) = \lambda_i \mu(I_1 \cap [\tau^*, \tau^* + \delta])$. For $t \in [\tau^*, \tau^* + \delta]$, we set

$$H(t, x) = \begin{cases} \underline{\operatorname{ext}\Gamma(t, x)} & \text{if } t \in I_0 \\ \operatorname{ext}\Gamma(t, x) \cap B[y_i, \epsilon] & \text{if } t \in J_i. \end{cases}$$

Then by (4), $y_i \in \operatorname{ext}\Gamma(\tau^*, u^*(\tau^*)) \subset B[\operatorname{ext}\Gamma(t, x), \epsilon]$ for $i = 1, 2, \ldots, m, t \in [\tau^*, \tau^* + \delta]$, and $x \in B[u^*(\tau^*), M\delta]$. It follows that $\operatorname{ext}\Gamma(t, x) \cap B[y_i, \epsilon] \neq \emptyset$ and hence, $H(t, x) \neq \emptyset$ for such pair (t, x). In view of Lemma 2, the restriction of H to each of the product spaces $I_0 \times B[u^*(\tau), M\delta]$ and $J_i \times B[u^*(\tau), M\delta]$ is lower hemi-continuous. Thus H is almost lower hemi-continuous and we have an absolutely continuous function $u_\delta : [\tau^*, \tau^* + \delta] \to$

$\mathfrak{X}$ satisfying the following conditions: $u_\delta(\tau^*) = u^*(\tau^*)$, and $\dot{u}_\delta(t) \in H(t, u_\delta(t))$ a.e. in $[\tau^*, \tau^* + \delta]$ (see, e.g., Deimling [3,Theorem 9.3]). Let us define $w^* : [0, \tau^* + \delta] \to \mathfrak{X}$ by

$$w^*(t) = \begin{cases} u^*(t) & \text{for } t \in [0, \tau^*] \\ u_\delta(t) & \text{for } t \in [\tau^*, \tau^* + \delta]. \end{cases}$$

Then, w^* can be seen to be an element of $\mathfrak{M}$ as follows. First, we verify that w^* satisfies the condition (2) for $\tau = \tau^* + \delta$. Setting $I_0^* = I_0 \cap [\tau^*, \tau^* + \delta], I_1^* = I_1 \cap [\tau^*, \tau^* + \delta]$, we have

$$\begin{aligned}
&\|w^*(\tau^* + \delta) - v(\tau^* + \delta)\|^w \\
&\le \|w^*(\tau^*) - v(\tau^*)\|^w + \| \int_{\tau^*}^{\tau^*+\delta} [\dot{u}_\delta(t) - \dot{v}(t)] dt\|^w \\
&\le \psi(\tau^*) + \int_{I_0^*} \|\dot{u}_\delta(t) - \dot{v}(t)\|^w dt + \| \sum_{i=1}^m \int_{J_i} \dot{u}_\delta(t) dt - \int_{I_1^*} \dot{v}(t) dt\|^w \\
&\le \psi(\tau^*) + 2M\mu(I_0^*) + \sum_{i=1}^m \int_{J_i} \|\dot{u}_\delta(t) - y_i\|^w dt + \mu(I_1^*) \| \sum_{i=1}^m \lambda_i y_i - y^*\|^w \\
&+ \delta \|y^* - \dot{v}(\tau^*)\|^w + \epsilon\mu(I_1^*) \\
&\le \psi(\tau^*) + 2M\mu(I_0^*) + 4\epsilon\mu(I_1^*) + \delta\|y^* - \dot{v}(\tau^*)\|^w \\
&\le \psi(\tau^*) + \int_{\tau^*}^{\tau^*+\delta} (2M\chi_{I_0}(t) + 4\epsilon t) dt + \delta\psi(\tau^*) \le \psi(\tau^* + \delta),
\end{aligned}$$

where the first inequality is a consequence of the fundamental theorem of calculus. In the second inequality, the interval $[\tau^*, \tau^* + \delta]$ splits into I_0^* and I_1^*. The third inequality uses the relation $\mu(J_i) = \lambda_i \mu(I_1^*)$.

Let us now turn to the condition (3): for each t with $\tau^* \le t \le \tau^* + \delta$, we have

$$\begin{aligned}
&\|w^*(t) - v(t)\|^w \\
&\le \|w^*(\tau^*) - v(\tau^*)\|^w + 2M(t - \tau) \\
&\le \psi(\tau^*) + 2M\epsilon \\
&\le \psi(t) + 2M\epsilon.
\end{aligned}$$

Step 6. Since w^* belongs to $\mathfrak{M}$, u^* is not a maximal element of $\mathfrak{M}$, which is a contradiction. Thus we conclude that u^* is defined on the whole interval $[0, T]$.

Step 7. We have shown, in the above various steps, that, for each solution v of differential inclusion (II) and each $\epsilon > 0$, there exists a solution u^* of the differential inclusion (I) such that $\|u^*(t) - v(t)\|^w \le \psi(t) + 2M\epsilon$ for all $t \in [0, T]$ and hence

$$\begin{aligned}
&\sup_{t \in [0,T]} \|u^*(t) - v(t)\|^w \\
&\le \psi(t) + 2M\epsilon \\
&\le \epsilon(2Me^T + 4(e^T - 1) + 2M).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 1.

References

[1] A. Bressan, *On differential relations with lower continuous right-hand side. An existence theorem*, J. Differential Equations **37** (1980), 89-97.
[2] A. Bressan, *A bang-bang principle for non-linear system*, Boll Un. Mat. Ital. Suppl., **1** (1980), 53-59.
[3] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.
[4] D.G. Larman and C.A. Rogers, *The normability of metrizable sets*, Bull. London Math. Soc. **5** (1973), 39-48.
[5] H. Tateishi, *Nonconvex-valued differential inclusions in a separable Hilbert space*, Proc. Japn Acad. **68** Ser.A (1992), 296-301.
[6] H. Tateishi, *A relaxation theorem for differential inclusions: Infinite dimensional case*, Math. Japon. **45** (1997), 411-421.
[7] H. Tateishi, *Remark on the relaxation for differential inclusions*, Far East J. Math. Sci. **40** (2010), 57-65.

Communicated by *Jun Kawabe*

SCHOOL OF ECONOMICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA 464-8601, JAPAN
E-mail address: tateishi@soec.nagoya-u.ac.jp

STRONGLY GRADED RINGS WHICH ARE MAXIMAL ORDERS

Hidetoshi Marubayashi[1], Sri Wahyuni[2],
Indah Emilia Wijayanti[3], Iwan Ernanto[4]

Received December 8, 2017

Abstract.

Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$. In [6], it is shown that if R_0 is a maximal order, then so is R. We define a concept of $\mathbb{Z}$-invariant maximal order and show R_0 is a $\mathbb{Z}$-invariant maximal order if and only if R is a maximal order. We provide examples of R_0 which are $\mathbb{Z}$-invariant maximal orders but not maximal orders.

1 Introduction Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. We always assume that R_0, the degree zero part, is a prime Goldie ring with its quotient ring Q_0 and $C_0 = \{c \in R_0 \mid c \text{ is regular in } R_0\}$, which is a regular Ore set of R and the ring of fractions Q^g of R at C_0 has the following properties:

(i). $Q^g = \oplus_{n\in\mathbb{Z}} Q_0R_n (Q_0R_n = R_nQ_0)$.

(ii). $Q^g = Q_0[X, X^{-1}, \sigma]$ for some automorphism σ of R_0 ([6, 1.3]) and so

(iii). Q^g is a left and right principal ideal ring.

We denote by the quotient ring of R by Q. We define a concept of $\mathbb{Z}$-invariant maximal order in order to get the following three conditions are equivalent: (i) R_0 is a $\mathbb{Z}$-invariant maximal order (ii) R is a maximal order (iii) R is a graded maximal order. We give examples of R_0 which are $\mathbb{Z}$-invariant maximal orders but not maximal orders. We refer the readers to [7] or [8] and [9] for some elementary properties and some definitions of order theory and graded ring theory which are not mentioned in the paper.

2 The proof of Theorem Since Q^g is the quotient ring of R at C_0, the following lemma follows from the proof of [2, Theorem 1.31].

Lemma 1 *Let A be an ideal of R. Then $AQ^g = Q^gA$.*

Lemma 2 *Let A_0 be an ideal of R_0. Then the right ideal A_0R is an ideal of R if and only if $R_nA_0 = A_0R_n$ for all $n \in \mathbb{Z}$. In this case, A_0R is a graded ideal.*

Proof. If A_0R is an ideal of R, then $R_nA_0R_{-n} \subseteq A_0$, that is, $R_nA_0 \subseteq A_0R_n$ for all $n \in \mathbb{Z}$ and so $R_{-n}A_0 \subseteq A_0R_{-n}$ also follows. Hence $R_nA_0 = A_0R_n$. Conversely if $R_nA_0 = A_0R_n$ for all $n \in \mathbb{Z}$, then it is easy to see that A_0R is an ideal of R. □

2010 *Mathematics Subject Classification.* 13A15, 13A18,13G05,13E05 .
Key words and phrases. Graded ring; maximal order; prime Goldie ring; hereditary Noetherian prime ring.

Definition 1

(1). A left and right R_0-submodule A_0 of Q_0 is called $\mathbb{Z}$-invariant if $R_nA_0 = A_0R_n$ for all $n \in \mathbb{Z}$.

(2). R_0 is called a $\mathbb{Z}$-invariant maximal order in Q_0 if $O_l(A_0) = R_0 = O_r(A_0)$ for any nonzero $\mathbb{Z}$-invariant ideal A_0 of R_0.

(3). (10, *p*.205) *R is a graded maximal order in Q^g if for each graded over-ring S such that $R \subseteq S \subseteq Q^g$ and $aSb \subseteq R$ for some regular homogeneous elements $a, b \in Q^g$, it follows $R = S$.*

Lemma 3

(1). Let A_0 and B_0 be $\mathbb{Z}$-invariant left and right R_0-submodules in Q_0. Then A_0B_0 is $\mathbb{Z}$-invariant.

(2). Let A_0 be a $\mathbb{Z}$-invariant left R_0-ideal which is a right R_0-submodule in Q_0 and B_0 be a $\mathbb{Z}$-invariant right R_0-ideal which is a left R_0-submodule in Q_0. Then $C_0 = \{r_0 \in R_0 \mid A_0r_0 \subseteq R_0\}$ and $D_0 = \{r_0 \in R_0 \mid r_0B_0 \subseteq R_0\}$) are both $\mathbb{Z}$-invariant.

Proof.

(1). It is clear.

(2). $R_0 \supseteq A_0C_0$ implies $R_0 \supseteq R_nA_0C_0R_{-n} = A_0R_nC_0R_{-n}$ for all $n \in \mathbb{Z}$ and so $R_nC_0R_{-n} \subseteq C_0$ and also $R_{-n}C_0R_n \subseteq C_0$. Hence $C_0R_n = R_nC_0$ for all $n \in \mathbb{Z}$, that is, C_0 is $\mathbb{Z}$-invariant. Similarly D_0 is $\mathbb{Z}$-invariant.

□

Lemma 4 *The following conditions are equivalent.*

(1). R_0 is a $\mathbb{Z}$-invariant maximal order.

(2). $O_l(A_0) = R_0$ for each $\mathbb{Z}$-invariant left R_0-ideal A_0 which is a right R_0-submodule in Q_0, and $O_r(B_0) = R_0$ for each $\mathbb{Z}$-invariant right R_0-ideal B_0 which is a left R_0-submodule in Q_0.

Proof.
(2) ⇒ (1) This is a special case.
(1) ⇒ (2) Let A_0 be a $\mathbb{Z}$-invariant left R_0-ideal which is a right R_0-submodule in Q_0 and let $C_0 = \{ r_0 \in R_0 \mid A_0r_0 \subseteq R_0 \}$. Then A_0C_0 is a $\mathbb{Z}$-invariant ideal of R_0 by Lemma 3. Thus $R_0 = O_l(A_0C_0) \supseteq O_l(A_0) \supseteq R_0$ and so $O_l(A_0) = R_0$ follows. Similarly if B_0 is a $\mathbb{Z}$-invariant right R_0-ideal which is a left R_0-submodule in Q_0, then $O_r(B_0) = R_0$.

□

Theorem 1 *Let $R = \oplus_{n\in\mathbb{Z}}R_n$ be a strongly graded ring of type $\mathbb{Z}$. Then the following conditions are equivalent:*

(1). R_0 is a $\mathbb{Z}$-invariant maximal order in Q_0.

(2). R is a maximal order in Q.

(3). R is a graded maximal order in Q^g.

Proof.
(1) $\Rightarrow$ (2) Let S be an over-ring of R such that $aSb \subseteq R$ for some regular $a, b \in Q$. We may assume that $a, b \in R$. Put $T = R + RaS$, an over-ring of R with $Tb \subseteq R$. We claim $T = R$. Since TbR is an ideal of R, it follows from Lemma 1 that $TbQ^g = TbRQ^g = uQ^g = Q^g u$ for some regular element $u \in Q^g$ since Q^g is a principal ideal ring. For any $t \in T$, $tu \in TbQ^g = Q^g u$ and so $t \in Q^g$. Thus $T \subseteq Q^g$ follows. For any $n \in \mathbb{Z}$, let $C_n(T) = \{a_n \in Q_0 R_n \mid \exists t = a_n + a_{n_1} + \cdots + a_{n_l} \in T \text{ such that } n > n_i (1 \leq i \leq l)\} \cup \{0\}$, which is a left and right R_0-submodule of Q^g. It is easy to see that $C_n(T) = R_n C_0(T) = C_0(T) R_n$. So, in particular, $C_0(T)$ is a $\mathbb{Z}$-invariant over-ring of R_0. To prove that $C_0(T)$ is a left R_0-ideal, write $b = b_k +$ (the lower degree parts). Since $Tb \subseteq R$, it follows that $R_0 \supseteq C_{-k}(T) b_k = C_0(T) R_{-k} b_k$ and so $R_0 \supseteq C_0(T) R_{-k} b_k R_0$. Hence $C_0(T)$ is a left R_0-ideal since $R_{-k} b_k R_0$ is a non-zero ideal of R_0 and is a right R_0-submodule. Thus, by Lemma 4, $O_l(C_0(T)) = R_0$ and $R_0 \subseteq C_0(T) \subseteq O_l(C_0(T)) = R_0$ since $C_0(T)$ is an over-ring of R_0, which implies $R_0 = C_0(T)$ and $R_n = R_n C_0(T) = C_n(T)$ for all $n \in \mathbb{Z}$. Hence $T = R$ follows. Since $aS \subseteq RaS \subseteq T = R$, the left version of the above proof shows that $S = R$. Hence R is a maximal order in Q.
(2) $\Rightarrow$ (3) This is a special case.
(3) $\Rightarrow$ (1) Let A_0 be a $\mathbb{Z}$-invariant ideal of R_0. By Lemma 2, $A_0 R$ is a graded ideal of R. Thus it follows from [6, Lemma 1.5] that $RO_l(A_0) = O_l(A_0 R) = R = O_r(RA_0) = O_r(A_0)R$ and so $O_l(A_0) = R_0 = O_r(A_0)$. Hence R_0 is a $\mathbb{Z}$-invariant maximal order in Q_0. $\square$

Finally, we give some examples of maximal orders R such that R_0 are $\mathbb{Z}$-invariant maximal orders but not maximal orders.

Let R_0 be a hereditary Noetherian prime ring (an HNP ring for short) with its quotient ring Q_0 satisfying the following conditions:

(a). There is a cycle $M_{01}, \ldots, M_{0n}$ $(n \geq 2)$ so that $X = M_{01} \cap \cdots \cap M_{0n}$ is a maximal invertible ideal of R_0.

(b). Any maximal ideal different from $M_{0i} (1 \leq i \leq n)$ is invertible.

See [1] and [5] for examples of HNP rings satisfying (a) and (b) (the simplest example is $\begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, where p is a prime number). Let

$$R = \oplus_{n \in \mathbb{Z}} X^n (R_n = X^n)$$

a strongly graded ring of type $\mathbb{Z}$, and A_0 be an eventually idempotent ideal of R_0. Then there are $M_{0i_1}, \ldots, M_{0i_r}$ $i_j \in \{1, \ldots, n\}$($r < n$) which are the full set of maximal ideals containing A_0. Thus A_0 is not a $\mathbb{Z}$-invariant ideal by [4, Theorem 14].

Hence R_0 is a $\mathbb{Z}$-invariant maximal order since an ideal of R_0 is $\mathbb{Z}$-invariant if and only if it is invertible by [3, Theorem 2.9 and 4.2].

Acknowledgements: This work was done while the first author was visiting Universitas Gadjah Mada (UGM) Yogyakarta, Indonesia by the support from the Department of Mathematics, UGM and The Ministry of Research and Higher Education. He would like to thank algebra's staff at the Department of Mathematics UGM for their kind hospitality during the visit in October 2017.

REFERENCES

[1] G.Q. Abbasi, S. Kobayashi, H. Marubayashi and A. Ueda, Non Commutative Unique Factorization Rings, Comm. in Algebra **19**(1) (1991) 167-198.

[2] A.W. Chatters and C.R. Hajarnavis, Rings with Chain Conditions, Research Notes in Mathematics **44**, Pitman, London, (1980).

[3] D. Eisenbud and J.C. Robson, Hereditary Noetherian Prime Rings, J. of Algebra, **16** (1) (1970) 86-104.

[4] K.R. Goodearl and R.B. Warfield, Simple Modules over Hereditary Noetherian Prime Rings, J. of Algebra **57** (1979) 82-100.

[5] M. R. Helmi, H. Marubayashi and A. Ueda, Ore-Rees rings which are maximal orders, J. Math. Soc. Japan, **68** (1) (2016) 405-423.

[6] H. Marubayashi, E. Nauwelaerts and F. Van Oystaeyen, Grade rings over arithmetical orders, Comm. in Algebra, **12** (6) (1984) 745-775.

[7] J.C. Mc Connell and J.C. Robson, Noncommutative Noetherian Rings, Pure and Applied Mathematics, Wiley-Interscienc, New York, 1987.

[8] H. Marubayashi and F. Van Oystaeyen, Prime Divisors and Noncommutative Valuation Theory, Lecture Notes in Mathematics **2059**, Springer, 2012.

[9] C. Nastasescu and F. Van Oystaeyen, Graded Ring Theory, North-Holland Mathematics Library **28**, North-Holland Amsterdam, 1982.

[10] F. Van Oystaeyen and A. Verschoren, Relative Invariants of Rings, Pure and Applied Mathematics **86**, Marcel Dekker, 1984.

Communicated by *Naruhiko Aizawa*

[1] Dept. Mathematics, Naruto University, Japan. E-mail: marubayahide@gmail.com,
[2,3,4] Dept. Mathematics, Universitas Gadjah Mada,Indonesia
[2] E-mail: swahyuni@ugm.ac.id
[3] E-mail: ind wijayanti@ugm.ac.id
[4] E-mail: iwan.ernanto@ugm.ac.id

VERMA MODULES OVER A $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ GRADED SUPERALGEBRA AND INVARIANT DIFFERENTIAL EQUATIONS

Naruhiko Aizawa

Received January 11, 2018 ; revised Feburuary 7, 2018

Abstract. Lowest weight representations of the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra introduced by Rittenberg and Wyler are investigated. We give a explicit construction of Verma modules over the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra and show their reducibility by using singular vectors. The explicit formula of singular vectors are given and are used to derive partial differential equations invariant under the color supergroup generated by the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra.

1 Introduction. The present work aims to study representations of a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Lie superalgebra (also called color superalgebra) and its application to differential equations. We focus on a simple example of color superalgebras for the sake of simplicity and investigate Verma modules over it. The result is used to derive the partial differential equations which are invariant under the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Lie supergroup generated by the color superalgebra.

Color superalgebras are a generalization of Lie superalgebras introduced by Rittenberg and Wyler [1, 2] (see also [3, 4]). The idea of generalization is to extend the $\mathbb{Z}_2$ graded structure of the underlying vector space of Lie superalgebra to more general abelian groups. The group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, discussed here, is the simplest non-trivial example of the generalization. During the last four decades, structure theory, classification of possible color superalgebras of given dimension *etc.* have been studied by many authors. See, for example, [5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein. However, applications of color superalgebras to mathematical and physical problems are very limited [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. In [22, 23] it is shown that symmetries of a first order linear partial differential equation, called the Lévy-Leblond equation [24], are generated by a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra. The authors seek symmetry operators of the Lévy-Leblond equation systematically and found that they do not close as a Lie algebra or superalgebra but do in a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra. Here, we reverse the argument. We start with a simple example of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra and then study its lowest weight representations and their irreducibility. Reducibility of the representations is detected by the existence of singular vectors. Explicit formulae of the singular vectors allows us to write down invariant partial differential equations. This is a generalization of the method developed for semi-simple Lie groups [25] to the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ setting. Thus we shall see the method is valid beyond the semi-simple Lie groups.

The plan of this paper is as follows. In the next section we give a definition of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra and present the one, denoted by $\mathfrak{g}$, investigated in this work. Lowest weight Verma modules over $\mathfrak{g}$ are studied in §3. After discussing the failure of the naive approach, we construct Verma modules over $\mathfrak{g}$ by using a trick. A list of all singular vectors in the Verma modules is presented with an explicit formulae of the singular vectors. The formulae of the singular vectors are used to find differential equations whose symmetries are generated by $\mathfrak{g}$ in §4. Summary and some remarks are given in §5.

2010 *Mathematics Subject Classification.* 17B75,76M60 .

Key words and phrases. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra, Verma modules, singular vectors, invariant PDEs .

2 Definition of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra. Let $\mathfrak{g}$ be a vector space over $\mathbb{C}$ (or $\mathbb{R}$) which is a direct sum of four subspaces labelled by an element of the group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$:

$$\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}. \tag{2.1}$$

An inner product of two $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is defined as usual:

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \alpha_1 \beta_1 + \alpha_2 \beta_2. \tag{2.2}$$

Now we give a definition of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra according to [1, 2].

Definition 2.1. If $\mathfrak{g}$ admits a bilinear form $[\![\ ,\ ]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following three relations, then $\mathfrak{g}$ is called a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded color superalgebra:

1. $[\![\mathfrak{g}_{\boldsymbol{\alpha}}, \mathfrak{g}_{\boldsymbol{\beta}}]\!] \subseteq \mathfrak{g}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$,
2. $[\![X_{\boldsymbol{\alpha}}, X_{\boldsymbol{\beta}}]\!] = -(-1)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} [\![X_{\boldsymbol{\beta}}, X_{\boldsymbol{\alpha}}]\!]$,
3. $[\![X_{\boldsymbol{\alpha}}, [\![X_{\boldsymbol{\beta}}, X_{\boldsymbol{\gamma}}]\!]]\!](-1)^{\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}} + \text{cyclic perm.} = 0$,

where $X_{\boldsymbol{\alpha}} \in \mathfrak{g}_{\boldsymbol{\alpha}}$ and the third relation is called the graded Jacobi identity.

When the inner product $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$ is an even integer the graded Lie bracket $[\![\ ,\ ]\!]$ is understood as a commutator, while it is an anticommutator if $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$ is an odd integer. If $N = 1$, then Definition 2.1 is identical to the definition of Lie superalgebras so that the color superalgebra is a natural generalization of Lie superalgebra.

The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra investigated in this work is eight dimensional and its basis and grading are given as follows:

$$\begin{array}{rcl} (0,0) & : & A_-,\ A_+,\ N \\ (1,0) & : & b_-,\ b_+ \\ (0,1) & : & a_-,\ a_+ \\ (1,1) & : & F \end{array} \tag{2.3}$$

The basis satisfy the relations (only non-vanishing ones are presented):

$$\begin{array}{llll} [A_-, A_+] = 4N, & [A_-, N] = 2A_-, & [A_+, N] = -2A_+, & \\ [A_-, b_+] = 2b_-, & [A_+, b_-] = -2b_+, & [N, b_-] = -b_-, & [N, b_+] = b_+, \\ [A_-, a_+] = 2a_-, & [A_+, a_-] = -2a_+, & [N, a_-] = -a_-, & [N, a_+] = a_+, \\ \{b_-, b_-\} = 2A_-, & \{b_-, b_+\} = 2N, & \{b_+, b_+\} = 2A_+, & \\ [b_-, a_+] = F, & [b_+, a_-] = -F, & \{b_-, F\} = 2a_-, & \{b_+, F\} = 2a_+, \\ \{a_-, a_-\} = 2A_-, & \{a_-, a_+\} = 2N, & \{a_+, a_+\} = 2A_+, & \\ \{a_-, F\} = 2b_-, & \{a_+, F\} = 2b_+. & & \end{array} \tag{2.4}$$

One may see from the defining relations that the color superalgebra has two $osp(1,2)$ subalgebras, $\langle\ A_\pm,\ N,\ a_\pm\ \rangle$ and $\langle\ A_\pm,\ N,\ b_\pm\ \rangle$. This color superalgebra was given in [1] as one of the non-trivial examples of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebras. It is also discussed in [13] as an $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ extension of the superalgebra generated by boson and fermion operators. We denote the color superalgebra (2.3) by $\mathfrak{g}$.

Lemma 2.2. *The superalgebra $\mathfrak{g}$ admits an algebra anti-involution $\omega : \mathfrak{g} \to \mathfrak{g}$ defined by*

$$\omega(X_\pm) = X_\mp, \qquad \omega(Y) = Y, \quad X_\pm = A_\pm,\ a_\pm,\ b_\pm, \quad Y = N,\ F \tag{2.5}$$

Proof. It is checked by straightforward computation. □

Next we give a definition of the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann numbers which was also introduced in [2].

Definition 2.3. Let $\boldsymbol{\alpha} \in \mathbb{Z}_2 \otimes \mathbb{Z}_2$ and $\zeta_{\boldsymbol{\alpha},i}$ be a basis of a vector space over the same field as $\mathfrak{g}$. If the basis satisfies the relations

$$[\![\zeta_{\boldsymbol{\alpha},i}, \zeta_{\boldsymbol{\beta},j}]\!] = 0, \tag{2.6}$$

then we call $\zeta_{\boldsymbol{\alpha},i}$ the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann numbers.

An extension of an integral and a derivative of the ordinary Grassmann numbers to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ setting is discussed in [11]. We use the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann numbers in the subsequent sections. The basis $\zeta_{\boldsymbol{\alpha},i}$ may be realized in terms of the ordinary Grassmann numbers and the Clifford algebra [13]. Let us recall that the Clifford algebra $Cl(p,q)$ is a unital algebra generated by γ_i $(i = 1, 2, \dots, N = p+q)$ subject to the relations:

$$\{\gamma_i, \gamma_j\} = 2\eta_{ij}, \qquad \eta = \mathrm{diag}(\underbrace{+1, \dots, +1}_{p}, \underbrace{-1, \dots, -1}_{q}) \tag{2.7}$$

The $Cl(p,q)$ is a 2^N dimensional algebra whose elements are given by a product of the generators:

$$1,\ \gamma_i,\ \gamma_i\gamma_j,\ \gamma_i\gamma_j\gamma_k,\ \dots,\ \gamma_1\gamma_2\cdots\gamma_N$$

Lemma 2.4. *The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann numbers $\zeta_{\boldsymbol{\alpha},i}$ are realized in terms of the ordinary Grassmann number ξ_μ and the Clifford algebra $Cl(p,q)$, $p+q=2$:*

$$\begin{aligned} \zeta_{(0,0),m} &= 1 \otimes x_m, \quad & \zeta_{(1,0),\mu} &= \gamma_1 \otimes \xi_\mu, \\ \zeta_{(0,1),\mu} &= \gamma_2 \otimes \xi_\mu, \quad & \zeta_{(1,1),m} &= \gamma_1\gamma_2 \otimes x_m, \end{aligned}$$

where $x_m \in \mathbb{R}$.

3 Verma modules and their reducibility.

We want to investigate the lowest weight representations of $\mathfrak{g}$ by employing the standard procedure of Lie theory [26]. However, it turns out that the naive approach is not successful in defining Verma modules over $\mathfrak{g}$. We begin with the failure of the naive construction of Verma modules and then consider a modified approach.

3.1 Failure of the naive construction.

The algebra $\mathfrak{g}$ has a natural triangular decomposition:

$$\begin{aligned} \mathfrak{g}_+ &= \langle\ A_+,\ a_+,\ b_+\ \rangle, \\ \mathfrak{g}_0 &= \langle\ N,\ F\ \rangle, \\ \mathfrak{g}_- &= \langle\ A_-,\ a_-,\ b_-\ \rangle. \end{aligned} \tag{3.1}$$

This is based on the eigenvalues of ad$N$ which are given as follows:

$+2$	$+1$	0	-1	-2
A_+	$a_+,\ b_+$	$N,\ F$	$a_-,\ b_-$	A_-

The anti-involution ω (2.5) acts on the subspaces as

$$\omega(\mathfrak{g}_\pm) = \mathfrak{g}_\mp, \qquad \omega(\mathfrak{g}_0) = \mathfrak{g}_0 \tag{3.2}$$

and it is easy to see that $[\![\mathfrak{g}_0, \mathfrak{g}_\pm]\!] \subseteq \mathfrak{g}_\pm$ and $[\![\mathfrak{g}_\pm, \mathfrak{g}_\pm]\!] \subseteq \mathfrak{g}_\pm$. Therefore, (3.1) is a natural triangular decomposition of $\mathfrak{g}$.

The decomposition (3.1) leads us to define Verma modules in a standard way. We first define the lowest weight state $|h,\varphi\rangle$ by

$$a_- |h,\varphi\rangle = 0, \qquad N|h,\varphi\rangle = h|h,\varphi\rangle, \qquad F|h,\varphi\rangle = \varphi|h,\varphi\rangle. \tag{3.3}$$

It follows that

$$A_- |h,\varphi\rangle = b_- |h,\varphi\rangle = 0. \tag{3.4}$$

Then the Verma modules over $\mathfrak{g}$ are defined by $M(h,\varphi) = \mathcal{U}(\mathfrak{g}_+) \otimes |h,\varphi\rangle$ where $\mathcal{U}(\mathfrak{g}_+)$ is the universal enveloping algebra of $\mathfrak{g}_+$. The basis of $M(h,\varphi)$ is obviously given by

$$|k,\ell\rangle = (a_+)^k (b_+)^\ell |h,\varphi\rangle, \quad k,\ell \in \mathbb{Z}_{\geq 0} \tag{3.5}$$

It is immediate to see that the relations $\{a_+,a_+\} = \{b_+,b_+\} = 2A_+$ are not realized on this basis. This shows the failure of the naive construction of Verma modules.

3.2 Verma modules and singular vectors. To overcome the difficulty in §3.1 we use a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann number ζ of degree $(1,1)$. Suppose further that $\zeta^2 = 1$. Legitimacy of the assumption is ensured by Lemma 2.4 since ζ may be realized by using $Cl(1,1)$. Now we define a new basis of $\mathfrak{g}$:

$$c_\pm = \frac{1}{\sqrt{2}}(a_- \pm b_-\zeta), \qquad d_\pm = \frac{1}{\sqrt{2}}(a_+ \pm \zeta b_+), \qquad \tilde{F} = \frac{1}{2}\zeta F. \tag{3.6}$$

The non-vanishing defining relations for the new basis are written as

$$\begin{aligned} &\{c_+,c_-\} = 2A_-, && \{d_+,d_-\} = 2A_+, && \{c_\pm,d_\pm\} = 2(N \mp \tilde{F}), \\ &[\tilde{F}, c_\pm] = \mp c_\pm, && [\tilde{F}, d_\pm] = \pm d_\pm. \end{aligned} \tag{3.7}$$

We remark that $c_\pm$ and $d_\pm$ are nilpotent and that $c_\pm$ anticommutes with $d_\mp$. The anti-involution ω (2.5) is extended to the new basis (3.6) by setting $\omega(\zeta) = 1$:

$$\omega(c_\pm) = d_\pm, \qquad \omega(\tilde{F}) = \tilde{F}. \tag{3.8}$$

In fact, this change of basis converts the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra to an ordinary superalgebra of $\mathbb{Z}_2$ grading. Due to the degree of ζ, the degree of $c_\pm$ and $d_\pm$ are all $(0,1)$, while the degree of $\tilde{F}$ is $(0,0)$. Thus one may consider representations of the $\mathbb{Z}_2$ graded algebra, then convert it to the ones for $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading.

The present choice of the basis of $\mathfrak{g}$ diagonalize $\mathrm{ad}N$ and $\mathrm{ad}\tilde{F}$. Their eigenvalues are summarized as

$$\begin{array}{llll} A_+\ (+2,0), & A_-\ (-2,0), & c_+\ (-1,-1), & c_-\ (-1,+1), \\ N\ (0,0), & \tilde{F}\ (0,0), & d_+\ (+1,+1), & d_-\ (+1,-1). \end{array} \tag{3.9}$$

We introduce the triangular decomposition of $\mathfrak{g}$ according to the eigenvalue of $\mathrm{ad}N$:

$$\mathfrak{g}_+ = \langle\, A_+,\ d_\pm \,\rangle, \qquad \mathfrak{g}_0 = \langle\, N,\ \tilde{F} \,\rangle, \qquad \mathfrak{g}_- = \langle\, A_-,\ c_\pm \,\rangle. \tag{3.10}$$

Define the lowest weight vector $|h,f\rangle$ by

$$c_\pm |h,f\rangle = 0, \qquad N|h,f\rangle = h|h,f\rangle, \qquad \tilde{F}|h,f\rangle = f|h,f\rangle. \tag{3.11}$$

It then follows that $A_- |h,f\rangle = 0$. We define the Verma modules over $\mathfrak{g}$ by a space induced from $|h,f\rangle$ by $M(h,f) = \mathcal{U}(\mathfrak{g}_+) \otimes |h,f\rangle$. The natural basis of $M(h,f)$ is given by

$$|k,\mu,\nu\rangle = (A_+)^k (d_+)^\mu (d_-)^\nu |h,f\rangle, \quad k \in \mathbb{Z}_{\geq 0}, \ \mu,\nu \in \{0,1\} \tag{3.12}$$

It is then straightforward to compute the action of $\mathfrak{g}$ on $M(h,f)$. The action of $\mathfrak{g}_0$ yields

$$\begin{aligned} N|k,\mu,\nu\rangle &= (h+2k+\mu+\nu)|k,\mu,\nu\rangle, \\ \tilde{F}|k,\mu,\nu\rangle &= (f+\mu-\nu)|k,\mu,\nu\rangle. \end{aligned} \tag{3.13}$$

The action of $\mathfrak{g}_+$ is given by

$$\begin{aligned} A_+|k,\mu,\nu\rangle &= |k+1,\mu,\nu\rangle, \\ d_+|k,\mu,\nu\rangle &= \delta_{\mu,0}|k,\mu+1,\nu\rangle, \\ d_-|k,\mu,\nu\rangle &= (-1)^\mu \delta_{\nu,0}|k,\mu,\nu+1\rangle + \delta_{\mu,1}2|k+1,\mu-1,\nu\rangle. \end{aligned} \tag{3.14}$$

The action of $\mathfrak{g}_-$ is summarized as

$$\begin{aligned} A_-|k,\mu,\nu\rangle &= 4k(h+k+\mu+\nu-1)|k-1,\mu,\nu\rangle + \delta_{\mu,1}\delta_{\nu,1}4(h+f)|k,\mu-1,\nu-1\rangle, \\ c+|k,\mu,\nu\rangle &= \delta_{\mu,1}2(h+2k+2\nu-f)|k,\mu-1,\nu\rangle + (-1)^\mu\delta_{\nu,0}2k|k-1,\mu,\nu+1\rangle, \\ c_-|k,\mu,\nu\rangle &= (-1)^\mu\delta_{\nu,1}2(h+f)|k,\mu,\nu-1\rangle + \delta_{\mu,0}2k|k-1,\mu+1,\nu\rangle. \end{aligned} \tag{3.15}$$

We have successfully obtained the Verma modules over $\mathfrak{g}$.

Let us now discuss reducibility of the Verma modules. This may be done by singular vectors [26]. The existence of the singular vector, by definition, means that $M(h,f)$ is reducible. The Verma module $M(h,f)$ has a natural grading as a vector space:

$$M(h,f) = \bigoplus_{m\in\mathbb{N}} M_m(h,f), \quad M_m(h,f) = \{\ |v\rangle \in M(h,f) \mid N|v\rangle = (h+m)|v\rangle\ \} \tag{3.16}$$

where $\mathbb{N}$ is the set of non-negative integers. In the basis (3.12), the integer m is given by $m = 2k+\mu+\nu$ and we call m the *level.* Any singular vector is an eigenvector of N so that it must be an element of a subspace $M_m(h,f)$. We give a complete list of singular vectors in $M(h,f)$.

Theorem 3.1. *$M(h,f)$ has precisely one singular vector if h,f satisfy one of the following conditions:*

(1) $h=f$: The singular vector is $|0,1,0\rangle$.

(2) $h=-f$: The singular vector is $|0,0,1\rangle$.

(3) $h=-n$ and $f\neq n$ for a positive integer n : The singular vector exists at level $2n$ subspace and given by

$$|n,0,0\rangle + \frac{n}{f-n}|n-1,1,1\rangle. \tag{3.17}$$

Proof. First of all, we note that $\dim M_m(h,f) = 2$ for any level m. If the level $m=2n+1$ is odd, then $M_m(h,f)$ is spanned by $|n,0,1\rangle$ and $|n,1,0\rangle$, while if $m=2n$, then $M_m(h,f)$ is spanned by $|n,0,0\rangle$ and $|n-1,1,1\rangle$. We study the odd and even level separately.

(i) $m = 2n+1$. Since $|n,0,1\rangle$ and $|n,1,0\rangle$ have distinct eigenvalues of $\tilde{F}$, singular vector is not a linear combination of the two vectors. Requirement that $|n,0,1\rangle$ is annihilated by the action of $\mathfrak{g}_-$ provides the relations

$$n = 0, \qquad h + f = 0. \tag{3.18}$$

The same argument for $|n,1,0\rangle$ provides the relations

$$n = 0, \qquad h - f = 0. \tag{3.19}$$

This proves (1) and (2) of the theorem.

(ii) $m = 2n$. A singular vector, if any, may be written as

$$|u\rangle = |n,0,0\rangle + \alpha\,|n-1,1,1\rangle \tag{3.20}$$

with a constant α. $|u\rangle$ is an eigenvector of $\tilde{F}$ with the eigenvalue f. The requirement that the action of $\mathfrak{g}_-$ annihilate $|u\rangle$ gives the relations

$$\begin{aligned} &n + \alpha(h + 2n - f) = 0,\\ &n - \alpha(h+f) = 0,\\ &n(h+n-1) + \alpha(h+f) = 0,\\ &\alpha(n-1)(h+n) = 0. \end{aligned} \tag{3.21}$$

The last equation implies that there exists three possibilities:

$$\text{(a) } \alpha = 0, \qquad \text{(b) } h = -n, \qquad \text{(c) } n = 1.$$

The case (a) means that $n = 0$ so that $|u\rangle = |0,0,0\rangle$. Thus this case is trivial. The case (b), the first three equations in (3.21) are reduced to single relation:

$$n + \alpha(n-f) = 0. \tag{3.22}$$

If $f = n$, then the relations means that $n = 0$. Thus we need $\alpha = 0$ and we have a only trivial solution. If $f \neq n$, then (3.22) is solved to α and give the unique singular vector (3.17). The case (c) is reduced to a subclass of the case (b) since we find the relation $h = -1 = -n$. We thus completed the proof. ☐

Corollary 3.2. *The Verma module $M(h,f)$ is irreducible if following conditions are true:*

(1) $h \neq \pm f$

(2) $h \neq -n$ or $f = n$

4 Invariant partial differential equations. In this section, we derive partial differential equations which are invariant under the color supergroup generated by $\mathfrak{g}$. This is done by employing the method of [25]. We give a brief outline of the method and refer to [25] for detail. The basic idea of the method is to realize the Verma modules, discussed in §3.2, by a space of differentiable functions defined on the color supergroup.

Let $\mathcal{G}$ be the color supergroup generated by $\mathfrak{g}$ which means that an element $g \in \mathcal{G}$ is written as $g = \exp(X),\ X \in \mathfrak{g}$. We consider a space of functions C_Λ on $\mathcal{G}$ having a special property. That is, $F \in C_\Lambda$ satisfies the relation:

$$F(gg_0g_-) = \exp(\Lambda(H))F(g), \quad g \in \mathcal{G},\ g_0 = \exp(H),\ H \in \mathfrak{g}_0,\ g_- \in \exp(\mathfrak{g}_-) \tag{4.1}$$

where $\Lambda(H)$ is an eigenvalue of H. This property means that F is, in fact, a function on $\mathcal{G}_+ = \exp(\mathfrak{g}_+)$. We then define the left (π_L) and right (π_R) actions of $\mathfrak{g}$ on C_Λ according to the standard way of Lie theory:

$$\pi_L(X)F(g) = \frac{d}{d\tau}F(e^{-\tau X}g)\bigg|_{\tau=0}, \tag{4.2}$$

$$\pi_R(X)F(g) = \frac{d}{d\tau}F(ge^{\tau X})\bigg|_{\tau=0}, \tag{4.3}$$

where $X \in \mathfrak{g}$. Note that the parameter τ is a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann number of degree same as X. A definition of derivative with respect to the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann numbers is found in [11]. It is then easy to see that, due to the property (4.1), the function $F \in C_\Lambda$ plays the role of a lowest weight vector $|h, f\rangle$ by the right action. Namely, $\pi_R(X)F = 0$ if $X \in \mathfrak{g}_-$ and $\pi_R(X)F = \Lambda(x)F$ if $X \in \mathfrak{g}_0$. While $\pi_R(X)$ with $X \in \mathfrak{g}_+$ becomes a differential operator so that a Verma module $M(h, f)$ is realized by the action of $\pi_R(X)$ with $X \in \mathfrak{g}_+$ on the function $F(g)$.

To be more explicit, we parametrize $g_+ \in \mathcal{G}_+$ as

$$g_+ = \exp(xA_+)\exp(\psi_+ d_+)\exp(\psi_- d_-). \tag{4.4}$$

Then we obtain from (4.3)

$$\begin{aligned} \pi_R(A_+) &= \frac{\partial}{\partial x}, \\ \pi_R(d_+) &= \frac{\partial}{\partial \psi_+} + 2\psi_- \frac{\partial}{\partial x}, \\ \pi_R(d_-) &= \frac{\partial}{\partial \psi_-} \end{aligned} \tag{4.5}$$

and by setting $\Lambda(N) = h,\ \Lambda(\tilde{F}) = f$ we have

$$\pi_R(N) = h, \qquad \pi_R(\tilde{F}) = f. \tag{4.6}$$

As shown in §3.2 the singular vectors in $M(h, f)$ is written as $\mathcal{P}\,|h, f\rangle$ with $\mathcal{P} \in U(\mathfrak{g}_+)$. It is shown in [25] that differential equations invariant under the color supergroup $\mathcal{G}$ are given by $\pi_R(\mathcal{P})\varphi = 0$. The symmetry transformations are generated by the left action (4.2). This is due to the fact that a singular vector is an intertwining operator of two representations $M(h, f)$ and $M(h', f')$. According to Theorem 3.1 we found a hierarchy of invariant differential equations.

Proposition 4.1. *The following equations are invariant under the color supergroup $\mathcal{G}$:*

$$\begin{aligned} &\frac{\partial}{\partial \psi_-}\varphi(x, \psi_\pm) = 0, \\ &\left(\frac{\partial}{\partial \psi_+} + 2\psi_- \frac{\partial}{\partial x}\right)\varphi(x, \psi_\pm) = 0, \\ &\left[\frac{\partial}{\partial x} + \frac{n}{f-n}\left(\frac{\partial}{\partial \psi_+} + 2\psi_- \frac{\partial}{\partial x}\right)\frac{\partial}{\partial \psi_-}\right]\left(\frac{\partial}{\partial x}\right)^{n-1}\varphi(x, \psi_\pm) = 0. \end{aligned} \tag{4.7}$$

Up to here our analysis of the representations of $\mathfrak{g}$ is done by converting the basis of $\mathfrak{g}$ into a $\mathbb{Z}_2$ grading ones. Thus the invariant equations are written in terms of the variables of degree $(0,0)$ and $(0,1)$. We now rewrite the invariant differential equations in a form also containing the variables of degree $(1,0)$ and $(1,1)$. Let ψ and θ be $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann number of degree $(0,1)$ and $(1,0)$, respectively. Taking the basis change (3.6) into account we set

$$\psi_\pm = \frac{1}{\sqrt{2}}(\psi \pm \zeta\theta). \tag{4.8}$$

and replace $\psi_\pm$ with ψ and θ. Then the independent variables of the invariant differential equations become x, ψ and θ and the equations yield as follows:

$$\left(\frac{\partial}{\partial\psi} + \zeta\frac{\partial}{\partial\theta}\right)\varphi = 0,$$

$$\left[\frac{\partial}{\partial\psi} - \zeta\frac{\partial}{\partial\theta} + 2(\psi - \zeta\theta)\frac{\partial}{\partial x}\right]\varphi = 0,$$

$$\left[\frac{\partial}{\partial x} + \frac{n}{f-n}\left(\psi\frac{\partial}{\partial\psi} + \theta\frac{\partial}{\partial\theta} - \zeta\Big(\psi\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial\psi}\Big)\right)\frac{\partial}{\partial x} - \frac{n\zeta}{f-n}\frac{\partial^2}{\partial\psi\partial\theta}\right]\left(\frac{\partial}{\partial x}\right)^{n-1}\varphi = 0.$$

5 Concluding remarks. We studied lowest weight representations of the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra given by Rittenberg and Wyler. Reducibility of the Verma modules over the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded superalgebra is shown by explicit construction of singular vectors. As an application of the present scheme invariant partial differential equations defined on the space of functions whose variables are $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Grassmann numbers are obtained. The present scheme may apply to other color superalgebras and one may find many related differential equations. Since differential equations are very basic objects in theoretical physics and many areas of mathematics, the present line of investigation will reveal a connection of color superalgebras with different areas of science.

We close this paper with two possible lines of further investigations on applications of color superalgebras. The first one is mathematical application. It is well known that representations of Lie algebras and Lie groups are closely related to orthogonal polynomials. We anticipate that representations of color algebras or groups also have such connection. The second one is physical application. One may use a nonlinear realization method to write down Lagrangians which are invariant under a color supergroup, since the method is originally defined for Lie groups then extended to supergroups. Color supergroups are natural generalization of supergroups, thus nonlinear realization methods will be generalized to any color supergroups. If this is done, one may find a dynamical system whose symmetry is governed by a color supergroup. This would open the way to physical applications of color supergroups.

Acknowledgment. The author is supported by the grants-in-aid from JSPS (Contract No. 26400209).

References

[1] V. Rittenberg and D. Wyler, *Generalized Superalgebras.* Nucl. Phys. **B 139** (1978), 189.

[2] V. Rittenberg and D. Wyler, *Sequences of $Z_2 \otimes Z_2$ graded Lie algebras and superalgebras.* J. Math. Phys. **19** (1978), 2193.

[3] M. Scheunert, *Generalized Lie algebras.* J. Math. Phys. **20** (1979), 712.

[4] H. S. Green and P. D. Jarvis, *Casimir invariants, characteristic identities, and Young diagrams for color algebras and superalgebras.* J. Math. Phys. **24** (1983), 1681.

[5] M. Scheunert, *Graded tensor calculus.* J. Math. Phys. **24** (1983), 2658.

[6] M. Scheunert, *Casimir elements of ϵ-Lie algebras.* J. Math. Phys. **24** (1983), 2671.

[7] S. D. Silvestrov, *On the classification of 3-dimensional coloured Lie algebras.* Banach Center Publications, **40** (1997), 159.

[8] X.-W. Chen, S. D. Silvestrov and F. Van Oystaeyen, *Representations and cocycle twists of color Lie algebras.* Algebr. Represent. Theor. **9** (2006), 633.

[9] R. Campoamor-Stursberg and M. Rausch de Traubenberg, *Color Lie algebras and Lie algebras of order F,.* J. Gen. Lie Thory Appl. **3** (2009), 113.

[10] G. Sigurdsson and S. D. Silvestrov, *Bosonic realizations of the colour Heisenberg Lie algebra.* J. Nonlinear Math. Phys. **13** (2006), 110.

[11] N. Aizawa and J. Segar, *$\mathbb{Z}_2 \times \mathbb{Z}_2$ generalizations of $\mathcal{N} = 2$ super Schrödinger algebras and their representations.* J. Math. Phys. **58** (2017), 113501.

[12] N.I. Stoilova, J. Van der Jeugt, *The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$-graded Lie superalgebra pso(2m+1—2n) and new parastatistics representations.* arXiv:1711.02136 [math-ph].

[13] N. Aizawa, *Generalization of Superalgebras to Color Superalgebras and Their Representations.* arXiv:1712.03008 [math-ph].

[14] J. Lukierski and V. Rittenberg, *Color-De Sitter and color-conformal superalgebras.* Phys. Rev. **D 18** (1978), 385.

[15] M. A. Vasiliev, *de Sitter supergravity with positive cosmological constant and generalized Lie superalgebras.* Class. Quantum Grav. **2** (1985), 645.

[16] P. D. Jarvis, M. Yang and B. G. Wybourne, *Generalized quasispin for supergroups.* J. Math. Phys. **28** (1987), 1192.

[17] A. A. Zheltukhin, *Para-Grassmann extension of the Neveu-Schwartz-Ramond algebra.* Theor. Math. Phys. **71** (1987), 491 (Teor. Mat. Fiz. **71** (1987), 218).

[18] L. A. Wills-Toro, *(I, q)-graded Lie algebraic extensions of the Poincaré algebra, constraints on I and q.* J. Math. Phys. **36** (1995), 2085.

[19] L. A. Wills-Toro, *Trefoil symmetries I. Clover extensions beyond Coleman-Mandula theorem.* J. Math. Phys. **42** (2001) 3915.

[20] V. N. Tolstoy, *Once more on parastatistics.* Phys. Part. Nucl. Lett. **11** (2014), 933.

[21] V. N. Tolstoy, *Super-de Sitter and Alternative Super-Poincaré Symmetries.* In: Dobrev V. (eds) Lie Theory and Its Applications in Physics. Springer Proceedings in Mathematics & Statistics, vol. 111, Springer, Tokyo, 2014.

[22] N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetries of the Lévy-Leblond equations.* Prog. Theor. Exp. Phys. **2016** (2016), 123A01.

[23] N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan, *Generalized supersymmetry and Lévy-Leblond equation.* in S. Duarte *et al* (eds), *Physical and Mathematical Aspects of Symmetries.* Springer, 2017.

[24] J.-M. Lévy-Leblond, *Nonrelativistic particles and wave equations.* Comm. Math. Phys. **6** (1967), 286.

[25] V. K. Dobrev, *Canonical construction of differential operators intertwining representations of real semisimple Lie groups.* Rep. Math. Phys. **25** (1988), 159.

[26] J. Dixmier, *Enveloping algebras.* North-Holland (1977).

Communicated by *Phillip Isaac*

Department of Physical Science
Graduate School of Science

Osaka Prefecture University
Nakamozu Campus, Sakai, Osaka 599-8531
Japan

CAUCHY'S THEOREM FOR B-ALGEBRAS

JOEMAR C. ENDAM[1] AND JENETTE S. BANTUG[2]

Received September 5, 2017 ; revised December 3, 2017

ABSTRACT. In this paper, we establish the Cauchy's Theorem for B-algebras. We also present some implications of Lagrange's Theorem and Cauchy's Theorem for B-algebras. In particular, the concept of B_p-algebras is introduced.

1 Introduction In [9], the notion of B-algebras was introduced by J. Neggers and H.S. Kim. A *B-algebra* is an algebra $(X; *, 0)$ of type (2, 0) (that is, a nonempty set X with a binary operation $*$ and a constant 0) satisfying the following axioms for all $x, y, z \in X$: (I) $x*x = 0$, (II) $x*0 = x$, (III) $(x*y)*z = x*(z*(0*y))$. A B-algebra $(X; *, 0)$ is *commutative* [9] if $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$. In [10], J. Neggers and H.S. Kim introduced the notions of a subalgebra and normality of B-algebras and some of their properties are established. A nonempty subset N of X is called a *subalgebra* of X if $x * y \in N$ for any x, $y \in N$. It is called *normal* in X if for any $x * y, a * b \in N$ implies $(x * a) * (y * b) \in N$. A normal subset of X is a subalgebra of X. There are several properties of B-algebras as established by some authors [1−12]. The following properties are used in this paper, for any $x, y, z \in X$, we have (P1) $0 * (0 * x) = x$ [9], (P2) $x * y = 0 * (y * x)$ [11], (P3) $x*(y*z) = (x*(0*z))*y$ [9], (P4) $x*y = x*z$ implies $y = z$ [3], (P5) $(0*x)*(y*x) = 0*y$ [9]. In [2], J.S. Bantug and J.C. Endam established the Lagrange's Theorem for B-algebras. In this paper, we provide some partial results on the converse of this theorem. In particular, we establish the Cauchy's Theorem for B-algebras. As a consequence, we also introduce the concept of B_p-algebras. Throughout this paper, X means a B-algebra $(X; *, 0)$.

2 Preliminaries This section presents some concepts and results needed in this paper. We start with some examples of B-algebras.

Example 2.1. [9] Let $X = \{0, 1, 2\}$ be a set with the following table of operation:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Example 2.2. [9] Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table of operation:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

2010 *Mathematics Subject Classification.* 08A05, 06F35.
Key words and phrases. B-algebras, Lagrange's Theorem, Cauchy's Theorem, B_p-algebras.

In [7], if S is a subset of X, then $\langle S\rangle_B$ is the intersection of all subalgebra H of X such that $S \subseteq H$, and the subalgebra $\langle S\rangle_B$ of X is called the *subalgebra generated by* S. If $X = \langle S\rangle_B$, then S is called a *set of generators* for X. Moreover, $\langle S\rangle_B$ is the smallest subalgebra of X containing S. If either $S = \varnothing$ or $S = \{0\}$, then $\langle S\rangle_B = \{0\}$. If S is a subalgebra of X, then $\langle S\rangle_B = S$. In particular, $\langle X\rangle_B = X$.

Let $x \in X$. In [9], J. Neggers and H.S. Kim defined $x^n = x^{n-1} * (0 * x)$ for $n \geq 1$ and $x^0 = 0$. Then $x^m * x^n = x^{m-n}$ if $m \geq n$ and $x^m * x^n = 0 * x^{n-m}$ otherwise. In [7], for each $x \in X$, N.C. Gonzaga and J.P. Vilela defined $-x = 0 * x$ and $x^{-n} = (-x)^n$ for each $n \geq 1$. In [5], J.C. Endam and R.C. Teves defined $x^m = 0 * x^{-m}$ for $m \leq -1$. If $m \geq 1$, then $x^m = 0 * (0 * x^m) = 0 * x^{-m}$. In effect, $x^m = 0 * x^{-m}$ for any $m \in \mathbb{Z}$. Furthermore, in [7], we have $x^m * x^n = x^{m-n}$, $(x^m)^n = x^{mn}$ for all $m, n \in \mathbb{Z}$, and $\langle x\rangle_B = \{x^n : n \in \mathbb{Z}\}$. If there exists a positive integer n such that $x^n = 0$, then the smallest such positive integer is denoted by $|x|_B$. If no such positive integer n exists, then we say that $|x|_B$ is infinite. If $A \subseteq X$, then we denote $|A|_B$ as the cardinality of A.

Let H and K be subalgebras of X. In [4], we define the subset HK of X to be the set $HK = \{x \in X : x = h * (0 * k) \text{ for some } h \in H, k \in K\}$. Clearly, we have $H \subseteq HK$, $H \subseteq KH$, $K \subseteq HK$, and $K \subseteq KH$. Moreover, if $H \subseteq K$, then $HK = KH = K$. Also, HK is a subalgebra of X if and only if $HK = KH$ if and only if $HK = \langle H \cup K\rangle_B$. A B-algebra X is called a *cyclic B-algebra* [7] if there exists $x \in X$ such that $X = \langle x\rangle_B$. Every cyclic B-algebra is commutative, but the converse need not be true. In [5], if $X = \langle x\rangle_B$ is a cyclic B-algebra with $|X|_B = m > 1$ and if H is a nontrivial subalgebra of X, then $H = \langle x^k\rangle_B$ for some integer $k > 1$ such that k divides m and $|H|_B$ divides m. Furthermore, for every positive divisor d of m, there exists a unique subalgebra H of X with $|H|_B = d$.

Let H be a subalgebra of X and $x \in X$. Let $xH = \{x * (0 * h) : h \in H\}$ and $Hx = \{h * (0 * x) : h \in H\}$, called the *left* and *right B-cosets* of H in X, respectively. If X is commutative, then $xH = Hx$ for all $x \in X$. Observe that $0H = H = H0$ and $x = x * (0 * 0) \in xH$ and $x = 0 * (0 * x) \in Hx$. It is easy to see that $xH = H$ if and only if $x \in H$.

Theorem 2.3. *[2] Let H be a subalgebra of X and $a, b \in X$. Then*
i. $aH = bH$ if and only if $(0 * b) * (0 * a) \in H$
ii. $Ha = Hb$ if and only if $a * b \in H$.

In [2], if H is a subalgebra of X, then $\{xH : x \in X\}$ forms a partition of X and there is a one-one correspondence of the set of all left B-cosets of H in X onto the set of all right B-cosets of H in X. Thus, we define the number of distinct left (or right) B-cosets, written $[X : H]_B$, of H in X as the *index* of H in X. If X is finite, then clearly $[X : H]_B$ is finite.

Theorem 2.4. *[2] (Lagrange's Theorem for B-algebras) Let H be a subalgebra of a finite B-algebra X. Then $|X|_B = [X : H]_B |H|_B$.*

Corollary 2.5. *[2] Let $|X|_B = p$, where p is prime. Then X is cyclic.*

Theorem 2.6. *[2] If H, K are finite subalgebras of X, then $|HK|_B = \dfrac{|H|_B |K|_B}{|H \cap K|_B}$.*

3 Some Implications of Lagrange's Theorem for B-algebras We now prove some results where Lagrange's Theorem plays a role.

Proposition 3.1. *Let X be a noncyclic B-algebra with $|X|_B = p^2$, where p is prime. Then $|x|_B = p$ for every nonzero $x \in X$.*

Proof. Let $x \in X$ and $x \neq 0$. By Lagrange's Theorem, $|x|_B$ divides $|X|_B = p^2$. Hence, $|x|_B$ is equal to 1, p, or p^2. If $|x|_B = p^2$, then $\langle x \rangle_B = X$ and so X is cyclic, a contradiction. Since $x \neq 0$, $|x|_B \neq 1$. Thus, $|x|_B = p$. □

Proposition 3.2. *If X is a B-algebra with prime order, then X has only the trivial subalgebras.*

Proof. Suppose that $|X|_B = p$, where p is prime. Let H be a subalgebra of X. By Lagrange's Theorem, $|H|_B$ is 1 or p. Thus, $H = \{0\}$ or $H = X$. □

Proposition 3.3. *Let $|X|_B = p^n$, where p is prime and $n \geq 1$. Then X contains an element of order p.*

Proof. Let $x \in X$ and $x \neq 0$. Then $H = \langle x \rangle_B$ is a cyclic subalgebra of X. By Lagrange's Theorem, $|H|_B$ divides $|X|_B = p^n$. Hence, $|H|_B = p^m$ for some $m \in \mathbb{Z}$, $0 < m \leq n$. It follows that for every divisor d of p^m, there exists a subalgebra of order d. In particular, for p, there exists a subalgebra K of H such that $|K|_B = p$. By Corollary 2.5, K is cyclic and so there exists $y \in K$ such that $K = \langle y \rangle_B$ and y is of order p. Hence, X contains an element of order p. □

Proposition 3.4. *Let X be a finite commutative B-algebra such that X contains two distinct elements of order 2. Then $|X|_B$ is a multiple of 4.*

Proof. Let x and y be two distinct elements of order 2. Let $H = \{0, x\}$ and $K = \{0, y\}$. Now, H and K are subalgebras of X. Since X is commutative, $HK = \{0, x, y, x * (0 * y)\}$ is a subalgebra of X of order 4. By Lagrange's Theorem, $|HK|_B = 4$ divides $|X|_B$. Thus, $|X|_B$ is a multiple of 4. □

The above result need not be true if X is not commutative. For instance, consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Note that X is not commutative. Now, 3 and 4 are elements of X with $|3|_B = 2$ and $|4|_B = 2$. However, 4 does not divide $|X|_B = 6$.

Proposition 3.5. *Let X be a B-algebra with $|X|_B = pq$, where p and q are prime numbers. Then every proper subalgebra of X is cyclic.*

Proof. Let H be a proper subalgebra of X. By Lagrange's Theorem, $|H|_B$ is 1, p, q, or pq. Since H is proper, $|H|_B$ is p or q. By Corollary 2.5, H is cyclic. □

Proposition 3.6. *Let H and K be subalgebras of a finite B-algebra X such that $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$. Then $|H \cap K|_B > 1$.*

Proof. Suppose that H and K are subalgebras of a finite B-algebra X such that $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$. By Theorem 2.6, $|H \cap K|_B = \dfrac{|H|_B |K|_B}{|HK|_B}$. Since $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$, it follows that $|H|_B |K|_B > |X|_B$. Since $|HK|_B \leq |X|_B$, it follows that $\dfrac{|X|_B}{|HK|_B} \geq 1$. Therefore, $|H \cap K|_B = \dfrac{|H|_B |K|_B}{|HK|_B} > \dfrac{|X|_B}{|HK|_B} \geq 1$. □

Proposition 3.7. *Let $|X|_B = pq$, where p and q are distinct primes with $p > q$. Then X has at most one subalgebra of order p.*

Proof. Suppose that H and K are subalgebras with $|H|_B = p = |K|_B$. Then $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$. By Proposition 3.6, $|H \cap K|_B > 1$. Thus, $|H \cap K|_B = p$ and so $H = K$. □

4 Cauchy's Theorem for B-algebras This section establishes the Cauchy's Theorem for B-algebras and it also provides some implications of this theorem. We start with a simple observation given in the following lemma.

Lemma 4.1. *Let $a \in X$. Then $a \in Z(X)$ if and only if $[X : C(a)]_B = 1$ if and only if $C(a) = X$.*

Let $a \in X$. An element $b \in X$ is said to be a *conjugate of* a in X if there exists $c \in X$ such that $b = c * (c * a)$. Let $R = \{(a, b) \in X \times X \colon b \text{ is a conjugate of } a\}$.

Theorem 4.2. *Let $a \in X$. Then the relation R on X is an equivalence relation.*

Proof. Since $a = 0 * (0 * a)$, a is conjugate to a. Thus, R is reflexive. Let $(a, b) \in R$. Then there exists $c \in X$ such that $b = c * (c * a)$. Multiplying both sides by $0 * c$ twice, we have $(0 * c) * ((0 * c) * b) = (0 * c) * [(0 * c) * (c * (c * a))]$. By (P2), (P3), (I), and (P1), we obtain

$$\begin{aligned}
(0 * c) * ((0 * c) * b) &= (0 * c) * [(0 * c) * (c * (c * a))] \\
&= (0 * c) * [((0 * c) * (0 * (c * a))) * c] \\
&= (0 * c) * [((0 * c) * (a * c)) * c] \\
&= (0 * c) * [(0 * a) * c)] \\
&= ((0 * c) * (0 * c)) * (0 * a) \\
&= 0 * (0 * a) \\
&= a.
\end{aligned}$$

Hence, a is conjugate to b. Thus, R is symmetric. Let $(a, b), (b, c) \in R$. Then there exist $u, v \in X$ such that $b = u * (u * a)$ and $c = v * (v * b)$. Now, by (P2) and (P3), we obtain

$$\begin{aligned}
c &= v * (v * b) \\
&= v * [v * (u * (u * a))] \\
&= v * [(v * (0 * (u * a))) * u] \\
&= v * [(v * (a * u)) * u] \\
&= (v * (0 * u)) * (v * (a * u)) \\
&= (v * (0 * u) * [(v * (0 * u)) * a]
\end{aligned}$$

Hence, $(a, c) \in R$ and so R is transitive. Therefore, R is an equivalence relation on X. □

The equivalence relation R in Theorem 4.2 is called *conjugacy* on X. The equivalence class of $a \in X$, denoted by $[a]_c$, of the relation R is called the *conjugacy class of* a in X.

Example 4.3. Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Then there are three distinct conjugacy classes in X, namely, $[0]_c = \{0\}, [1]_c = [2]_c = \{1, 2\}, [3]_c = [4]_c = [5]_c = \{3, 4, 5\}$.

Remark 4.4. *Let $a \in X$. Then $a \in Z(X)$ if and only if $[a]_c = \{a\}$.*

The following theorem shows that the number of conjugates of a is equal to the index of $C(a)$ in X.

Theorem 4.5. *Let $a \in X$. Then $|[a]_c|_B = [X : C(a)]_B$.*

Proof. Let $a \in X$. Let $\mathcal{L}$ denote the set of all distinct left B-cosets of $C(a)$ in X. Then $|\mathcal{L}|_B = [X : C(a)]_B$. By definition, $b * (b * a) \in [a]_c$ for all $b \in X$. Define $f : \mathcal{L} \to [a]_c$ by $f(bC(a)) = b * (b * a)$. Suppose that $f(bC(a)) = f(cC(a))$. Then by (P2), (P3), (P5), (I), (III), and Theorem 2.3(i), we have

$$\begin{aligned}
f(bC(a)) = f(cC(a)) &\Rightarrow b * (b * a) = c * (c * a)\\
&\Rightarrow 0 * (b * (b * a)) = 0 * (c * (c * a))\\
&\Rightarrow (b * a) * b = (c * a) * c\\
&\Rightarrow (0 * c) * ((b * a) * b) = (0 * c) * ((c * a) * c)\\
&\Rightarrow (0 * c) * ((b * a) * b) = ((0 * c) * (0 * c)) * (c * a)\\
&\Rightarrow (0 * c) * ((b * a) * b) = 0 * (c * a)\\
&\Rightarrow (0 * c) * ((b * a) * b) = a * c\\
&\Rightarrow [(0 * c) * ((b * a) * b)] * (0 * b) = (a * c) * (0 * b)\\
&\Rightarrow [((0 * c) * (0 * b)) * (b * a)] * (0 * b) = a * ((0 * b) * (0 * c))\\
&\Rightarrow ((0 * c) * (0 * b)) * [(0 * b) * (0 * (b * a))] = a * [0 * ((0 * c) * (0 * b))]\\
&\Rightarrow ((0 * c) * (0 * b)) * ((0 * b) * (a * b)) = a * [0 * ((0 * c) * (0 * b))]\\
&\Rightarrow ((0 * c) * (0 * b)) * (0 * a) = a * [0 * ((0 * c) * (0 * b))]\\
&\Rightarrow (0 * c) * (0 * b) \in C(a)\\
&\Rightarrow bC(a) = cC(a).
\end{aligned}$$

Therefore, f is a one-one function. Let $y \in [a]_c$. Then there exists $x \in X$ such that $y = x * (x * a) = f(xC(a))$. Hence, f is onto. Therefore,f is a one-one function from $\mathcal{L}$ onto $[a]_c$. Consequently, $|[a]_c|_B = |\mathcal{L}|_B = [X : C(a)]_B$. □

Corollary 4.6. *Let X be a finite B-algebra. Then $|X|_B = \sum_a [X : C(a)]_B$, where the summation is over a complete set of distinct conjugacy class representatives.*

Proof. By Theorem 4.2, $X = \bigcup_a [a]_c$, where the union runs over a complete set of distinct conjugacy class representatives. Since the distinct conjugacy classes are mutually disjoint, we have $|X|_B = \left|\bigcup_a [a]_c\right|_B = \sum_a |[a]_c|_B$. By Theorem 4.5, it follows that $|X|_B = \sum_a [X : C(a)]_B$, where the summation is over a complete set of distinct conjugacy class representatives. □

Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Then $|X|_B = 6 = 1+2+3 = |[0]_c|_B + |[1]_c|_B + |[3]_c|_B = \sum_a |[a]_c|_B = \sum_a [X : C(a)]_B$.

Corollary 4.7. *If X is a finite B-algebra, then $|X|_B = |Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B$, where the summation runs over a complete set of distinct conjugacy class representatives, which do not belong to $Z(X)$.*

Proof. By Corollary 4.6, $|X|_B = \sum_a [X : C(a)]_B$, where the summation is over a complete set of distinct conjugacy class representatives. Thus, we have $|X|_B = \sum_{a \in Z(X)} [X : C(a)]_B +$

$\sum_{a \notin Z(X)} [X : C(a)]_B$. By Lemma 4.1, we have $\sum_{a \in Z(X)} [X : C(a)]_B = |Z(X)|_B$. Hence, $|X|_B = |Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B$, where the summation runs over a complete set of distinct conjugacy class representatives which do not belong to $Z(X)$. □

Example 4.8. Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Then $Z(X) = \{0\}$. Hence, $|Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B = 1 + |[1]_c|_B + |[3]_c|_B = 1 + 2 + 3 = 6 = |X|_B$.

We now prove a partial converse of Lagrange's Theorem.

Lemma 4.9. *If X is a finite commutative B-algebra with $|X|_B = n$ such that n is divisible by a prime p, then X contains an element of order p and hence a subalgebra of order p.*

Proof. We proceed by induction on the order of X. If $|X|_B = p$ where p is prime, then every element of X (except 0) has order p. Thus, in particular, the lemma is true when $|X|_B = 2$. Suppose that the lemma is true for all B-algebras of order r, where $2 \leq r < n$. Suppose that X is a B-algebra of order n. Let $a \in X$ with $a \neq 0$ and let $|a|_B = m$. Then either $p|m$ or $p \nmid m$. If $p|m$, then $m = pk$ for some $k \in \mathbb{Z}^+$. In this case, $(a^k)^p = a^m = 0$. Hence, $a^k \neq 0$ and $|a^k|_B = p$. Suppose $p \nmid m$. Since X is commutative, the cyclic subalgebra $H = \langle a \rangle_B$ of X is a normal subalgebra of X. By Lagrange's Theorem, $|X|_B = m[X : H]_B$. Since $p \nmid m$, we have $p|[X : H]_B = |X/H|_B$. Since $|X/H|_B < n$, there exists $bH \in X/H$ s.t. $|bH|_B = p$. Now, $b^p H = (bH)^p = H$. Hence, $b^p \in H$. Thus, $(b^m)^p = (b^p)^m = 0$ and so either $b^m = 0$ or $|b^m|_B = p$. If $b^m = 0$, then $(bH)^m = H$ which implies $p|m$, a contradiction. Therefore, $|b^m|_B = p$ and so b^m is the desired element of X. □

Theorem 4.10. *(Cauchy's Theorem for B-algebras) Let X be a finite B-algebra with $|X|_B = n$ such that n is divisible by a prime p. Then X contains an element of order p and hence a subalgebra of order p.*

Proof. We proceed by induction on the order of X. If $n = 2$, then X is commutative and the result follows from Lemma 4.9. Suppose that the theorem is true for all B-algebras of order m s.t. $2 \leq m < n$. By Corollary 4.7, $|X|_B = |Z(X)|_B + \sum_{a \notin Z(x)} [X : C(a)]$. If $X = Z(X)$, then X is commutative and the result follows from Lemma 4.9. If $X \neq Z(X)$, then there exists $a \in X$ s.t. $a \notin Z(X)$. Then $X \neq C(a)$ and so $[X : C(a)]_B > 1$. By Lagrange's Theorem, $|X|_B = [X : C(a)]_B |C(a)|_B > |C(a)|_B$. If $p||C(a)|_B$, then $C(a)$ has an element of order p and so X has an element of order p. If $p \nmid |C(a)|_B$ for all $a \notin Z(X)$, then $p|[X : C(a)]_B$ for all $a \notin Z(X)$. Since p divides each term of the summation and also divides $|X|_B$, we have $p||Z(X)|$. By Lemma 4.9, X contains an element of order p and hence a subalgebra of order p □

The following theorem proves that the converse of Lagrange's Theorem for B-algebras hold for finite commutative B-algebras.

Theorem 4.11. *Let X be a finite commutative B-algebra with $|X|_B = n$. If $m \in \mathbb{Z}^+$ such that $m|n$, then X has a subalgebra of order m.*

Proof. If $m = 1$, then $\{0\}$ is the required subalgebra of order m. If $n = 1$, then $m = n = 1$ and the result follows easily. Assume that $m > 1$ and $n > 1$. We proceed by induction on n. If $n = 2$, then $m = 2$ and X is the required subalgebra of order m. Suppose that the theorem is true for all finite commutative B-algebras of order k s.t. $2 \leq k < n$. Let

p be a prime integer s.t. $p|m$. Then there exists $m_1 \in \mathbb{Z}^+$ s.t. $m = pm_1$. By Cauchy's Theorem, X has a subalgebra H of order p. Since X is commutative, H is normal and X/H is a B-algebra. Now, $1 \leq |X/H|_B = \frac{|X|_B}{|H|_B} < |X|_B$ and $|X/H|_B = \frac{n}{p}$. Now, $n = mm_2$ for some $m_2 \in \mathbb{Z}^+$. Thus, $|X/H|_B = \frac{pm_1m_2}{p} = m_1m_2$ and so m_1 divides $|X/H|_B$. Hence, X/H has a subalgebra K/H s.t. $|K/H|_B = m_1$, where K is a subalgebra of X. Now, $|K|_B = |K/H|_B|H|_B = m_1p = m$. Hence, K is a subalgebra of order m. □

As a consequence of Cauchy's Theorem, we now introduce the concept of B_p-algebras.

Definition 4.12. Let p be a prime number. A B-algebra X is called a *B_p-algebra* if the order of each element of X is a power of p. A subalgebra H of a B-algebra X is called *B_p-subalgebra* if H is a B_p-algebra.

The B-algebra in Example 2.1 is B_3-algebra. We now prove some results where Cauchy's Theorem plays a role. The following theorem provides a necessary and sufficient condition for a finite B-algebra to be a B_p-algebra.

Theorem 4.13. *Let X be a nontrivial B-algebra. Then X is a finite B_p-algebra if and only if $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$.*

Proof. Suppose that X is a finite B_p-algebra. If $q||X|_B$ for some prime $q \neq p$, then by Cauchy's Theorem, X has an element of order q, a contradiction. Thus, p is the only prime divisor of $|X|_B$, that is, $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$. Conversely, suppose that $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$. Then by Lagrange's Theorem, the order of each element of X is a power of p. Therefore, X is a finite B_p-algebra. □

The following theorem shows that the center of a B_p-algebra is nontrivial.

Theorem 4.14. *If X is a finite B_p-algebra with $|X|_B > 1$, then $|Z(G)|_B > 1$.*

Proof. Suppose that X is a finite B_p-algebra with $|X|_B$ >1. If $X = Z(X)$, then $|Z(X)|_B = |X|_B > 1$. Suppose that $Z(X) \subset X$ and consider $a \in X$ such that $a \notin Z(X)$. Then $C(a)$ is a proper subalgebra of a B_p-algebra X. By Theorem 4.13, $p||X|_B$. It follows that $p|[X : C(a)]_B$ for all $a \notin Z(X)$. Thus, p divides $\sum\limits_{a\notin Z(X)} [X : C(a)]_B$. By Corollary 4.7, $|X|_B = |Z(X)|_B + \sum\limits_{a\notin Z(X)} [X : C(a)]_B$. Since $p||X|_B$ and $p|\sum\limits_{a\notin Z(X)} [X : C(a)]_B$, it follows that $p||Z(X)|_B$. Therefore, $|Z(X)|_B > 1$. □

Corollary 4.15. *If $|X|_B = p^2$, where p is prime, then X is commutative.*

Proof. Suppose that $|X|_B = p^2$, where p is prime. By Theorem 4.14, $|Z(X)|_B > 1$. Since $Z(X)$ is a subalgebra, $|Z(X)|_B$ divides p^2 by Lagrange's Theorem. Hence, $|Z(X)|_B$ is p or p^2. If $|Z(X)|_B = p$. Then $Z(X) \neq X$ and so there exists $a \in X$ such that $a \notin Z(X)$. In [6], $C(a)$ is a subalgebra of X with $a \in C(a)$. Hence, $Z(X) \subset C(a)$. This implies that $|C(a)|_B = p^2$. Thus, $X = C(a)$ and so $a \in Z(X)$, a contradiction. Therefore, $|Z(X)|_B = p^2$ and so $X = Z(X)$. Consequently, X is commutative. □

Proposition 4.16. *Let H and K be subalgebras of a commutative B-algebra X. If $|H|_B = m$ and $|K|_B = n$, then X has a subalgebra of order $lcm(m,n)$.*

Proof. Let H and K be subalgebras of a commutative B-algebra X with $|H|_B = m$ and $|K|_B = n$. Since $HK = KH$, HK is a subalgebra of X. Since H and K are finite, H and K are subalgebras of a finite B-algebra HK. By Lagrange's Theorem, $m||HK|_B$ and $n||HK|_B$. Hence, $lcm(m,n)||HK|_B$. By Theorem 4.11, HK has a subalgebra of order $lcm(m,n)$ and so X has a subalgebra of order $lcm(m,n)$. □

The version of Lagrange's Theorem for B-algebras in [2] is analogue to the Lagrange's Theorem for groups, and the version of Cauchy's Theorem for B-algebras in this paper is analogue to the Cauchy's Theorem for groups. It is then natural to seek an analogue results to the Sylow Theorems for groups.

Acknowledgement. The authors would like to thank the referee for the remarks and suggestions which were incorporated into this revised version. This research is partially funded by the Commission on Higher Education (CHED), Philippines.

REFERENCES

[1] Allen, P.J., Neggers, J., and Kim, H.S., *B-algebras and groups*, Sci. Math. Japn., **9**(2003), 159-165.

[2] Bantug, J.S. and Endam, J.C., *Lagrange's Theorem for B-algebras*, Int. J. of Alg., **11**(1)(2017), 15-23.

[3] Cho, J.R. and Kim, H.S., *On B-algebras and quasigroups*, Quasigroups and Related Systems, **8**(2001), 1-6.

[4] Endam, J.C. and Vilela, J.P., *The Second Isomorphism Theorem for B-algebras*, App. Math. Sci., **8**(38)(2014), 1865-1872.

[5] Endam, J.C. and Teves, R.C., *Some Properties of Cyclic B-algebras*, Int. Math. Forum, **11**(8)(2016), 387-394.

[6] Endam, J.C., *Centralizer and Normalizer of B-algebras*, Sci. Math. Japn., (2016).

[7] Gonzaga, N.C. and Vilela, J.P., *On Cyclic B-algebras*, App. Math. Sci., **9**(111)(2015), 5507-5522.

[8] Kim, H.S. and Park, H.G., *On 0-commutative B-algebras*, Sci. Math. Japn., (2005), 31-36.

[9] Neggers, J. and Kim, H.S., *On B-algebras*, Mat. Vesnik, **54**(2002), 21-29.

[10] Neggers, J. and Kim, H.S., *A fundamental theorem of B-homomorphism for B-algebras*, Int. Math. J., **2**(3)(2002), 207-214.

[11] Walendziak, A., *Some Axiomatizations of B-algebras*, Math. Slovaca, **56**(3)(2006), 301-306.

[12] Walendziak, A., *A note on normal subalgebras in B-algebras*, Sci. Math. Japn., (2005), 49-53.

Communicated by *Klaus Denecke*

[1]Department of Mathematics
Negros Oriental State University
Kagawasan Ave., Dumaguete City, PHILIPPINES
jc_endam@yahoo.com

[2]Department of Mathematics
Silliman University
Hibbard Ave., Dumaguete City, PHILIPPINES
jenettesbantug@su.edu.ph

TRANSFORMS ON OPERATOR MONOTONE FUNCTIONS

Masaru Nagisa

Received August 31, 2017 ; revised December 7, 2017

Abstract. Let f be an operator monotone function on $[0,\infty)$ with $f(t) \geq 0$ and $f(1)=1$. If $f(t)$ is neither the constant function 1 nor the identity function t, then

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))} \qquad t \geq 0$$

is also operator monotone on $[0,\infty)$, where $a, b \geq 0$ and

$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \geq 0.$$

Moreover, we show some extensions of this statement.

1 Introduction and History We call a real continuous function $f(t)$ on an interval I operator monotone on I (in short, $f \in \mathbb{P}(I)$), if $A \leq B$ implies $f(A) \leq f(B)$ for any self-adjoint matrices A, B with their spectrum containd in I. In this paper, we consider only the case $I=[0,\infty)$ or $I=(0,\infty)$. We denote $f \in \mathbb{P}_+(I)$ if $f \in \mathbb{P}(I)$ satisfies $f(t) \geq 0$ for any $t \in I$.

Let $\mathbb{H}_+$ be the upper-half plain of $\mathbb{C}$, that is,

$$\mathbb{H}_+ = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\} = \{z \in \mathbb{C} \mid |z| > 0,\ 0 < \arg z < \pi\},$$

where $\mathrm{Im} z$ (resp. $\arg z$) means the imaginary part (resp. the argument) of z. As Loewner's theorem, it is known that f is operator monotone on I if and only if f has an analytic continuation to $\mathbb{H}_+$ that maps $\mathbb{H}_+$ into its closure $\overline{\mathbb{H}_+}$ and also has an analytic continuation to the lower half-plane $\mathbb{H}_-$, obtained by the reflection across I. (see [1],[3],[5]).

D. Petz [11] proved that an operator monotone function $f : [0,\infty) \longrightarrow [0,\infty)$ satisfying the functional equation

$$f(t) = tf(t^{-1}) \qquad t \geq 0$$

2010 *Mathematics Subject Classification.* 15A45, 15A60, 47A63.

Key words and phrases. operator monotone functions, Pick functions, Petz-Hasegawa's functions.

is related to a Morozova-Chentsov function [9] which gives a monotone metric on the manifold of $n \times n$ density matrices. In the work [12], the concrete functions (Petz-Hasegawa's functions)

$$f_a(t) = a(1-a)\frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \qquad (-1 \le a \le 2)$$

appeared and their operator monotonicity was proved (see also [2]). V.E.S. Szabo introduced an interesting idea for checking their operator monotonicity in [13], but his idea was something strange. We use a similar idea in our argument. M. Uchiyama [14] proved the operator monotonicity of the following extended functions:

$$\frac{(t-a)(t-b)}{(t^p-a^p)(t^{1-p}-b^{1-t})} \in \mathbb{P}_+[0.\infty)$$

for $0 < p < 1$ and $a, b > 0$. The main result of this paper is as follows:

Theorem 1. *Let a and b be non-negative real numbers. If $f \in \mathbb{P}_+[0,\infty)$ and both f and $f^\sharp$ are not constant, then*

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))} \in \mathbb{P}_+[0.\infty),$$

where

$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \ge 0.$$

The proof of this statement was given in [7] by the author and his student, M. Kawasaki. We also made its revised version as [8]. But these manuscripts have been unpublished. F. Hansen [4] has inspired by [8] and given the different proof of the above statement in the case $a = b = 1$. The proof in this paper was based on the theory of Complex Analysis and we have considered this method useful. The same method was introduced for the proof of the operator monotonicity of Petz-Hasegawa's functions in [6]. Also the autor and S. Wada have extended the method and succeeded to justify Szabo's result in a sense [10]. The last section we will prove main theorem and give some applications.

2 Main result For $f \in \mathbb{P}[0,\infty)$, we have the following integral representation:

$$f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{z+\lambda} dw(\lambda),$$

where $\beta \geq 0$ and

$$\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < \infty$$

(see [1]). When $f(0) \geq 0$ (i.e., $f \in \mathbb{P}_+[0,\infty)$), it holds that $0 < \arg f(z) \leq \arg z$ whenever $0 < \arg z < \pi$.

For any $f \in \mathbb{P}_+[0,\infty)$ ($f \neq 0$), we define $f^\sharp$ as follows:

$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \in [0,\infty).$$

Then it is well-known $f^\sharp \in \mathbb{P}_+[0,\infty)$.

Proposition 2. *Let f be an operator monotone function on $(0,\infty)$ and a be a positive real number.*

(1) *When $f(t)$ is not constant, we have*

$$g_1(t) = \frac{t-a}{f(t)-f(a)} \in \mathbb{P}_+[0,\infty).$$

(2) *When $f(t) \geq 0$ for $t \geq 0$, we have*

$$g_2(t) = \frac{f(t)(t-a)}{tf(t)-af(a)} \in \mathbb{P}_+[0,\infty).$$

Proof. (1) It has proved in [14]. We state the outline of the proof. For $f \in \mathbb{P}_+(0,\infty)$, we have

$$f(z) = \alpha + \beta z + \int_0^\infty \left(-\frac{1}{x+z} + \frac{x}{x^2+1}\right) d\nu(x) \quad (\alpha \in \mathbb{R}, \beta \geq 0)$$

for $z \in \mathbb{H}_+ \cup (0,\infty)$ by Loewner's theorem ([1], [3], [6]). Since

$$g_1(z) = \frac{z-a}{f(z)-f(a)} = \frac{1}{\beta + \int_0^\infty \frac{1}{(x+z)(x+a)} d\nu(x)},$$

we have

$$\begin{aligned} \mathrm{Im} g_1(z) &= \frac{-1}{|g_1(z)|^2} \mathrm{Im}\left(\beta + \int_0^\infty \frac{1}{(x+z)(x+a)} d\nu(x)\right) \\ &= \frac{-1}{|g_1(z)|^2} \int_0^\infty \frac{1}{x+a} \mathrm{Im} \frac{1}{x+z} d\nu(x) > 0 \end{aligned}$$

for $z \in \mathbb{H}_+$. This implies $g_1 \in \mathbb{P}_+[0,\infty)$.

(2) Since $g_2([0,\infty)) \subset [0,\infty)$, it suffices to show that $g_2(\mathbb{H}_+) \subset \mathbb{H}_+$. By the calculation

$$\begin{aligned} g_2(z) &= \frac{zf(z) - af(a) + af(a) - f(z)a}{zf(z) - af(a)} = 1 - \frac{a(f(z) - f(a))}{zf(z) - af(a)} \\ &= 1 - \frac{a}{\dfrac{zf(z) - af(a)}{f(z) - f(a)}} = 1 - \frac{a}{z + f(a)g_1(z)}, \end{aligned}$$

we have

$$\mathrm{Im} g_2(z) = -\mathrm{Im}\frac{a}{z + f(a)g_1(z)} = \mathrm{Im}\frac{a(z + f(a)g_1(z))}{|z + f(a)g_1(a)|^2}.$$

When $z \in \mathbb{H}_+$, $\mathrm{Im} g_1(z) > 0$ by (1) and $\mathrm{Im} g_2(z) > 0$. So $g_2(t)$ belongs to $\mathbb{P}_+[0,\infty)$. □

Lemma 3. *Let $a \geq 0$ and $0 < p < 1$. If f is a non-constant operator monotone function on $(0,\infty)$, then we have*

$$f(t^p) - f(a^p) \neq 0 \text{ and } tf(t^p) - af(a^p) \neq 0$$

for any $t \in (-\infty, 0)$.

Proof. When f is operator monotone and not constant, we have $\mathrm{Im} f(z) > 0$ for any $z \in \mathbb{H}_+$ by the maximum principle for the harmonic function $\mathrm{Im} f$ on $\mathbb{H}_+$. For any $t = |t|e^{i\pi} \in (-\infty, 0)$, we have $t^p \in \mathbb{H}_+$ and $\mathrm{Im} f(t^p) > 0$. This implies

$$\mathrm{Im} f(t^p) \neq \mathrm{Im} f(a^p) = 0 \text{ and } \mathrm{Im} f(t^p) \neq \mathrm{Im}\frac{af(a^p)}{t} = 0,$$

that is, $f(t^p) - f(a^p) \neq 0$ and $tf(t^p) - af(a^p) \neq 0$. □

Lemma 4. *For any $z \in \mathbb{H}_+$ and a positive integer n $(n \geq 2)$, we have*

$$\arg z < \arg(z - l) < \frac{\pi + (n-1)\arg z}{n} \quad \text{if} \quad 0 < l \leq \frac{|z|}{n-1}.$$

Proof. It is clear that $\arg z < \arg(z - l)$ for $z \in \mathbb{H}_+$ and $l > 0$.

It suffices to show that, for $z = e^{i\theta}$ $(0 < \theta < \pi)$,

$$\arg(z - l) < \frac{\pi + (n-1)\theta}{n} \quad \text{if } 0 < l \leq \frac{1}{n-1}.$$

We set

$$w = \frac{\sin\theta}{\sin\frac{\pi+(n-1)\theta}{n}} e^{i(\pi+(n-1)\theta)/n}.$$

Then we have $\mathrm{Im}z = \mathrm{Im}w$ and

$$0 < z - w = \cos\theta - \frac{\sin\theta}{\sin\frac{\pi+(n-1)\theta}{n}} \cos\frac{\pi+(n-1)\theta}{n} = \frac{\sin\frac{\pi-\theta}{n}}{\sin\frac{\pi+(n-1)\theta}{n}}.$$

By the estimation

$$\inf\{z - w \mid 0 < \theta < \pi\} = \inf\{\frac{\sin\frac{\pi-\theta}{n}}{\sin\frac{\pi+(n-1)\theta}{n}} \mid 0 < \theta < \pi\}$$
$$= \inf\{\frac{\sin t}{\sin(\pi-(n-1)t)} \mid 0 < t < \frac{\pi}{n}\} = \inf\{\frac{\sin t}{\sin(n-1)t} \mid 0 < t < \frac{\pi}{n}\} = \frac{1}{n-1},$$

we can get the desired result. □

Now we can prove the following theorem and remark that Theorem 1 follows from this statement because $f(t)f^\sharp(t)/t = 1 \in \mathbb{P}_+[0,\infty)$:

Theorem 5. *Let n be a positive integer, $a, b, b_1, \ldots, b_n \geq 0$ and $f, g, g_1, \ldots, g_n$ be non-constant, non-negative operator monotone functions on $[0,\infty)$.*

(1) *If $\dfrac{f(t)g(t)}{t}$ is operator monotone on $[0,\infty)$, then the function*

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(g(t)-g(b))}$$

is operator monotone on $[0,\infty)$ for any $a, b \geq 0$.

(2) *If $\dfrac{f(t)}{\prod_{i=1}^n g_i(t)}$ is operator monotone on $[0,\infty)$, then the function*

$$h(t) = \frac{(t-a)}{(f(t)-f(a))} \prod_{i=1}^n \frac{g_i(t)(t-b_i)}{tg_i(t) - b_i g_i(b_i)}$$

is operator monotone on $[0,\infty)$ for any $a, b \geq 0$.

Proof. (1) By $f, g \in \mathbb{P}_+[0,\infty)$ and Proposition 2 (1),

$$\frac{t-a}{f(t)-f(a)} \text{ and } \frac{t-b}{g(t)-g(b)}$$

are operator monotone on $[0,\infty)$. Therefore

$$h(z)=\frac{(z-a)(z-b)}{(f(z)-f(a))(g(z)-g(b))}$$

is holomorphic on $\mathbb{H}_+$, continuous on $\mathbb{H}_+\cup[0,\infty)$ and satisfies $h([0,\infty))\subset[0,\infty)$ and

$$\arg h(z)=\arg\frac{z-a}{f(z)-f(a)}+\arg\frac{z-b}{g(z)-g(b)}>0 \text{ for } z\in\mathbb{H}_+.$$

We assume that $f(z)$ and $g(z)$ are continuous on the closure $\overline{\mathbb{H}_+}$ of $\mathbb{H}_+$ and

$$f(t)-f(a)\neq 0 \text{ and } g(t)-g(b)\neq 0 \text{ for any } t\in(-\infty,0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

In the case $z\in(-\infty,0)$, i.e., $|z|>0$ and $\arg z=\pi$, we have

$$\begin{aligned}&\arg h(z)\\ =&\arg(z-a)-\arg(f(z)-f(a))+\arg(z-b)-\arg(g(z)-g(b))\\ \leq&\pi-\arg f(z)+\pi-\arg g(z)\\ \leq&2\pi-\arg z=\pi \qquad (\text{since } \arg f(z)+\arg g(z)-\arg z\geq 0).\end{aligned}$$

So it holds $0\leq\arg h(z)\leq\pi$.

In the case that $z\in\mathbb{H}_+$ satisifying $|z|>\max\{a,b\}$, it holds that

$$\arg(z-a),\arg(z-b)<\frac{\pi+\arg z}{2}$$

by Lemma 4 (as $l=\max\{a,b\}$ and $n=2$). Since

$$\begin{aligned}\arg h(z)&=\arg(z-a)-\arg(f(z)-f(a))+\arg(z-b)-\arg(g(z)-g(b))\\ &<\frac{\pi+\arg z}{2}-\arg f(z)+\frac{\pi+\arg z}{2}-\arg g(z)\\ &=\pi+\arg z-\arg f(z)-\arg g(z)\leq\pi,\end{aligned}$$

we have $0<\arg h(z)<\pi$.

For $r>0$, we define $H(r)=\{z\in\mathbb{C}\mid |z|\leq r,\mathrm{Im}z\geq 0\}$. Whenever $r>l=\max\{a,b\}$, we can get

$$0\leq\arg h(z)\leq\pi$$

on the boundary of $H(r)$. Since $h(z)$ is holomorphic on the interior $H(r)^\circ$ of $H(r)$ and continuous on $H(r)$, $\mathrm{Im}h(z)$ is harmonic on $H(r)^\circ$ and continuous on

$H(r)$. Because $\mathrm{Im}h(z) \geq 0$ on the boundary of $H(r)$, we have $h(H(r)) \subset \overline{\mathbb{H}_+}$ by the minimum principle of harmonic functions. This implies

$$h(\overline{\mathbb{H}_+}) = h(\bigcup_{r>l} H(r)) \subset \bigcup_{r>l} h(H(r)) \subset \overline{\mathbb{H}_+},$$

and $h \in \mathbb{P}_+[0,\infty)$.

In general case, we define f_p and g_p as follows $(0 < p < 1)$:

$$f_p(t) = f(t^p) \text{ and } g_p(t) = t^{1-p}g(t^p).$$

It is clear $f_p \in \mathbb{P}_+[0,\infty)$. When $0 < \arg z < \pi$, we have

$$\begin{gathered}\arg g_p(z) \geq (1-p)\arg z > 0 \text{ and}\\ \arg g_p(z) = (1-p)\arg z + \arg g(z^p) \leq (1-p)\arg z + p\arg z < \pi.\end{gathered}$$

So $g_p \in \mathbb{P}_+[0,\infty)$. For any $t \in (-\infty, 0)$, we have $f_p(t) - f_p(a) \neq 0$ by Lemma 3 and $g_p(t) - g_p(a) \neq 0$ because $t = |t|e^{\pi i}$ and $(1-p)\pi < \arg g_p(t) \leq \pi$. So we have

$$h_p(z) = \frac{(z-a)(z-b)}{(f_p(z) - f_p(a))(g_p(z) - g_p(b))}$$

is holomorphic on $\mathbb{H}_+$ and continuous on $\overline{\mathbb{H}_+}$. Since $\frac{f(t)g(t)}{t} \in \mathbb{P}_+[0,\infty)$,

$$\frac{f_p(t)g_p(t)}{t} = \frac{f(t^p)t^{1-p}g(t^p)}{t} = \frac{f(t^p)g(t^p)}{t^p}$$

also belongs to $\mathbb{P}_+[0,\infty)$. By the above argument, we have $h_p \in \mathbb{P}_+[0,\infty)$.

Since

$$\begin{aligned} h_p(t) &= \frac{(t-a)(t-b)}{(f_p(t) - f_p(a))(g_p(t) - g_p(b))} \\ &= \frac{(t-a)(t-b)}{(f(t^p) - f(a^p))(t^{1-p}g(t^p) - b^{1-p}g(b^p))} \qquad \text{for } t \geq 0, \end{aligned}$$

we have

$$\lim_{p \to 1-0} h_p(t) = h(t).$$

So we can get the operator monotonicity of $h(t)$.

(2) We show this by the similar way as (1). By Proposition 2,

$$\frac{t-a}{f(t) - f(a)} \text{ and } \frac{g_i(t)(t-b_i)}{tg_i(t) - b_ig_i(b_i)} \quad (i = 1, 2, \ldots, n)$$

are operator monotone on $[0,\infty)$. So we have that

$$h(z) = \frac{z-a}{f(z)-f(a)} \prod_{i=1}^{n} \frac{g_i(z)(z-b_i)}{zg_i(z)-b_ig_i(b_i)}$$

is holomorphic on $\mathbb{H}_+$, continuous on $\mathbb{H}_+ \cup [0,\infty)$ and satisfies $h([0,\infty)) \subset [0,\infty)$ and

$$\arg h(z) = \arg \frac{z-a}{f(z)-f(a)} + \sum_{i=1}^{n} \arg \frac{g_i(z)(z-b_i)}{zg_i(z)-b_ig_i(b_i)} > 0$$

for $z \in \mathbb{H}_+$.

We assume that $f(z)$ and $g_i(z)$ $(i = 1, 2, \ldots, n)$ are continuous on $\overline{\mathbb{H}_+}$ and

$$f(t) - f(a) \neq 0 \text{ and } tg_i(t) - b_ig_i(b_i) \neq 0 \text{ for any } t \in (-\infty, 0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned}
&\arg h(z) \\
=&\arg(z-a) + \sum_{i=1}^{n} \arg g_i(z)(z-b_i) - \arg(f(z)-f(a)) - \sum_{i=1}^{n} \arg(zg_i(z)-b_ig_i(b_i)) \\
\leq&\pi + \sum_{i=1}^{n} \arg g_i(z) + n\pi - \arg f(z) - n\pi \quad (\text{since } \arg(zg_i(z)-b_ig_i(b_i)) \geq \pi) \\
\leq&\pi \qquad (\text{since } \arg f(z) - \sum_{i=1}^{n} \arg g_i(z) \geq 0).
\end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case $z \in \mathbb{H}_+$ satisifying $|z| > n \max\{a, b_1, b_2, \ldots, b_n\}$, it holds that

$$\arg(z-a), \arg(z-b_i) < \frac{\pi + n \arg z}{n+1} \qquad (i = 1, 2, \ldots, n)$$

by Lemma 4. We may assume that there exists a number k $(1 \leq k \leq n)$ such that

$$\arg(zg_i(z)) \leq \pi \quad (i \leq k), \qquad \arg(zg_i(z)) > \pi \quad (i > k).$$

Since

$$\begin{aligned}
&\arg h(z) \\
=&\arg(z-a)+\sum_{i=1}^{n}\arg(z-b_i)+\sum_{i=1}^{n}\arg g_i(z) \\
&\quad-\arg(f(z)-f(a))-\sum_{i=1}^{n}\arg(zg_i(z)-b_ig_i(b_i)) \\
\le&\frac{\pi+n\arg z}{n+1}\times(n+1)+\sum_{i=1}^{n}\arg g_i(z) \\
&\quad-\arg f(z)-\sum_{i=1}^{k}\arg zg_i(z)-(n-k)\pi \\
=&\pi+n\arg z+\sum_{i=k+1}^{n}\arg g_i(z)-\arg f(z)-k\arg z-(n-k)\pi \\
\le&\pi+(n-k)\arg z-(n-k)\pi\le\pi,
\end{aligned}$$

we have $0 \le \arg h(z) \le \pi$.

This means that it holds

$$0 \le \arg h(z) \le \pi$$

if z belongs to the boundary of $H(r) = \{z \in \mathbb{C} \mid |z| \le r, \mathrm{Im} z \ge 0\}$ for a sufficiently large r. Using the same argument in (1), we can prove the operator monotonicity of h.

In general case, we define functions, for p $(0 < p < 1)$, as follows:

$$f_p(t) = f(t^p), \quad g_{i,p}(t) = g_i(t^p) \quad (i = 1, 2, \ldots, n).$$

Since $f, g_i \in \mathbb{P}_+[0, \infty)$,

$$0 < \arg f_p(z) < \pi, \quad 0 < \arg zg_{i,p}(z) < 2\pi$$

for $z \in \mathbb{H}_+$. This means that $f_p(z)$ and $g_{i,p}(z)$ are continuous on $\overline{\mathbb{H}_+}$ and

$$f_p(t) - f_p(a) \ne 0 \text{ and } tg_{i,p}(t) - b_i g_{i,p}(b_i) \ne 0 \text{ for any } t \in (-\infty, 0)$$

by Lemma 3. Since

$$\frac{f_p(t)}{\prod_{i=1}^{n} g_{i,p}(t)} = \frac{f(t^p)}{\prod_{i=1}^{n} g_i(t^p)} \qquad (0 < p < 1)$$

is operator monotone on $[0,\infty)$, we can get the operator monotonicity of

$$\begin{aligned} h_p(t) &= \frac{t-a}{f_p(t)-f_p(a)} \prod_{i=1}^{n} \frac{g_{i,p}(t)(t-b_i)}{tg_{i,p}(t)-b_i g_{i,p}(b_i)} \\ &= \frac{t-a}{f(t^p)-f(a^p)} \prod_{i=1}^{n} \frac{g_i(t^p)(t-b_i)}{tg_i(t^p)-b_i g_i(b_i^p)}. \end{aligned}$$

So we can see that

$$h(t) = \lim_{p\to 1-0} h_p(t)$$

is operator monotone on $[0,\infty)$. □

Remark 6. *Using Proposition 2 and Theorem 5, we can prove the operator monotonicity of the concrete functions in [12]. Since* t^a $(0<a<1)$ *and* $\log t$ *is operator monotone on* $(0,\infty)$*,*

$$\begin{aligned} f_a(t) &= a(1-a)\frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \qquad (-1\le a\le 2) \\ &= \begin{cases} a(a-1)\dfrac{t^{-a}(t-1)^2}{(t^{-a}-1)(t\cdot t^{-a}-1)} & -1\le a<0 \\ \dfrac{t-1}{\log t} & a=0,1 \\ a(1-a)\dfrac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} & 0<a<1 \\ a(a-1)\dfrac{t^{a-1}(t-1)^2}{(t^{a-1}-1)(t\cdot t^{a-1}-1)} & 1<a\le 2 \end{cases} \end{aligned}$$

becomes operator monotone.

We can also prove this remark and the first part of Example 10 using some formula stated in [10] as Theorem 1.2.

Corollary 7. *Let* $f\in\mathbb{P}_+(0,\infty)$ *and both* f *and* $f^\sharp$ *be not constant. For any* $a>0$*, we define*

$$h_a(t) = \frac{(t-a)(t-a^{-1})}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1}))} \qquad t\in(0,\infty).$$

Then we have

(1) h_a *is operator monotone on* $(0,\infty)$.

(2) $f(t) = t \cdot f(t^{-1})$ *implies* $h_a(t) = t \cdot h_a(t^{-1})$.

(3) $a = 1$ *and* $f(t^{-1}) = f(t)^{-1}$ *imply* $h_1(t) = t \cdot h_1(t^{-1})$.

Proof. We can directly prove (1) from Theorem 5. Because

$$\begin{aligned} t \cdot h_a(t^{-1}) &= \frac{t(t^{-1} - a)(t^{-1} - a^{-1})}{(f(t^{-1}) - f(a))(f^\sharp(t^{-1}) - f^\sharp(a^{-1}))} \\ &= \frac{(t - a)(t - a^{-1})}{t(f(t^{-1}) - f(a))(f^\sharp(t^{-1}) - f^\sharp(a^{-1}))}, \end{aligned}$$

we can compute

$$\begin{aligned} &t(f(t^{-1}) - f(a))(f^\sharp(t^{-1}) - f^\sharp(a^{-1})) - (f(t) - f(a))(f^\sharp(t) - f^\sharp(a^{-1})) \\ =&(f(t^{-1}) - f(a))(1/f(t^{-1}) - t/af(a^{-1})) - (f(t) - f(a))(t/f(t) - 1/af(a^{-1})) \\ =&0 \end{aligned}$$

if it holds $f(t) = t \cdot f(t^{-1})$ or $a = 1$, $f(t^{-1}) = f(t)^{-1}$. So we have (2) and (3). □

The function h is called symmetric if it satisfies the following condition:

$$h(t) = th(t^{-1}), \qquad t \geq 0.$$

We can define a symmetric operator mean using a symmetric operator monotone function in the sense of Kubo-Ando ([5], [6]). Corollary 7 says that we can repeatedly construct a symmetric operator monotone function from a symmetric operator monotone function. We can give the following examples.

Example 8. *If we choose* t^p $(0 < p < 1)$ *as* $f(t)$ *in Corollary* 7(3),

$$h(t) = \frac{(t-1)^2}{(t^p - 1)(t^{1-p} - 1)}.$$

If we choose $t^p + t^{1-p}$ $(0 < p < 1)$ *as* $f(t)$ *in Corollary* 7(2),

$$\begin{aligned} h(t) &= \frac{t-a}{t^p + t^{1-p} - a^p - a^{1-p}} \times \frac{t - a^{-1}}{\dfrac{1}{t^{p-1} + t^{-p}} - \dfrac{1}{a^p + a^{1-p}}} \qquad (a > 0) \\ &= \frac{\sqrt{t}(\cosh(\log t) - \cosh(\log a))}{\cosh(\log \sqrt{t}) - \cosh(\log \sqrt{t} + \log(t^p + t^{1-p}) - \log(a^p + a^{1-p}))}. \end{aligned}$$

These functions, $h \in \mathbb{P}_+[0, \infty)$, *are symmetric.*

3 Extension of Theorem 4 Let m and n be positive integers and $f_1, f_2, \dots, f_m$, $g_1, g_2, \dots, g_n$ be non-constant, non-negative operator monotone functions on $[0, \infty)$. We assume that the function

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{t^{m-1} \prod_{j=1}^n g_j(t)}$$

is operator monotone on $[0, \infty)$. For non-negative numbers $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$, we define the function $h(t)$ as follows:

$$h(t) = \prod_{i=1}^m \frac{t - a_i}{f_i(t) - f_i(a_i)} \prod_{j=1}^n \frac{g_j(t)(t - b_j)}{t g_j(t) - b_j g_j(b_j)} \qquad (t \geq 0).$$

Then it follows from Proposition 2 that $h(z)$ is holomorphic on $\mathbb{H}_+$, $h([0,\infty)) \subset [0, \infty)$ and $\arg h(z) > 0$ for any $z \in \mathbb{H}_+$.

Theorem 9. *In the above setting, we have the following:*

(1) *When f_i and g_j $(1 \leq i \leq m,\ 1 \leq j \leq n)$ are continuous on $\overline{\mathbb{H}_+}$ and*

$$f_i(t) - f_i(a_i) \neq 0, \quad t g_j(t) - b_j g_j(b_j) \neq 0, \quad t \in (-\infty, 0),$$

$h(t)$ is operator monotone on $[0, \infty)$.

(2) *When there exists a positive number α such that $\alpha \arg z \leq \arg F(z)$ for all $z \in \mathbb{H}_+$,$h(t)$ is operator monotone on $[0, \infty)$.*

Proof. (1) Using the same argument of proof of Theorem 5 (1), it suffices to show that $0 \leq \arg h(z) \leq \pi$ for $z \in \mathbb{R}$ or $z \in \mathbb{H}_+$ whose absolutely value is sufficiently large.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned}
&\arg h(z) \\
=&\sum_{i=1}^m \arg(z - a_i) + \sum_{j=1}^n \arg(g_j(z)(z - b_j)) \\
&\qquad - \sum_{i=1}^n \arg(f_i(z) - f_i(a_i)) - \sum_{j=1}^n (z g_j(z) - b_j g_j(b_j)) \\
\leq& m\pi + n\pi + \sum_{j=1}^n \arg g_j(z) - \sum_{i=1}^m \arg f_i(z) - n\pi \\
=&\pi - \arg \frac{\prod_{i=1}^n f_i(z)}{z^{m-1} \prod_{j=1}^n g_j(z)} \leq \pi.
\end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case that $z \in \mathbb{H}_+$ satisifies

$$|z| > (m+n-1)\max\{a_i, b_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then it holds that

$$\arg(z-a_i), \arg(z-b_j) < \frac{\pi + (m+n-1)\arg z}{m+n}$$

by Lemma 4. We may assume that there exists k $(1 \leq k \leq n)$ such that

$$\arg(zg_j(z)) \leq \pi \quad (j \leq k), \qquad \arg(zg_j(z)) > \pi \quad (j > k).$$

Since

$$\begin{aligned}
&\arg h(z) \\
\leq& \frac{\pi + (m+n-1)\arg z}{m+n} \times m + \frac{\pi + (m+n-1)\arg z}{m+n} \times n + \sum_{j=1}^{n} \arg g_j(z) \\
&\quad - \sum_{i=1}^{m} \arg f_i(z) - \sum_{j=1}^{k} \arg zg_j(z) - (n-k)\pi \\
=& \pi + (m+n-k-1)\arg z + \sum_{j=k+1}^{n} \arg g_j(z) - \sum_{i=1}^{m} \arg f_i(z) - (n-k)\pi \\
\leq& \pi + (n-k)(\arg z - \pi) - \arg \frac{\prod_{i=1}^{m} f_i(z)}{z^{m-1}\prod_{j=1}^{m} g_j(z)} \\
\leq& \pi - \arg F(z) \leq \pi,
\end{aligned}$$

we have $0 \leq \arg h(z) \leq \pi$. So $h(t)$ is operator monotone on $[0, \infty)$.

(2) We choose a positive number p as follows:

$$\frac{m-1}{\alpha + m - 1} < p < 1.$$

We define functions $f_{i,p}, g_{j,p}$ as follows:

$$f_{i,p}(z) = f_i(z^p), \quad g_{j,p}(z) = g_j(z^p) \qquad (z \in \mathbb{H}_+).$$

Since $f_i, g_j \in \mathbb{P}_+[0, \infty)$, $f_{i,p}, g_{j,p}$ are continuous on $\overline{\mathbb{H}_+}$ and satisfy the condition

$$f_{i,p}(t) - f_{i,p}(a_i) \neq 0, \quad tg_{j,p}(t) - b_j g_{j,p}(b_j) \neq 0, \quad t \in (-\infty, 0)$$

by Lemma 3. We put

$$F_p(t) = \frac{\prod_{i=1}^{m} f_{i,p}(t)}{t^{m-1}\prod_{j=1}^{n} g_{j,p}(t)} = F(t^p)t^{-(m-1)(1-p)}.$$

Then F_p is holomorphic on $\mathbb{H}_+$ and satisfies $F_p((0,\infty)) \subset (0,\infty)$. For any $z \in \mathbb{H}_+$, we have

$$\arg F_p(z) = \arg F(z^p) - (m-1)(1-p)\arg z \le \arg F(z^p) \le \pi$$

and

$$\begin{aligned}\arg F_p(z) &\ge \alpha \arg z^p - (m-1)(1-p)\arg z\\ &= (\alpha p - (m-1)(1-p))\arg z\\ &= ((\alpha + m - 1)p - (m-1))\arg z > 0.\end{aligned}$$

So we can see $F_p \in \mathbb{P}_+[0,\infty)$. By (1), we can show that

$$h_p(t) = \prod_{i=1}^m \frac{(t-a_i)}{f_{i,p}(t) - f_{i,p}(a_i)} \prod_{j=1}^n \frac{g_{j,p}(t)(t-b_j)}{tg_{j,p}(t) - b_j g_{j,p}(b_j)}$$

is operator monotone on $[0,\infty)$. When p tends to 1, $h_p(t)$ also tends to $h(t)$. Hence $h(t)$ is operator monotone on $[0,\infty)$. □

Example 10. *Let* $0 < p_i \le 1$ $(i = 1, 2, \ldots, m)$ *and* $0 \le q_j \le 1$ $(j = 1, 2, \ldots, n)$. *We put*

$$f_i(t) = t^{p_i}, \quad g_j(t) = t^{q_j} \qquad (t \ge 0).$$

By the calculation

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{t^{m-1}\prod_{j=1}^n g_j(t)} = t^{\sum_{i=1}^m p_i - \sum_{j=1}^n q_j - (m-1)},$$

we have, for real numbers a_i, $b_j \ge 0$,

$$h(t) = t^{\sum_{j=1}^n q_j} \frac{(t-a_1)\cdots(t-a_m)(t-b_1)\cdots(t-b_n)}{(t^{p_1} - a_1^{p_1})\cdots(t^{p_m} - a_m^{p_m})(t^{1+q_1} - b_1^{1+q_1})\cdots(t^{1+q_n} - b_n^{1+q_n})}$$

is operator monotone on $[0,\infty)$ *by Theorem 9 if it holds*

$$0 \le \sum_{i=1}^m p_i - \sum_{j=1}^n q_j - (m-1) \le 1,$$

i.e., $F(t)$ *is operator monotone on* $[0,\infty)$.

When $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j + (m-1)$, *we can see that*

$$h(t) = \frac{t^{\sum_{j=1}^n q_j}(t-1)^{m+n}}{\prod_{i=1}^m (t^{p_i} - 1)\prod_{j=1}^n (t^{1+q_j} - 1)}$$

is operator monotone on $[0, \infty)$ *and symmetric.*

We can easily check that, if h_1*,* $h_2 \in \mathbb{P}_+[0, \infty)$ *are symmetric, then the functions*

$$f(t) = h_1(t)^{1/p} h_2(t)^{1-1/p} \qquad (p > 1)$$
$$g(t) = \frac{t}{h_1(t)}$$

are also operator monotone on $[0, \infty)$ *and symmetric.*

Combining these facts, for r_i, s_i $(i = 1, 2, \dots, n)$ *with*

$$0 < r_1, \dots, r_c \le 1, \quad 1 \le r_{c+1}, \dots, r_n \le 2$$
$$0 < s_1, \dots, s_d \le 1, \quad 1 \le s_{d+1}, \dots, s_n \le 2$$
$$\sum_{i=1}^{c} r_i = \sum_{i=c+1}^{n} r_i - 1, \quad \sum_{i=1}^{d} s_i = \sum_{i=d+1}^{n} s_j - 1,$$

we can see that the function

$$h(t) = \sqrt{t^{\gamma} \prod_{i=1}^{n} \frac{r_i(t^{s_i} - 1)}{s_i(t^{r_i} - 1)}}$$

is operator monotone on $[0, \infty)$ *and symmetric with* $h(1) = 1$*, where* $\gamma = 1 - c + d + \sum_{i=1}^{c} r_i - \sum_{i=1}^{d} s_i$.

By such a way, we can also construct from a symmetric operator monotone function to new one.

Ackowledgemant. The author is grateful to the refree for his careful comments.

References

[1] R. Bhatia, *Matrix Analysis* , Springer-Verlag, New York, Berlin , Heiderberg 1996.

[2] L. Cai and F. Hansen, Metric-adjusted skew information: Convexity and restricted forms of superadditivity, Lett. Math. Phys. 93(2010) 1–13.

[3] W. F. Donoughue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Springer-Verlag, Berlin , Heiderberg, New York 1974.

[4] F. Hansen, WYD-like skew information measures, J. Stat. Phys. 151(2013), 974–979.

[5] F. Hiai, Matrix Analysis: Matrix monotone functions, matrix means, and majorization, Interdecip. Inform. Sci. 16 (2010) 139–248.

[6] F. Hiai, D. Petz, *Introduction to Matrix Analysis and Applications*, Universitext, Springer, 2014.

[7] M. Kawasaki, M. Nagisa, Some Operator monotone functions related to Petz-Hasegawa's functions, Technical Reports Math. Sci. Chiba Univ. 2012.

[8] M. Kawasaki, M. Nagisa, Transforms on operator monotone functions, arXiv:1206.5452v1.

[9] E.A. Morozova, N.N. Chentsov, Markov invariant geometry on state manifolds, Itogi Nauki i Techniki 36 (1990), 69–102, Translated in J. Soviet Math. 56(1991) 2648–2669.

[10] M. Nagisa, S. Wada, Operator monotonicity of some functions, Linear Algebra Appl. 486, (2015), 389–408.

[11] D. Petz, Monotone metric on matrix spaces, Linear Algebra Appl. 244 (1996) 81–96.

[12] D. Petz, H. Hasegawa, On the Riemannian metric of α-entropies of density matrices, Lett. Math. Phys. 38 (1996) 221–225.

[13] V.E.S. Szabo, A class of matrix monotone functions, Linear Algebra Appl. 420 (2007) 79–85.

[14] M. Uchiyama, Majorization and some operator monotone functions, Linear Algebra Appl. 432(2010) 1867–1872.

Communicated by *Hiroyuki Osaka*

Masaru NAGISA
Graduate School of Science
Chiba University
Chiba 263-8522
Japan
e-mail address: nagisa@math.s.chiba-u.ac.jp

JI-DISTRIBUTIVE, DUALLY QUASI-DE MORGAN SEMI-HEYTING AND HEYTING ALGEBRAS

HANAMANTAGOUDA P. SANKAPPANAVAR

Dedicated to Professor P.N. Shivakumar
A Great Humanitarian who changed the course of my life

Abstract. The variety **DQD** of dually quasi-De Morgan semi-Heyting algebras and several of its subvarieties were investigated in the series [26] - [31]. In this paper we define and investigate a new subvariety **JID** of **DQD**, called "JI-distributive, dually quasi-De Morgan semi-Heyting algebras", defined by the identity: $x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$, as well as the (closely related) variety **DSt** of dually Stone semi-Heyting algebras. Firstly, we prove that **DSt** and **JID** are discriminator varieties of level 1 and level 2 respectively. Secondly, we give a characterization of subdirectly irreducible algebras of the subvariety $\mathbf{JID_1}$ of **JID** of level 1. As applications, we derive that the variety $\mathbf{JID_1}$ is the join of the variety **DSt** and the variety of De Morgan Boolean semi-Heyting algebras, give a concrete description of the subdirectly irreducible algebras in the subvariety $\mathbf{JIDL_1}$ of $\mathbf{JID_1}$ defined by the linear identity: $(x \to y) \vee (y \to x) \approx 1$, and deduce that the variety $\mathbf{JIDL_1}$ is the join of the variety **DStHC** generated by the dually Stone Heyting chains and the variety generated by the 4-element De Morgan Boolean Heyting algebra. Furthermore, we present an explicit description of the lattice of subvarieties of $\mathbf{JIDL_1}$ and equational bases for all subvarieties of $\mathbf{JIDL_1}$. Finally, we prove that the amalgamation property holds for all subvarieties of **DStHC**.

1. Introduction

The De Morgan (strong) negation and the pseudocomplement are two of the fairly well known negations that generalize the classical negation. A common generalization of these two negations led to a

2010 *Mathematics Subject Classification. Primary* : $03G25, 06D20, 08B26$; *Secondary* : $08B15$, 06D30, 06D15.

Key words and phrases. JI-distributive, dually quasi-De Morgan semi-Heyting algebra, De Morgan semi-Heyting algebra, De Morgan Heyting algebra, dually Stone semi-Heyting algebra, dually Stone Heyting algebra, discriminator variety, simple algebra, subdirectly irreducible algebra, equational base.

new variety of algebras, called "semi-De Morgan algebras", which was investigated in [24]. Several subvarieties of this variety, including a subvariety called "(upper) quasi-De Morgan algebras" were also studied in [24].

In a different vein, semi-Heyting algebras were introduced in [25] as an abstraction of Heyting algebras. Using the dual version of quasi-De Morgan negation, an expansion of semi-Heyting algebras, called "dually quasi-De Morgan semi-Heyting algebras (**DQD**, for short)" was defined and investigated in [26], as a common generalization of De Morgan (or symmetric) Heyting algebras [23] (see also [19]) and dually pseudocomplemented Heyting algebras [22]. It may also be mentioned here that [8] has proposed recently a propositional logic, called "Dually quasi-De Morgan semi-Heyting logic", which has dually quasi-De Morgan semi-Heyting algebras as an equivalent algebraic semantics.

Several new subvarieties of **DQD** were studied in [26]- [31], including the variety **DStHC** generated by the dually Stone Heyting chains (i.e., the expansion of the Gödel variety by the dual Stone operation), the variety **DMB** of De Morgan Boolean semi-Heyting algebras and the variety **DMBH** generated by the 4-element De Morgan Boolean Heyting algebra. These investigations led us naturally to the problem of equational axiomatization for the join of the variety **DStHC** and the variety **DMBH**. Our investigations into this problem led us to the results of the present paper that include a solution to the just mentioned problem.

In this paper we define and investigate a new subvariety of **DQD**, called "JI-distributive, dually quasi-De Morgan semi-Heyting algebras (**JID**, for short)", defined by the identity: $x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$, as well as the (closely related) variety **DSt** of dually Stone semi-Heyting algebras. We first prove that **DSt** and **JID** are discriminator varieties of level 1 and level 2 respectively (see Section 2 for definitions). Secondly, we prove that the lattice of subvarieties of **DStHC** is an $\omega + 1$-chain. Thirdly, we give a characterization of subdirectly irreducible algebras of the subvariety $\mathbf{JID_1}$ of level 1. As a first application of it, we derive that the variety $\mathbf{JID_1}$ is the join of the variety **DSt** and the variety **DMB**. As a second application, we give a concrete description of the subdirectly irreducible algebras in the subvariety $\mathbf{JIDL_1}$ of $\mathbf{JID_1}$ defined by the linear identity: $(x \to y) \vee (y \to x) \approx 1$, and deduce that the variety $\mathbf{JIDL_1}$ is the join of the variety **DStHC** generated by the dually Stone Heyting chains and the variety **DMBH**. Other applications include a description of the lattice of subvarieties of $\mathbf{JIDL_1}$, equational bases of all subvarieties of $\mathbf{JIDL_1}$, and the fact that the amalgamation property holds in all subvarieties of **DStHC**.

More explicitly, the paper is organized as follows: In Section 2 we recall definitions, notations and results from [26], [27] and [28] and also prove some new results needed in the rest of the paper. In Section 3, we define the variety **JID** of JI-distributive, dually quasi-De Morgan semi-Heyting algebras and give some arithmetical properties of **JID**. In particular, we show that **JID** satisfies the $\vee$-De Morgan law and the level 2 identity: $(x \wedge x'^{*})'^{*} \approx (x \wedge x'^{*})'^{*'*}$. These two propertes allow us to apply [26, Corollary 8.2(a)] to deduce that **JID** is a discriminator variety. These properties also play a crucial role in the rest of the paper. Section 4 will prove that the variety **DSt** is a discriminator variety of level 1. It will also present some properties of **DSt**, which, besides being of interest in their own right, will also be useful in the later sections. It is also proved that the lattice of subvarieties of **DStHC** is an $\omega + 1$-chain. In Section 5, we give a characterization of subdirectly irreducible (= simple) algebras in the variety $\mathbf{JID_1}$ of level 1 and deduce that $\mathbf{JID_1}$ is the join of **DSt** and the variety **DMB** of De Morgan Boolean semi-Heyting algebras. Several applications of this characterization are given in Section 6 and Section 7. We investigate, in Section 6, the variety $\mathbf{JIDL_1}$ of JI-distributive, dually quasi-De Morgan, linear semi-Heyting algebras of level 1. An explicit description of subdirectly irreducible algebras in $\mathbf{JIDL_1}$ is given, and from this description it is deduced that $\mathbf{JIDL_1} = \mathbf{DStHC} \vee \mathbf{DMBH}$, which solves the aforementioned problem of axiomatizing the join of **DStHC** and **DMBH**. In Section 7, some applications of the just-mentioned result are given. It is shown that the lattice of subvarieties of $\mathbf{JIDL_1}$ is isomorphic to $\mathbf{1} \oplus [(\omega + \mathbf{1}) \times \mathbf{2}]$, where $\mathbf{1}$ and $\mathbf{2}$ are the 1-element and the 2-element lattices, respectively. Also, (small) equational bases for all subvarieties of $\mathbf{JIDL_1}$ are given. Finally, it is shown that all subvarieties of **DStHC** have the amalgamation property.

2. **Preliminaries**

In this section we recall some notions and known results needed to make this paper as self-contained as possible. However, for other information used but not mentioned here, we refer the reader to [5], [7] and [20].

An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a *semi-Heyting algebra* ([25]) if $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded lattice and $\mathbf{L}$ satisfies:

(SH1) $x \wedge (x \to y) \approx x \wedge y$,
(SH2) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(SH3) $x \to x \approx 1$.

Semi-Heyting algebras are distributive and pseudocomplemented, with $a^* := a \to 0$ as the pseudocomplement of an element a.

Let $\mathbf{L}$ be a semi-Heyting algebra. $\mathbf{L}$ is a *Heyting algebra* if $\mathbf{L}$ satisfies:

(H) $(x \wedge y) \to y \approx 1$.

$\mathbf{L}$ is a *Boolean semi-Heyting algebra* if $\mathbf{L}$ satisfies:

(Bo) $x \vee x^* \approx 1$.

$\mathbf{L}$ is a *Boolean Heyting algebra* if $\mathbf{L}$ is a Heyting algebra and satisfies (Bo).

The following definition, taken from [26], is central to this paper.

DEFINITION 2.1. *An algebra* $\mathbf{L} = \langle L, \vee, \wedge, \to, ', 0, 1\rangle$ *is a semi-Heyting algebra with a dual quasi-De Morgan operation or dually quasi-De Morgan semi-Heyting algebra* (**DQD**-algebra, for short) if $\langle L, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra, and $\mathbf{L}$ satisfies:

(a) $0' \approx 1$ and $1' \approx 0$,
(b) $(x \wedge y)' \approx x' \vee y'$,
(c) $(x \vee y)'' \approx x'' \vee y''$,
(d) $x'' \leq x$.

Let $\mathbf{L}$ be a **DQD**-algebra. $\mathbf{L}$ is a *dually pseudocomplemented semi-Heyting algebra* (**DPC**-algebra) (see [24]) if $\mathbf{L}$ satisfies:

(e) $x \vee x' \approx 1$.

$\mathbf{L}$ is a *dually Stone semi-Heyting algebra* (**DSt**-algebra) if $\mathbf{L}$ satisfies the dual Stone identity:

(DSt) $x' \wedge x'' \approx 0$.

It should be noted that if (DSt) holds in a **DQD**-algebra $\mathbf{L}$, then (e) holds in $\mathbf{L}$ as well, and hence $'$ is indeed the dual pseudocomplement satisfying the dual Stone identity, and so $\mathbf{L}$ has, indeed, a dual Stone algebra as a reduct. $\mathbf{L}$ is a *De Morgan semi-Heyting algebra* (**DM**-algebra) if $\mathbf{L}$ satisfies:

(DM) $x'' \approx x$.

The varieties of **DQD**-algebras, **DPC**-algebras, **DSt**-algebras, **DM**-algebras are denoted, respectively, by **DQD**, **DPC**, **DSt**, and **DM**. If the underlying semi-Heyting algebra of a **DQD**-algebra is a Heyting algebra, then we add "**H**" at the end of the names of the varieties that will be considered in the sequel. Thus, for example, **DStH** denotes the variety of dually Stone Heyting algebras.

The following lemmas are basic to this paper. The proof of the first lemma is straightforward and is left to the reader.

LEMMA 2.2. *Let* $\mathbf{L} \in \mathbf{DQD}$ *and let* $x, y, z \in L$. *Then*

(i) $1'^* = 1$, *and* $1 \to x = x$,
(ii) $x \leq y$ *implies* $x' \geq y'$,
(iii) $(x \wedge y)'^* = x'^* \wedge y'^*$,
(iv) $x''' = x'$,
(v) $(x \vee y)' = (x'' \vee y)'$,
(vi) $x \wedge [y \vee (x \to z)] = x \wedge (y \vee z)$,
(vii) $x \wedge (x \to y)'' \leq y$.

LEMMA 2.3. *Let* $L \in \mathbf{DQD}$ *and* $x, y \in L$. *Then*

(1) $(x \vee y)' \leq x' \to (x \vee y)'$,
(2) $[x \vee (y \vee z)']' = (x \vee y')' \vee (x \vee z')'$,
(3) $x \wedge [(x \to y) \vee z] = x \wedge (y \vee z)$,
(4) $y \wedge [x \to (y \wedge z)] = y \wedge (x \to z)$,
(5) $x \to (y \wedge z) \ \geq \ y \wedge (x \to z)$,
(6) $x \ \leq \ y \to (x \wedge y)$,
(7) $(x \vee y)' = x' \wedge [(x \vee y)' \vee \{x' \to (x \vee y)'\}'']$,
(8) $x \leq (x \to y) \to y$.

Proof.
(1) is straightforward to verify since $(x \vee y)' \leq x'$.

$$
\begin{aligned}
(2)\text{: } [(x \vee (y \vee z)']' &= [x'' \vee (y \vee z)']' && \text{by Lemma 2.2 (v)}\\
&= [x' \wedge (y \vee z)]''\\
&= [(x' \wedge y) \vee (x' \wedge z)]''\\
&= (x' \wedge y)'' \vee (x' \wedge z)''\\
&= (x'' \vee y')' \vee (x'' \vee z')'\\
&= (x \vee y')' \vee (x \vee z')' && \text{by Lemma 2.2 (v)}
\end{aligned}
$$

(3) and (4) are easy to verify.

$$
\begin{aligned}
(5)\text{: } [x \to (y \wedge z)] \wedge y \wedge (x \to z) &= y \wedge [x \to (y \wedge z)] \wedge (x \to z)\\
&= y \wedge (x \to z) \text{ by (4).}
\end{aligned}
$$

(6): $x = x \wedge (y \to y) \leq y \to (x \wedge y)$ by (5).

(7): $(x \vee y)' = x' \wedge (x \vee y)'$.

$= x' \wedge [x' \to (x \vee y)']$

$= x' \wedge [\{x' \to (x \vee y)'\} \vee \{x' \to (x \vee y)'\}'']$

$= [x' \wedge (x \vee y)'] \vee [x' \wedge \{x' \to (x \vee y)'\}'']$

$= x' \wedge [(x \vee y)' \vee \{x' \to (x \vee y)'\}'']$,

(8): $x \wedge [(x \to y) \to y] = x \wedge [\{x \wedge (x \to y)\} \to (x \wedge y)] = x \wedge [(x \wedge y) \to (x \wedge y)] = x \wedge 1 = x$, completing the proof. □

The following three 4-element algebras, called $\mathbf{D_1}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ (following the notation of [26]), in **DQD**, play an important role in the sequel. All three of them have the Boolean lattice reduct with the universe $\{0, a, b, 1\}$, where b is the Boolean complement of a, and the operation $'$ is defined as follows: $a' = a$, $b' = b$, $0' = 1$, $1' = 0$, while the operation $\to$ is defined in Figure 1.

$\mathbf{D_1}$:

$\to$	0	1	a	b
0	1	0	b	a
1	0	1	a	b
a	b	a	1	0
b	a	b	0	1

$\mathbf{D_2}$:

$\to$	0	1	a	b
0	1	1	1	1
1	0	1	a	b
a	b	1	1	b
b	a	1	a	1

$\mathbf{D_3}$:

$\to$	0	1	a	b
0	1	a	1	a
1	0	1	a	b
a	b	a	1	0
b	a	1	a	1

Figure 1

Let **DQB** and **DMB** denote respectively the subvarieties of **DQD** and **DM** defined by (Bo). Also, by an earlier convention, **DQBH** and **DMBH** denote, respectively, the subvarieties of **DQB** and **DMB** defined by (H). $\mathbf{V}(\mathbf{K})$ denotes the variety generated by the class K of algebras in **DQD**. The following proposition is proved in ([26]) and is needed later in this paper.

PROPOSITION 2.4.

(a) $\mathbf{DQB} = \mathbf{DMB} = \mathbf{V}(\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\})$,

(b) $\mathbf{DQBH} = \mathbf{DMBH} = \mathbf{V}(\mathbf{D_2})$.

The following definition is from [26].

DEFINITION 2.5. *Let* $\mathbf{L} \in \mathbf{DQD}$ *and* $x \in \mathbf{L}$. *For* $n \in \omega$, *we define* $t_n(x)$ *recursively as follows:*

$$x^{0('*)} := x;$$
$$x^{(n+1)('*)} := (x^{n('*)})'^*, \text{ for } n \geq 0;$$
$$t_0(x) := x,$$
$$t_{n+1}(x) := t_n(x) \wedge x^{(n+1)('*)}, \text{ for } n \geq 0.$$

Let $n \in \omega$. *The subvariety* $\mathbf{DQD_n}$ *of level* n *of* $\mathbf{DQD}$ *is defined by the identity:*

(lev n) $$t_n(x) \approx t_{(n+1)}(x);$$

For a subvariety $\mathbf{V}$ *of* $\mathbf{DQD}$, *we let* $\mathbf{V_n} := \mathbf{V} \cap \mathbf{DQD_n}$.

Recall from [26] (or [27]) that $\mathbf{BDQDSH}$ is the subvariety of $\mathbf{DQD}$ (= $\mathbf{DQDSH}$) defined by the identity:

(BL) $(x \vee y^*)' \approx x' \wedge y^{*\prime}$.

We will abbreviate $\mathbf{BDQDSH}$ by $\mathbf{BDQD}$.

The following "simplicity condition", (SC), is crucial in the rest of the paper.

(SC) For every $x \in L$, if $x \neq 1$, then $x \wedge x'^* = 0$.

The following theorem, which was proved in [27, Corollary 4.1] (which is, in turn, a consequence of Corollaries 7.6 and 7.7 of [26]), will play a fundamental role in this paper.

THEOREM 2.6. [27, Corollary 4.1] *Let* $\mathbf{L} \in \mathbf{BDQD_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is simple,*
(2) $\mathbf{L}$ *is subdirectly irreducible,*
(3) $\mathbf{L}$ *satisfies* (SC).

3. JI-distributive, dually quasi-De Morgan semi-Heyting algebras

The identity, $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$, was shown in [28, Corollary 3.55] to be an equational base for the variety generated by $\mathbf{D_2}$, relative to $\mathbf{DQD}$. Let us refer to this identity as "strong

JI-distributive identity". We now introduce a slightly weaker identity, called "JI-distributive identity" (by restricting the first variable to "primed" elements). The subvariety **JID** of **DQD** defined by this identity and some of its subvarieties are the subject of our investigation in the rest of this paper.

DEFINITION 3.1. *The subvariety* **JID** *of* **DQD** *is defined by:*

(JID) $x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$ (restricted **D**istribution of **J**oin over **I**mplication).

Members of the variety **JID** are called "JI-distributive, dually quasi-De Morgan semi-Heyting algebras" and will be referred to as **JID**-algebras. Examples of **JID**-algebras come from a surprising source to which we shall now turn. But, first we need some notation.

A **DQD**-algebra is a **DQD**-chain if its lattice reduct is a chain. Let **DQDC** [**DPCC**] denote the variety generated by the **DQD**-chains [**DPC**-chains]. The following lemma provides an important class of examples of **JID**, which is partly the motivation for our interest in **JID**.

LEMMA 3.2. **DPCC** $\subseteq$ **JID**.

Proof. It suffice to show that **DPCC** $\models$ (JID). Let **A** be a **DPC**-chain and let $a \in A \setminus \{1\}$. Since **A** is a chain, we have $a' \leq a$ or $a \leq a'$, from which we get that $a \vee a' \leq a$ or $a \vee a' \leq a'$. Since **A** is dually pseudocomplemented, we have $a \vee a' = 1$, implying $a' = 1$, as $a \neq 1$. Now, it is routine to verify (JID) holds in **A**. □

For **L** a **DPC**-chain, it was observed in the proof of the preceding lemma that the dual pseudocomplement $'$ satisfies: $a' = 1$, if $a \neq 1$, and hence **L** $\models$ (DSt). Thus, we have the following corollary, where **DStC** denotes the variety generated by the dually Stone semi-Heyting chains.

COROLLARY 3.3. **DPCC** = **DStC**.

From now on, we use **DPCC** and **DStC** interchangeably. We note that $\mathbf{D_1}$, $\mathbf{D_2}$, and $\mathbf{D_3}$ are also examples of **JID**-algebras.

In the rest of this section we present several useful arithmetical properties of **JID**. Following our convention made earlier, **JIDH** denotes the subvariety of **JID** defined by the identity (H).

Throughout this section, we assume that L $\in$ JID.

LEMMA 3.4. *Let* $x, y, z \in$ **L**. *Then*

(1) $x' \to (x' \vee y) = x' \vee (x' \to y)$,
(2) $x' \to (x' \vee y) = x' \vee (0 \to y)$,

(3) $x' \vee (x' \to y) = x' \vee (0 \to y)$; *in particular,* $x' \vee x'^* = 1$,
(4) $(x' \vee y) \to x' = x' \vee y^*$,
(5) $(x' \vee y) \to x' = x' \vee (y \to x')$,
(6) $x' \vee (y \to x') = x' \vee y^*$,
(7) $x' \to (x \vee y)' = x'^* \vee (x \vee y)'$.

Proof. Observe that $x' \to (x' \vee y) = (x' \vee x') \to (x' \vee y) = x' \vee (x' \to y)$ by (JID), which proves (1). To prove (2), again using (JID), we get $x' \vee (0 \to y) = (x' \vee 0) \to (x' \vee y) = x' \to (x' \vee y)$. (3) is immediate from (1) and (2). For (4), $(x' \vee y) \to x' = (x' \vee y) \to (x' \vee 0) = x' \vee (y \to 0) = x' \vee y^*$, in view of (JID). Next, $(x' \vee y) \to x' = (x' \vee y) \to (x' \vee x') = x' \vee (y \to x')$, proving (5), and (6) is immediate from (4) and (5). For (7), we have

$$\begin{aligned} x' \to (x \vee y)' &= (x \vee y)' \vee [x' \to (x \vee y)'] \text{ by Lemma 2.3 (1)} \\ &= (x \vee y)' \vee x'^* \text{ by (6).} \end{aligned}$$

□

We now prove an important property of the variety **JID**, namely the $\vee$-De Morgan law. We denote by **Dms** the subvariety of **DQD** (called "dually ms semi-Heyting algebras") defined by

$$(x \vee y)' \approx x' \wedge y' \qquad (\vee\text{-De Morgan Law}).$$

THEOREM 3.5. **JID** $\subseteq$ **Dms**.

Proof. Let $x, y \in \mathbf{L}$. As $x' \wedge x'^{*\prime\prime} \leq x' \wedge x'^* = 0$, we get $x' \wedge y' = (x' \wedge x'^{*\prime\prime}) \vee (x' \wedge y')$. Hence,

$$\begin{aligned} x' \wedge y' &= x' \wedge (x'^{*\prime\prime} \vee y') \\ &= x' \wedge [(x \vee y)' \vee x'^{*\prime\prime} \vee y'] && \text{since } (x \vee y)' \leq y' \\ &= x' \wedge [(x \vee y)' \vee x'^{*\prime\prime} \vee y'''] \\ &= x' \wedge [(x \vee y)' \vee (x'^* \vee y')''] \\ &= x' \wedge [(x \vee y)' \vee \{(x'^* \vee x')' \vee (x'^* \vee y')'\}'] && \text{by Lemma 3.4 (3.4)} \\ &= x' \wedge [(x \vee y)' \vee \{x'^* \vee (x \vee y)'\}''] && \text{by Lemma 2.3 (2)} \\ &= x' \wedge [(x \vee y)' \vee \{x' \to (x \vee y)'\}''] && \text{by Lemma 3.4 (7)} \\ &= (x \vee y)' && \text{by Lemma 2.3 (7).} \end{aligned}$$

Hence, **JID** $\subseteq$ **Dms**. □

The following lemma is useful in this and later sections.

LEMMA 3.6. *Let* $x, y, z \in \mathbf{L}$. *Then*

(1) $x'^{*\prime\prime} = x'^*$,
(2) $x''^* = x'^{*\prime}$,
(3) $x \to (x \wedge y') = x^* \vee y'$,
(4) $(x \wedge y'^*)^* = y' \vee x^*$,

(5) $(x' \vee y''^{*})^{*\prime} = (x'' \wedge y'^{*})^{*}$.

Proof. (1): From Lemma 3.4 (3) we have $x' \vee x'^{*} = 1$, which yields $x''' \vee x'^{*\prime\prime} = 1$, implying $x' \vee x'^{*\prime\prime} = 1$, leading to $x'^{*} \leq x'^{*\prime\prime}$; thus, $x'^{*} = x'^{*\prime\prime}$.

(2): From $x' \vee x'^{*} = 1$ and Theorem 3.5 we get $x'' \wedge x'^{*\prime} = 0$, implying $x'^{*\prime} \leq x''^{*}$. To prove the reverse inequality, from $x' \wedge x'^{*} = 0$, we get $x'' \vee x'^{*\prime} = 1$, from which it follows that $x''^{*} \leq x'^{*\prime}$.

(3):
$$\begin{aligned} x^{*} \vee y' &= (y' \vee x) \to y' && \text{by (JID)} \\ &= (y' \vee x) \to [y' \vee (x \wedge y')] \\ &= y' \vee [x \to (x \wedge y')] && \text{by (JID)} \\ &= x \to (x \wedge y') && \text{by Lemma 2.3 (6).} \end{aligned}$$

(4):
$$\begin{aligned} (x \wedge y'^{*})^{*} &= (x \wedge y'''^{*})^{*} && \text{by Lemma 2.2 (v)} \\ &= (x \wedge y'^{*\prime\prime})^{*} && \text{by (2) (twice)} \\ &= (x \wedge y'^{*\prime\prime}) \to (y' \wedge x \wedge y'^{*\prime\prime}) && \text{as } y' \wedge y'^{*\prime\prime} = 0 \\ &= y' \vee (x \wedge y'^{*\prime\prime})^{*} && \text{by (3)} \\ &= y' \vee (x \wedge y'''^{*})^{*} && \text{by (2) (twice)} \\ &= y' \vee (x \wedge y'^{*})^{*} \\ &= y' \vee [(x \wedge y'^{*}) \to 0] \\ &= [y' \vee (x \wedge y'^{*})] \to y' && \text{by (JID)} \\ &= [(y' \vee x) \wedge (y' \vee y'^{*})] \to y' \\ &= [(y' \vee x) \wedge 1] \to y' && \text{by Lemma 3.4 (3)} \\ &= (y' \vee x) \to y' \\ &= y' \vee (x \to 0) && \text{by (JID)} \\ &= y' \vee x^{*}. \end{aligned}$$

$$
\begin{aligned}
(5)\colon\ (x' \vee y''^{*})^{*\prime} &= (x' \vee y'^{*\prime})^{*\prime} && \text{by } (2)\\
&= (x \wedge y'^{*})'^{*\prime}\\
&= (x \wedge y'^{*})''^{*} && \text{by } (2)\\
&= (x' \vee y'^{*\prime})'^{*}\\
&= (x'' \wedge y'^{*\prime\prime})^{*} && \text{by Theorem 3.5}\\
&= (x'' \wedge y'''^{*})^{*} && \text{by (2) (twice)}\\
&= (x'' \wedge y'^{*})^{*}.
\end{aligned}
$$

This completes the proof. □

3.1. An Alternate Definition of "level n", for $n \geq 1$.

The following lemmas enable us to give an alternate definition of "Level n", for $n \geq 1$.

LEMMA 3.7. *Let $x \in \mathbf{L}$. Then $x'^{**} = x'$.*

Proof. Since $x' \vee x'^{*} = 1$ by Lemma 3.4, and $x' \wedge x'^{*} = 0$, we get $x'^{**} = x'$. □

LEMMA 3.8. *Let $x \in \mathbf{L}$. Then $x \wedge x'^{*} \wedge x'^{*\prime *} = (x \wedge x'^{*})'^{*}$.*

Proof.
$$
\begin{aligned}
x \wedge x'^{*} \wedge x'^{*\prime *} &= x \wedge x'^{*} \wedge x''^{**} && \text{by Lemma 3.6 (2)}\\
&= x \wedge x'^{*} \wedge x'' && \text{by Lemma 3.7}\\
&= x'^{*} \wedge x''\\
&= x'^{*} \wedge x''^{**} && \text{by Lemma 3.7}\\
&= x'^{*} \wedge x'^{*\prime *} && \text{by Lemma 3.6 (2)}\\
&= (x \wedge x'^{*})'^{*}.
\end{aligned}
$$
□

Since $\mathbf{JID_n} = \mathbf{JID} \cap \mathbf{DQD_n}$, the above lemma allows us to make the following alternate (but equivalent) definition for $\mathbf{JID_n}$, for $n \in \omega$ such that $n \geq 1$.

DEFINITION 3.9. *Let n be an integer ≥ 1. The variety $\mathbf{JID_n}$ is the subvariety of $\mathbf{JID}$ defined by*

(Lev n) $\quad (x \wedge x'^{*})^{(n-1)('*)} \approx (x \wedge x'^{*})^{n('*)}$.

Thus, in particular, $\mathbf{JID_1}$ and $\mathbf{JID_2}$ are, respectively, defined, relative to $\mathbf{JID}$, by

(Lev 1) $\quad x \wedge x'^{*} \approx (x \wedge x'^{*})'^{*}$, (Lev 2) $\quad (x \wedge x'^{*})'^{*} \approx (x \wedge x'^{*})'^{*}{}'^{*}.$

In the rest of the paper we will use these definitions for the levels of $\mathbf{JID_1}$ and $\mathbf{JID_2}$.

3.2. **The Level of JID.**

Next, we wish to prove that **JID** is at Level 2.

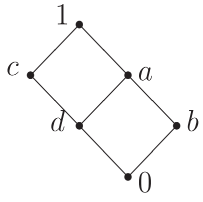
THEOREM 3.10. *We have*

(1) $\mathbf{JID_1} \subset \mathbf{JID}$,
(2) $\mathbf{JID} = \mathbf{JID_2}$.

Proof. First, we prove (2). That is, we need to prove that the "level 2" identity holds in **JID**. Let $x \in \mathbf{L}$.

$$\begin{aligned}
(x \wedge x'^{*})'^{*}{}'^{*} &= (x' \vee x'^{*}{}')^{*}{}'^{*} && \\
&= (x' \vee x''^{*})^{*}{}'^{*} && \text{by Lemma 3.6 (2)}\\
&= (x'' \wedge x'^{*})^{**} && \text{by Lemma 3.6 (5)}\\
&= (x' \vee x''^{*})^{*} && \text{by Lemma 3.6 (4)}\\
&= (x' \vee x'^{*}{}')^{*} && \text{by Lemma 3.6 (2)}\\
&= (x \wedge x'^{*})'^{*}.
\end{aligned}$$

Hence (2) is proved. For (1), we consider the following algebra **SIX** with its lattice reduct, $\to$ and $'$ as given in Figure 2. We note that $\mathbf{SIX} \in \mathbf{JID}$; but it is not of level 1 (at a). $\square$



$\to$:	0	1	a	b	c	d
0	1	1	1	1	1	1
1	0	1	a	b	c	d
a	0	1	1	b	c	c
b	c	1	1	1	c	c
c	b	1	a	b	1	a
d	b	1	1	b	1	1

$'$:	0	1	a	b	c	d
	1	0	b	b	c	1

Figure 2

The following corollary is immediate from the above theorem and [26, Corollary 8.2(a)].

COROLLARY 3.11. JID *is a discriminator variety of* level 2.

4. Dually Stone Semi-Heyting algebras

The study of dually Stone Heyting algebras goes back to [22], while the investigations into the variety **DSt** of dually Stone semi-Heyting algebras were initiated in [26]. In this section we will prove that the variety **DSt** is a discriminator variety of level 1 and also present some of its properties that, besides being of interest in their own right, will be needed in the later sections. We will also consider the subvariety **DStHC** of **DStH** generated by dually Stone Heyting chains and prove that the lattice of subvarieties of **DStHC** is an $\omega + 1$-chain–a result which was implicit in [26, Section 13].

It is well-known that the identity $(x \wedge y)^* \approx x^* \vee y^*$ holds in Stone algebras. The following lemma is just its dual.

LEMMA 4.1. *Let* $\mathbf{L} \in \mathbf{DSt}$. *Then* $\mathbf{L}$ *satisfies:* $(x \vee y)' \approx x' \wedge y'$.

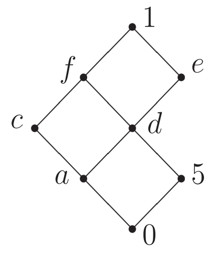
See Section 2 for the definition of the condition (SC). The following theorem will be useful in the sequel.

THEOREM 4.2. *Let* $\mathbf{L} \in \mathbf{DSt}$. *Then*

(a) $\mathbf{L} \models x'' \approx x'^*$;
(b) $\mathbf{L} \models$ (Lev 1);
(c) *If* $\mathbf{L} \models$ (SC), *then* $\mathbf{L} \in \mathbf{JID_1}$.

Proof. We note that (a) is the dual of a well known property of Stone algebras. From (a) we have $(x \wedge x'^*)'^* = (x \wedge x'')'^* = x'''^* = x'^* = x'' = x \wedge x'^*$, implying that (b) holds. Finally, let $\mathbf{L} \in \mathbf{DSt}$ and satisfy (SC), and let $a \in L \setminus \{1\}$. Then, by (SC) and (a), we have $a'' = a \wedge a'' = a \wedge a'^* = 0$, implying $a' = 1$. Then it is straightforward to verify that $\mathbf{L} \models$ (JID). Hence, (c) holds, in view of (b). □

REMARK 4.3. In contrast to **DSt**, **DPC** is not, however, at level 1. For example, the algebra **EIGHT** with its lattice reduct, $\to$ *and* $'$ *as given below, is, in fact, in the subvariety of* **DPC**, *defined by:* $(x \vee y)' \wedge (x' \vee y)' \wedge (x \vee y')' = 0$*; but it fails to satisfy* (Lev 1) *identity.*



$'$:	0	1	e	c	a	b	f	d
	1	0	c	e	1	1	e	1

$\to$:	0	1	e	c	a	b	f	d
0	1	0	0	b	b	c	0	0
1	0	1	e	c	a	b	f	d
e	0	1	1	c	c	b	f	f
c	b	c	a	1	e	0	c	a
a	b	c	c	1	1	0	c	c
b	c	b	b	0	0	1	b	b
f	0	1	e	c	a	b	1	e
d	0	1	1	c	c	b	1	1

The following corollary is immediate from Lemma 4.1, Theorem 4.2 and [26, Corollary 8.2(a)].

COROLLARY 4.4. **DSt** *is a discriminator variety of level* 1.

Observe that Lemma 4.1 impies that **DSt** satisfies (BL). The following corollary is, therefore, immediate from Theorem 4.2(b) and Theorem 2.6.

COROLLARY 4.5. *Let* $\mathbf{L} \in \mathbf{DSt}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) **L** *is simple,*
(2) **L** *is subdirectly irreducible,*
(3) **L** *satisfies* (SC).

4.1. **The variety DStHC.**

Recall that **DStHC** is the variety generated by dually Stone Heyting chains. We now give an application of Corollary 4.5.

DEFINITION 4.6. *For* $n \in \mathbb{N}$, *let* $\mathbf{C^{dp}_n}$ *denote the* n*-element* **DStH**-*chain such that*
$C^{dp}_n = \{0, a_1, a_2, \ldots, a_{n-2}, 1\}$, *where* $0 < a_1 < a_2 < \cdots < a_{n-2} < 1$.

We denote by $\mathbf{V}(\mathbf{C^{dp}_n})$ the variety generated by $\mathbf{C^{dp}_n}$. (Note that $\mathbf{C^{dp}_3}$ is the same as $\mathbf{L^{dp}_1}$ given in [26].)

It follows from Corollary 3.3 that **DPCHC** = **DStHC**. The following theorem was implicit in [26, Section 13].

THEOREM 4.7. *The lattice of subvarieties of* $\mathbf{DStHC}$ *is the following* $\omega + 1$*-chain:*

$$\mathbf{V}(\mathbf{C_1^{dp}}) < \mathbf{V}(\mathbf{C_2^{dp}}) < \cdots < \mathbf{V}(\mathbf{C_n^{dp}}) < \cdots < \mathbf{DStHC}.$$

Proof. We claim that subdirectly irreducible algebras in $\mathbf{DStHC}$ are precisely the $\mathbf{DStH}$-chains. For, let $\mathbf{C^{dp}}$ be a $\mathbf{DStHC}$-chain and let $x \in \mathbf{C^{dp}}$. Since $x \leq x'$ or $x' \leq x$, it follows that $x = 1$ or $x' = 1$, for every $x \in \mathbf{C^{dp}}$, which implies that $\mathbf{C^{dp}}$ satisfies (SC). On the other hand, let $\mathbf{A} \in \mathbf{DStHC}$ satisfy (SC). Let $a \in A \setminus \{1\}$. By Theorem 4.2 (a) we have $a'^* \leq a$; hence by (SC), we get $a'^* = 0$, implying $a' = 1$, again by Theorem 4.2 (a). Since each $\mathbf{DStHC}$-chain satisfies the identity: (L) $(x \to y) \vee (y \to x) \approx 1$, it follows that $\mathbf{DStHC}$ satisfies it too, implying that $\mathbf{A} \models$ (L). Hence, any two elements of $\mathbf{A}$ are comparable in $\mathbf{A}$, so $\mathbf{A}$ is a $\mathbf{DStH}$-chain. Thus, $\mathbf{A} \in \mathbf{DStHC}$ is subdirectly irreducible iff $\mathbf{A}$ is a $\mathbf{DStH}$-chain. Now it is not hard to observe that if an identity fails in an infinite $\mathbf{DStHC}$-chain, then it fails in a finite $\mathbf{DStHC}$-chain. Thus $\mathbf{DStHC}$ is generated by finite $\mathbf{DStH}$-chains. Hence, the conclusion of the theorem follows. □

Note, however, that if we consider $\mathbf{DStC}$-chains with semi-Heyting reducts that are not Heyting algebras, the situation gets more complicated, since the structure of the lattice of subvarieties of $\mathbf{DStC}$ is quite complex, as shown by the following class of examples: Let A be a semi-Heyting algebra. Let A^e be the expansion of A by adding a unary operation $'$ as follows:

$$x' = 0, \text{ if } x = 1, \text{ and } x' = 1, \text{ otherwise.}$$

Then it is clear that A^e is a $\mathbf{DSt}$-algebra and is simple. In particular, if A is a semi-Heyting-chain, then $A^e \in \mathbf{DStC}$ and is simple. Furthermore, the number of semi-Heyting chains even for a small size is large; for example, there are 160 semi-Heyting chains of size 4 and, therefore, there are 160 $\mathbf{DStC}$-chains of size 4. If we denote the 2-element, non-Boolean, dually Stone semi-Heyting algebra by $\bar{\mathbf{2}}^{\mathbf{e}}$, then it is interesting to observe that $\bar{\mathbf{2}}^{\mathbf{e}} \in \mathbf{DStC} \setminus \mathbf{DStHC}$, and $\mathbf{DStHC}$ is only a "small" subvariety of $\mathbf{DStC}$. These observations naturally suggest that the following open problem is of interest:

Problem: Investigate the structure of the lattice of subvarieties of DStC.

5. SUBDIRECTLY IRREDUCIBLE ALGEBRAS IN $\mathbf{JID_1}$

Recall that the variety $\mathbf{JID_1}$ is the subvriety of $\mathbf{JID}$ defined by

$$\text{(Lev 1)} \qquad x \wedge x'^* \approx (x \wedge x'^*)'^*.$$

In this section we give a somewhat concrete characterization of subdirectly irreducible (=simple) algebras in the variety $\mathbf{JID_1}$.

The following theorem follows immediately from Theorem 3.10 and Theorem 2.6.

THEOREM 5.1. *Let* $\mathbf{L} \in \mathbf{JID_1}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is simple,*
(2) $\mathbf{L}$ *is subdirectly irreducible,*
(3) $\mathbf{L}$ *satisfies* (SC).

We now wish to refine further the above characterization of the subdirectly irreducible algebras in $\mathbf{JID_1}$. In view of the above theorem, it suffices to characterize the algebras in $\mathbf{JID_1}$ satisfying the condition (SC).

Unless otherwise stated, in the rest of this section we assume that $\mathbf{L} \in \mathbf{JID_1}$ with $|L| \geq 2$ and satisfies the simplicity condition (SC).

LEMMA 5.2. *Let* $a, b \in L$ *such that* $a' = a$. *Then*

$$a \vee b \vee b^* = 1.$$

Proof. From Lemma 3.4 (4) and $a' = a$, we have

$$(1) \qquad (a \vee b) \to a = a \vee b^*.$$

Now,

$$\begin{aligned} a \vee (a \vee b)'^* &= a' \vee [(a \vee b)' \to 0] \\ &= [a' \vee (a \vee b)'] \to (a' \vee 0), \text{ by (JID)} \\ &= a' \to a' \qquad\qquad \text{as } a' \geq (a \vee b)' \\ &= 1. \end{aligned}$$

Thus, we have

$$(2) \qquad a \vee (a \vee b)'^* = 1.$$

If $a \vee b = 1$, then clearly the lemma is true. So, we assume that $a \vee b \neq 1$. Then $(a \vee b) \wedge (a \vee b)'^{*} = 0$ by (SC), and hence, we have

$$\begin{aligned}
a &= a \vee (b \wedge b^*) \\
&= (a \vee b) \wedge (a \vee b^*) \\
&= [(a \vee b) \wedge (a \vee b)'^{*}] \vee [(a \vee b) \wedge (a \vee b^*)] && \text{by (SC)} \\
&= (a \vee b) \wedge [(a \vee b)'^{*} \vee (a \vee b^*)] \\
&= (a \vee b) \wedge [(a \vee b)'^{*} \vee \{(a \vee b) \to a\}] && \text{by (1)} \\
&= (a \vee b) \wedge [\{(a \vee b) \to a\} \vee (a \vee b)'^{*}] \\
&= (a \vee b) \wedge [a \vee (a \vee b)'^{*}] && \text{by Lemma 2.3(3)} \\
&= a \vee b && \text{by (2).}
\end{aligned}$$

Hence, $a \vee b = a$, which implies, by (1), that $a \vee b^* = 1$. The conclusion of the lemma is now immediate. □

LEMMA 5.3. *Let $x \in L \setminus \{1\}$. Then $x \leq x'$.*

Proof. Since $x \neq 1$, we have $x \wedge x'^{*} = 0$ by (SC), from which we get $(x' \vee x) \wedge (x' \vee x'^{*}) = x'$, whence $x' \vee x = x'$, as $x' \vee x'^{*} = 1$ by Lemma 3.4 (3), proving the lemma. □

LEMMA 5.4. *Let $|L| > 2$ and let $a \in L$ such that $a' = a$. Then the height of L is at most 2.*

Proof. Suppose there are $b, c \in L$ such that $0 < b < c < 1$. We wish to arrive at a contradiction.
From Lemma 5.3 we have $c \leq c'$, from which it follows that

$$b \leq c'. \tag{3}$$

Claim 1: $b' = 1$.

Suppose $b' \neq 1$. Then, by Lemma 5.3, we get $b' \leq b'' \leq b \leq c$; thus $b' \leq c$. Next, $b \leq c$ implies $c' \leq b'$; and also $c \leq c'$ from Lemma 5.3, whence $c \leq b'$. Thus we conclude that $b' = c$, whence $c' = b'' \leq b$, implying $c' = b$, by (3). Then, in view of Lemma 5.3. we have $c \leq c' = b$; thus $c \leq b$, which is a contradiction, proving the claim.

From Lemma 5.2 we have $a \vee b \vee b^* = 1$. Hence, $a' \wedge b' \wedge b^{*\prime} = 0$ by Theorem 3.5, implying $a \wedge b^{*\prime} = 0$ by **Claim 1** and the hypothesis. Thus

$$a \wedge b^{*\prime} = 0. \tag{4}$$

Therefore, $a \vee b^* \geq a \vee b^{*\prime\prime} = 1$ as $a' = a$, yielding $b \leq a$. Hence, again from (4), we obtain

$$b \wedge b^{*\prime} = 0. \tag{5}$$

Claim 2: $b \vee b^* = 1$.

Suppose the claim is false. Then $b \leq b \vee b^* \leq (b \vee b^*)'$ by Lemma 5.3, whence $b \leq b' \wedge b^{*\prime}$, which implies $b = b \wedge b' \wedge b^{*\prime} = 0$ by the equation (5), contrary to $b > 0$, proving the claim.

From **Claim 2** and Theorem 3.5 we have $b' \wedge b^{*\prime} = 0$, Since $b' = 1$ by **Claim 1**, it follows that $b^{*\prime} = 0$, whence $b^* \geq b^{*\prime\prime} = 1$; so $b \leq b^{**} = 0$, contradicting $b > 0$, proving the lemma. □

LEMMA 5.5. *For every $x \in \mathbf{L}$, $x = 1$ or $x' = 1$ or $x = x'$.*

Proof. Suppose $x \in L$ such that $x \neq 1$ and $x' \neq 1$. Then by Lemma 5.3, we have $x \leq x'$. Also, since $x' \neq 1$, we have $x' \leq x'' \leq x$, again by Lemma 5.3. So, $x = x'$, proving the lemma. □

LEMMA 5.6. *Let $a \in L$ such that $a' = a$. Then $a^{*\prime} = a^*$.*

Proof. First, observe that $a \neq 0$ and $a \neq 1$, since $a = a'$. Suppose $a^{*\prime} \neq a^*$. The following claims will lead to a contradiction.

Claim 1: $a^* = a^{*\prime\prime}$.

$a \vee a^{*\prime\prime} = a'' \vee a^{*\prime\prime} = (a \vee a^*)'' = [a' \vee (a' \to 0)]'' = 1$ by Lemma 3.4(3). Hence, $a \vee a^{*\prime\prime} = 1$, implying $a^* \leq a^{*\prime\prime}$, and so $a^* = a^{*\prime\prime}$, proving the claim.

Claim 2: $a^* = 0$.

We have, by Lemma 5.5, that $a^{*\prime} = 1$ or $a^{*\prime\prime} = 1$ or $a^{*\prime} = a^{*\prime\prime}$. So, by **Claim 1**, we get $a^* = a^{*\prime\prime} = 0$ or $a^* = 1$ (as $a^* \geq a^{*\prime\prime}$) or $a^{*\prime} = a^*$. But, we know, by our assumption, that $a^* \neq a^{*\prime}$. Hence, $a^* = 0$ or $a^* = 1$, which clearly implies $a^* = 0$ or $a = 0$. Since we know that $a \neq 0$, the claim is proved.

Now, in view of (JID) and **Claim 2**, we have $a = a \vee 0 = a \vee a^* = a' \vee (a \to 0) = (a' \vee a) \to (a' \vee 0) = a \to a = 1$, implying $a = 1$, which is a contradiction, proving the lemma. □

PROPOSITION 5.7. *Let $|L| > 2$. Suppose there is an $a \in L$ such that $a' = a$. Then $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism.*

Proof. In view of Lemma 5.4 and $|L| > 2$, the height of L is exactly 2. Since the lattice reduct of L is distributive, L is either a 3-element chain or a 4-element Boolean lattice. We know from Lemma 5.6 that $a^{*\prime} = a^*$. Thus a and a^* are complementary, implying that the lattice reduct of $\mathbf{L}$ is a 4-element Boolean lattice; so $\mathbf{L} \models (Bo)$, and hence $\mathbf{L} \in \mathbf{DQB}$. Then, from Proposition 2.4 (a) it follows that $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism. □

PROPOSITION 5.8. *Suppose $x' \neq x$, for every $x \in L$. Then* $\mathbf{L} \in \mathbf{DSt}$.

Proof. Let $x \in L$. Without loss of generalty, we can assume that $x \neq 1$. Then we claim that $x' = 1$. For, assume that $x' \neq 1$. Then, by Lemma 5.3 we get $x \leq x'$ and $x' \leq x''$, which implies $x = x'$, as $x'' \leq x$, contradicting the hypothesis. So, we have $x' = 1$, which implies $x' \wedge x'' = 0$, Hence $\mathbf{L}$ is a dually Stone semi-Heyting algebra, completing the proof. □

We are now ready to prove our main theorem of this section.

THEOREM 5.9. *Let* $\mathbf{L} \in \mathbf{DQD}$ *with* $|L| \geq 2$. *Then the following are equivalent:*

(a) $\mathbf{L}$ *is a subdirectly irreducible algebra in* $\mathbf{JID_1}$,
(b) $\mathbf{L}$ *is a simple algebra in* $\mathbf{JID_1}$,
(c) $\mathbf{L} \in \mathbf{JID_1}$ *such that* (SC) *holds in* $\mathbf{L}$,
(d) $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, *up to isomorphism, or* $\mathbf{L} \in \mathbf{DSt}$ *such that* $\mathbf{L}$ *satisfies* (SC).

Proof. In view of Theorem 5.1, we only need to prove (c) $\Leftrightarrow$ (d). Now, suppose (d) holds. First, let us suppose $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism. Then it is routine to verify that $\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\} \subseteq \mathbf{JID_1}$ and $\{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$ satisfies (SC), implying (c). Next, suppose $\mathbf{L} \in \mathbf{DSt}$ such that $\mathbf{L}$ satisfies (SC). Then $\mathbf{L} \in \mathbf{JID_1}$, in view of Theorem 4.2(c), implying that (c) holds. Thus (d) $\Rightarrow$ (c). To prove the converse, suppose (c) holds. We consider two cases. First, suppose there is an $a \in L$ such that $a' = a$. Then, by Proposition 5.7, $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, up to isomorphism, implying (d).

Next, suppose there is no element $a \in L$ such that $a' = a$. Hence, $\mathbf{L}$ satisfies:

$$\text{For every } x \in L,\ x' \neq x. \tag{6}$$

Then, using Proposition 5.8, we obtain that $\mathbf{L}$ is dually Stone, which, together with the hypothesis, leads us to conclude (c) $\Rightarrow$ (d). □

We have the following important consequence of Theorem 5.9.

COROLLARY 5.10. $\mathbf{JID_1} = \mathbf{DSt} \ \vee \ \mathbf{V}(\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3})$.

Recall that $\mathbf{JIDH}$ is the subvariety of $\mathbf{JID}$ defined by the identity: $(x \wedge y) \to x \approx 1$, and $\mathbf{DStH}$ is the variety of dually Stone Heyting algebras. Now, we focus on the subvariety $\mathbf{JIDH_1}$ of $\mathbf{JIDH}$. Note that the variety of Boolean algebras is the only atom in the lattice of subvarieties of $\mathbf{JIDH_1}$. For $\mathbf{V}$ a subvariety of $\mathbf{JIDH_1}$, let $\mathcal{L}(\mathbf{V})$ and $\mathcal{L}^+(\mathbf{V})$ denote, respectively, the lattice of subvarieties of $\mathbf{V}$ and the lattice of nontrivial subvarieties of $\mathbf{V}$. Let $\mathbf{1} \oplus \mathbf{L}$ denote the ordinal sum of the trivial lattice $\mathbf{1}$ and a lattice $\mathbf{L}$.

Restricting the semi-Heyting reduct in the above corollary to Heyting algebras, we obtain the following interesting corollary, where $\mathbf{2}$ denotes a 2-element lattice.

COROLLARY 5.11. *We have*

(1) $\mathbf{JIDH_1} = \mathbf{DStH} \vee \mathbf{V}(\mathbf{D_2})$,
(2) $\mathcal{L}(\mathbf{JIDH_1}) \cong \mathbf{1} \oplus (\mathcal{L}^+(\mathbf{DStH}) \times \mathbf{2})$.

The preceding corollary leads to the following open problem.

PROBLEM: Investigate the structure of $\mathcal{L}^+(\mathbf{DStH})$.

6. JI-distributive, dually quasi-De Morgan, linear Semi-Heyting Algebras

In this section we focus on the linear identity:

(L) $\qquad (x \to y) \vee (y \to x) \approx 1.$

Let **DQDL** [**JIDL**] denote the subvariety of **DQD** [**JID**] defined by (L), and let **JIDLH** denote the subvariety of **JIDL** consisting of JI-distributive, dually quasi-De Morgan, linear Heyting algebras.

The following result is needed later in this section. Part (a) of it is proved in [26, Lemma 12.1(f)], and (b) follows immediately from (a).

PROPOSITION 6.1. [26, Lemma 12.1(f)] *Let* $\mathbf{L}$ *be a linear semi-Heyting algebra* (i.e., $\mathbf{L} \models$ (L)). Then

(a) $\mathbf{L} \models$ (H),
(b) $\mathbf{JIDL} = \mathbf{JIDLH}$.

LEMMA 6.2. *Let* $\mathbf{L} \in \mathbf{DQDL}$ *and let* $x, y \in L$. *Then*

(a) $(x \to y) \vee (y \to x)'' = 1$,
(b) $x \ \leq \ y \vee (y \to x)''$.

Proof. $(x \to y) \vee (y \to x)'' \geq (x \to y)'' \vee (y \to x)'' = [(x \to y) \vee (y \to x)]'' = 1$ by (L), proving (a). Using (a), we get $x \wedge [y \vee (y \to x)''] = (x \wedge y) \vee [x \wedge (y \to x)''] = [x \wedge (x \to y) \vee [x \wedge (y \to x)''] = x \wedge [(x \to y) \vee (y \to x)''] = x \wedge 1 = x$, implying (b). $\square$

Note that the algebra **SIX** described earlier in Section 3 is actually an algebra in **JIDL**. Hence **JIDL** does not satisfy (Lev 1); but **JIDL** is at level 2, in view of Theorem 3.10.

In this section, our goal is to present, as an application of Theorem 5.9, an explicit description of subdirectly irreducible (= simple) algebras in the variety $\mathbf{JIDL_1}$ of JI-distributive, dually quasi-De Morgan, linear semi-Heyting algebras of level 1.

Recall that $\mathbf{DPCC} = \mathbf{DStC}$ and $\mathbf{DPCHC} = \mathbf{DStHC}$. So, we use these names interchangeably.

LEMMA 6.3. $\mathbf{DPCC} \models$ (Lev1).

Proof. Let $\mathbf{L}$ be a $\mathbf{DPC}$-chain and let $x \in L$. Since x, x' are comparable, we have $x \vee x' = x$ or $x \vee x' = x'$, implying $x = 1$ *or* $x' = 1$, as $x \vee x' = 1$. Then it is easy to see that (Lev 1) holds in $\mathbf{L}$, and hence in $\mathbf{DPCC}$. □

PROPOSITION 6.4. $\mathbf{DPCHC} \vee \mathbf{V(D_2)} \subseteq \mathbf{JIDL_1}$.

Proof. It follows from Lemma 3.2, and Lemma 6.3 that $\mathbf{DPCHC}$ satisfies (JID) and (Lev 1), and it is easy to see that $\mathbf{DPCHC} \models$ (L). Also, it is routine to verify that (JID), (L) and (Lev 1) hold in $\mathbf{D_2}$. □

Our goal in this section is to prove that, in fact, the equality holds in the statement of the above Proposition.

Unless otherwise stated, in the rest of this section we assume that $\mathbf{L} \in \mathbf{JIDL_1}$ with $|L| > 2$ and satisfies (SC).

LEMMA 6.5. *Let $x, y \in L$ such that $x \vee y \neq 1$. Then, $x \leq y'$.*

Proof. Let $x \vee y \neq 1$. Since $y' \vee (x \vee y)'^{*} \geq y' \vee y'^{*} = 1$ by Lemma 3.4 (3), we get, using (SC), that $x = x \wedge (x \vee y) \wedge [y' \vee (x \vee y)'^{*}] = x \wedge [\{(x \vee y) \wedge y'\} \vee \{(x \vee y) \wedge (x \vee y)'^{*}\}] = x \wedge (x \vee y) \wedge y' = x \wedge y'$, whence $x \leq y'$. □

LEMMA 6.6. *Let $a, b \in L$ such that $a' \neq a$, $a \neq 1$, and $a \nleq b$. Then $(a \to b)'' = 0$.*

Proof. First, we claim that $a \nleq (a \to b)''$. For, suppose $a \leq (a \to b)''$; then $a = a \wedge (a \to b)'' \leq b$ by Lemma 2.2 (vii), implying $a \leq b$, which is a contradiction to the hypothesis $a \nleq b$. Hence $a \nleq (a \to b)''$. Then $a \vee (a \to b)' = 1$ by (the contrapositive of) Lemma 6.5, whence $a'' \vee (a \to b)''' = 1$. Since $a \neq 1$ and $a' \neq a$ by hypothesis, we get $a' = 1$ by Lemma 5.5, whence $a'' = 0$. Then we conclude that $(a \to b)'' = 0$, proving the lemma. □

We are now ready to give an explicit description of subdirectly irreducible (=simple) algebras in $\mathbf{JIDL_1}$.

THEOREM 6.7. *Let* $\mathbf{L} \in \mathbf{DQD_1}$ *with* $|L| > 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is a subdirectly irreducible algebra in* $\mathbf{JIDL_1}$,
(2) $\mathbf{L}$ *is a simple algebra in* $\mathbf{JIDL_1}$,
(3) $\mathbf{L} \in \mathbf{JIDL_1}$ *such that* (SC) *holds in* $\mathbf{L}$,
(4) $\mathbf{L} \cong \mathbf{D_2}$, *or* $\mathbf{L}$ *is a* $\mathbf{DStH}$*-chain.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 5.9. So we need to prove (3) $\Rightarrow$ (4) $\Rightarrow$ (3). Suppose (3) holds. Then, by Theorem 5.9, either $\mathbf{L} \in \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, or $\mathbf{L} \in \mathbf{DSt}$ and satisfies (SC). In the former case, since $\mathbf{L} \models$ (L), it follows from Proposition 6.1 that $\mathbf{L} \models$ (H). Hence $\mathbf{L} \cong \mathbf{D_2}$. Next, we assume the latter case. So, $\mathbf{L} \in \mathbf{DSt}$ and satisfies (SC). Since $\mathbf{L} \models$ (L) by hypothesis, we get, by Proposition 6.1, that $\mathbf{L} \in \mathbf{DStH}$. So, we need only prove that $\mathbf{L}$ is a chain. Let $a, b \in L \setminus \{1\}$ such that $a \not\leq b$. Then, from Lemma 6.2(b), we have that $b \leq a \vee (a \to b)''$. Also, since $\mathbf{L} \models$ (DSt), it is clear that $a' \neq a$. Hence, by Lemma 6.6, we get $(a \to b)'' = 0$, implying $b \leq a$. Thus, the lattice reduct of $\mathbf{L}$ is a chain, and so, (3) $\Rightarrow$ (4). Finally, assume (4) holds. First, if $\mathbf{L} \cong \mathbf{D_2}$, then it is routine to verify that (3) holds. Next, suppose $\mathbf{L}$ is a $\mathbf{DStH}$-chain and let $x \in L$. Then, $x' \leq x''$ or $x'' \leq x'$, implying $x' \wedge x'' = x'$ or $x' \wedge x'' = x''$. Hence, by (DSt), we get $x' = 0$ or $x' = 1$, from which it is easy to see that $\mathbf{L}$ satisfies (SC). So, from Theorem 4.2 (c), we conclude that $\mathbf{L} \in \mathbf{JID_1}$. Also, it is well known that Heyting chains satisfy (L). Thus, $\mathbf{L} \in \mathbf{JIDL_1}$ and $\mathbf{L}$ satisfies (SC), implying (3). □

The following corollary is immediate from Theorem 6.7.

COROLLARY 6.8. $\mathbf{JIDL_1} = \mathbf{DStHC} \vee \mathbf{V}(\mathbf{D_2})$.

We would like to mention here that the attempt to solve the problem of axiomatization of $\mathbf{DStHC} \vee \mathbf{V}(\mathbf{D_2})$ led to the results of this paper, with Corollary 6.8 yielding a solution to that problem.

We conclude this section with an axiomatization of $\mathbf{DStHC}$.

THEOREM 6.9. $\mathbf{DStHC} = \mathbf{DStL}$.

Proof. We know from the proof of Theorem 4.7 that the subdirectly irreducible algebras in $\mathbf{DStHC}$ are precisely the $\mathbf{DStH}$-chains. So, to complete the proof, it suffices to prove that the subdirectly irreducible algebras in $\mathbf{DStL}$ are precisely the $\mathbf{DStH}$-chains. For this, first note that from Proposition 6.1 we have that a linear semi-Heyting algebra satisfies (L) and hence $\mathbf{DStL} \models$ (H), implying $\mathbf{DStL} \subseteq \mathbf{DStH}$. Now, let $\mathbf{L}$ be a subdirectly irreducible (= simple) algebra in $\mathbf{DStL}$. We wish to show that $\mathbf{L}$ is a Heyting chain. Let $a, b \in L$ be arbitrary. By

Corollary 4.5, $\mathbf{L}$ satisfies (SC); and $\mathbf{L} \models$ (H), as $\mathbf{DStL} \models$ (H). Hence, by Lemma 6.2 (a), we have

$$\mathbf{L} \models (x \to y) \vee (y \to x)'' = 1. \tag{7}$$

Suppose that $a \neq 1$. Then from (SC) we have $a \wedge a'^{*} = 0$, whence $a \wedge a'' = 0$ by Lemma 4.2 (a), implying $a' = 1$. Thus we have proved

$$\text{For every } x \in L,\ x = 1 \text{ or } x' = 1. \tag{8}$$

Hence, by (8), we get $(a \to b)' = 1$ or $(a \to b)'' = 1$, implying $(a \to b)'' = 0$ or $(a \to b) = 1$. So,, by (7), we have $b \to a = 1$ or $a \to b = 1$, implying $b \leq a$ or $a \leq b$, as $\mathbf{L} \models$ (H). Thus $\mathbf{L}$ is a **DStH**-chain, completing the proof. □

7. More Consequences of Theorem 6.7

In this section we present some more consequences of Theorem 6.7.

As mentioned earlier, the axiomatizations of the variety **DPCHC** (= **DStHC**) and all of its subvarieties were given in [26].

The following corollary is immediate from Corollary 6.8 and Theorem 4.7.

COROLLARY 7.1.

(1) $\mathcal{L}(\mathbf{JIDL_1}) \cong \mathbf{1} \oplus [(\omega + \mathbf{1}) \times \mathbf{2}]$.

(2) $\mathbf{JIDL_1}$ *and* **DStHC** *are the only two elements of infinite height in the lattice* $\mathcal{L}(\mathbf{JIDL_1})$.

(3) $\mathbf{V} \in \mathcal{L}^{+}(\mathbf{JIDL_1})$ *is of finite height iff* $\mathbf{V}$ *is either* $\mathbf{V}(\mathbf{D_2})$, *or* $\mathbf{V}(\mathbf{C_n^{dp}})$ *for some* $n \in \mathbb{N} \setminus \{1\}$, *or* $\mathbf{V}(\mathbf{C_m^{dp}}) \vee \mathbf{V}(\mathbf{D_2})$ *for some* $m \in \mathbb{N} \setminus \{1\}$.

In Corollaries 7.2-7.5, we give equational bases to all subvarieties of $\mathbf{JIDL_1}$.

COROLLARY 7.2. *The variety* **DStHC** *is defined, modulo* $\mathbf{JIDL_1}$, *by*

$$x \vee x' \approx 1.$$

Proof. Observe that $\mathbf{DStHC} \models x \vee x' \approx 1$, but $\mathbf{V}(\mathbf{D_2}) \not\models x \vee x' \approx 1$, and then apply Theorem 6.7. □

The variety $\mathbf{V}(\mathbf{D_2})$ was axiomatized in [26]. Here is a new one.

COROLLARY 7.3. *The variety* $\mathbf{V}(\mathbf{D_2})$ *is defined, modulo* $\mathbf{JIDL_1}$, *by*

$x'' \approx x$.

Proof. Observe that $\mathbf{DStHC} \not\models x'' \approx x$, but $\mathbf{V(D_2)} \models x'' \approx x$, and then use Theorem 6.7. □

COROLLARY 7.4. *Let $n \geq 2$. The variety $\mathbf{V(C_n^{dp})} \vee \mathbf{V(D_2)}$ is defined, modulo $\mathbf{JIDL_1}$, by*

(C_n) $x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee \cdots \vee (x_{n-1} \to x_n) = 1.$

Proof. We now prove that $\mathbf{C_n^{dp}} \models (\mathrm{C_n})$. Let $\langle c_1, c_2, \ldots, c_n \rangle \in C_n^{dp}$ be an arbitrary assignment in C_n^{dp} for the variables x_i such that c_i is the value of x_i, for $i = 1, \cdots, n$. If $c_i \leq c_{i+1}$ for some i, then $c_i \to c_{i+1} = 1$, as $\mathbf{C_n^{dp}}$ has a Heyting algebra reduct, and hence, the identity holds in $\mathbf{C_n^{dp}}$. So, we assume that $c_i > c_{i+1}$, for $i = 1, 2, \cdots, n$. Then, $c_1 = 1$ since $|C_n^{dp}| = n$, implying that (C_n) holds in $\mathbf{C_n^{dp}}$. Also, it is routine to check that that $\mathbf{D}_2 \models (\mathrm{C}_2)$ and (C_i) implies (C_{i+1}), for $i = 2, \cdots, n-1$. So, $\mathbf{D}_2 \models (\mathrm{C_n})$, implying that $\mathbf{V(C_n^{dp})} \vee \mathbf{V(D_2)} \models (\mathrm{C_n})$.

Next, suppose that $\mathbf{V}$ is the subvariety of $\mathbf{JIDL_1}$ satisfying (C_n). Then, by Corollary 3.11, $\mathbf{V}$ is a discriminator variety. Let $\mathbf{L}$ be a simple (= subdirectly irreducible) algebra in $\mathbf{V}$. Then, it follows from Corollary 6.8 (or Theorem 6.7) that $\mathbf{L}$ is a $\mathbf{DStH}$-chain or $\mathbf{L} \cong \mathbf{D_2}$. Suppose that $\mathbf{L}$ is a $\mathbf{DStH}$-chain. Assume, if possible, $|L| > n$. Then, there exist $b_1, b_2 \cdots, b_{n-1} \in L$ such that $0 < b_1 < \cdots, < b_{n-1} < 1$. Since $\mathbf{L} \models (\mathrm{C}_n)$, we can assign $\langle b_{n-1}, b_{n-2}, \cdots, b_1, 0 \rangle$ for $\langle x_1, x_2, \cdots, x_{n-1}, x_n \rangle$. Then, $b_{n-1} \vee (b_{n-1} \to b_{n-2}) \vee \cdots, \vee (b_1 \to 0) = 1$, yielding $b_{n-1} \vee b_{n-2} \vee \cdots \vee b_1 \vee 0 = 1$, implying that $b_{n-1} = 1$, which is a contradiction. Thus we have $|L| \leq n$, from which it follows that $\mathbf{V} \subseteq \mathbf{V(C_n^{dp})} \vee \mathbf{V(D_2)}$, completing the proof. □

COROLLARY 7.5. *The variety $\mathbf{V(C_n^{dp})}$ is defined, modulo $\mathbf{JIDL_1}$, by*

(1) $x \vee x' \approx 1$,
(2) $x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \to x_2) \vee (x_2 \to x_3) \vee \cdots \vee (x_{n-1} \to x_n) = 1.$

For a different base for $\mathbf{V(C_n^{dp})}$, see [26]. Regularity was studied in [26], [27], [28] and [29]. Here is another use of it.

COROLLARY 7.6. *The variety $\mathbf{V(C_3^{dp})} \vee \mathbf{V(D_2)}$ is defined, modulo $\mathbf{JIDL_1}$, by*

$x \wedge x^+ \leq y \vee y^*$ (Regularity), where $x^+ := x'^{*'}$.

It is also defined, modulo $\mathbf{JIDL_1}$, by

$x \wedge x' \leq y \vee y^*$.

The variety $\mathbf{V(C_3^{dp})}$ is axiomatized in [26]. Here is another axiomatization for it.

COROLLARY 7.7. *The variety* $\mathbf{V}(\mathbf{C_3^{dp}})$ *is defined, modulo* $\mathbf{JIDL_1}$*, by*

(1) $x \wedge x^+ \leq y \vee y^*$ (Regularity),
(2) $x' = x^+$.

7.1. **Amalgamation Property.** We now examine the Amalgamation Property for subvarieties of the variety **DStHC**. For this purpose, we need the following lemma whose proof is straightforward.

We use "$\leq$" to abbreviate "is a subalgebra of" in the next lemma.

Recall from Theorem 4.7 (see also [26]) that the proper, nontrivial subvarieties of **DStHC** are precisely the subvarieties of the form $\mathbf{V}(\mathbf{C_n^{dp}})$, for $n \in \mathbb{N}$.

LEMMA 7.8. *Let* $m, n \in \mathbb{N}$*. Then*

$\mathbf{C_m^{dp}} \leq \mathbf{C_n^{dp}}$, *for* $m \leq n$.

COROLLARY 7.9. *Every subvariety of* **DStHC** *has Amalgamation Property.*

Proof. It follows from Corollary 4.4 that **DStHC** is a discriminator variety; and hence has CEP. Also, from Theorem 6.7 we obtain that every subalgebra of each subdirectly irreducible (= simple) algebra in **DStHC** is subdirectly irreducible. Let **V** be a subvariety of **DStHC**. Then, using a result from [11] that we need only consider an amalgam (**A**: **B**, **C**), where **A**, **B**, **C** are simple in **V** and **A** a subalgebra of **B** and **C**. First, suppose $\mathbf{V} = \mathbf{V}(\mathbf{C_n^{dp}})$ for some n. Then **B** and **C** are **DStHC**-chains. Then, in view of the preceding lemma, it is clear that the amalgam (**A**: **B**, **C**) can be amalgamated in **V**. Next, suppose $\mathbf{V} = \mathbf{DStHC}$, then it is clear that the amalgamation can be achieved as in the previous case. □

We conclude this section with the following remark: Since every subvariety **V** of **DStHC** has Congruence Extension Property and Amalgamation Property, it follows from Banachewski [6] that all subvarieties of **DStHC** have enough injectives (see [6] for the definition of this notion).

References

[1] M. Abad, J.M. Cornejo and J.P. Díaz Varela, *The variety Generated by semi-Heyting chains*, Soft computing, 15 (2010), no. 4, 721-728.

[2] M. Abad, J.M. Cornejo and J.P. Díaz Varela, *The variety of semi-Heyting algebras satisfying the equation* $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$, Rep. Math. Logic 46 (2011), 75-90.

[3] M. Abad, J.M. Cornejo and J. P. Díaz Varela. Semi-Heyting algebras term-equivalent to gödel algebras. *Order*, (2):625–642, 2013.

[4] M. Abad and L. Monteiro, *Free symmetric boolean algebras*, Revista de la U.M.A.(1976), 207–215

[5] R. Balbes and PH. Dwinger, **Distributive lattices**, Univ. of Missouri Press, Columbia, 1974.

[6] B. Banachewski, *Injectivity and essential extensions in equational classes of algebras*, Proceedings of the conference on Universal Algebra, Oct 1969 (Queen's University, Kingston, (1970), 131-147.

[7] S. Burris and H.P. Sankappanavar, **A course in universal algebra**, Springer-Verlag, New York, 1981. The free, corrected version (2012) is available online as a PDF file at math.uwaterloo.ca/∼snburris.

[8] J. M. Cornejo and H.P. Sankappanavar *A logic for dually hemimorphic semi-Heyting algebras and axiomatic extensions*, Arxiv:1908.02403v1 [math.LO] 7 Aug 2019, pp. 1-66.

[9] J. M. Cornejo and H. J. San Martin, *A categorical equivalence between semi-Heyting algebras and centered semi-Nelson algebras*, Logic Journal of IGPL, 28 , issue 4, (2018), 408-428.

[10] M. Dummett, *A propositional calculus with denumerable matrix*, J. Symbolic Logic 24 (1959), 97-106.

[11] G. Grätzer and H. Lakser, *The structure of pseudocomplemented distributive lattices II: Congruence extension and amalgamation*, Trans. Amer. Math. Soc., 156 (1971), 343-358.

[12] A. Horn, *Logic with truth values in a linearly ordered Heyting algebras*, J. Symbolic. Logic 34 (1969), 395–408.

[13] T. Katrinâk, *Remarks on the W.C.Nemitz's paper "Semi-Boolean lattices"* , Notre Dame Journal of Formal Logic, Volume XI, Number 4, (1970), 425-430.

[14] T. Hecht and T. Katrinâk, *Equational classes of relative Stone algebras*, Notre Dame Journal of Formal Logic Volume XIII, Number 2, (1972), 395–408.

[15] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. 21 (1967), 110–121.

[16] W. McCune, *Prover9 and Mace 4, http://www.cs.unm.edu/mccune/prover9/*

[17] V. Yu. Meskhi, *A discriminator variety of Heyting algebras with involution*, Algebra i Logika 21 (1982), 537–552.

[18] Gr. C. Moisil, Logique modale, Disquisitiones Mathematicae et Physica, 2 (1942), 3-98.

[19] A. Monteiro, *Sur les algebres de Heyting symetriques*, Portugaliae Mathemaica 39 (1980), 1–237.

[20] H. Rasiowa, **An algebraic approach to non-classical logics**, North–Holland Publ. Comp., Amsterdam, (1974).

[21] H. Rasiowa and R. Sikorski, **The Mathematics of Metamathematics**, Warsazawa, (1970).

[22] H. P. Sankappanavar, *Heyting algebras with dual pseudocomplementation*, Pacific J. Math. 117 (1985), 405–415.

[23] H. P. Sankappanavar, *Heyting algebras with a dual lattice endomorphism*, Math. Logic Quarterly (Zeitschr. f. math. Logik und Grundlagen d. Math.) 33 (1987), 565–573.

[24] H. P. Sankappanavar, *Semi-De Morgan algebras*, J. Symbolic. Logic 52 (1987), 712–724.

[25] H. P. Sankappanavar, *Semi–Heyting algebras: An abstraction from Heyting algebras*. In: Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish), Actas Congr. "Dr. Antonio A. R. Monteiro", pages 33–66, Bahía Blanca, 2008. Univ. Nac. del Sur.

[26] H. P. Sankappanavar, *Expansions of Semi-Heyting algebras. I: Discriminator varieties*, Studia Logica 98 (1-2) (2011), 27-81.

[27] H. P. Sankappanavar, *Dually quasi-De Morgan Stone Semi-Heyting algebras I. Regularity*, Categories and General Algebraic Structures with Applications, 2 (2014), 55-75.

[28] H. P. Sankappanavar, *Dually quasi-De Morgan Stone Semi-Heyting algebras II. Regularity*, Categories and General Algebraic Structures with Applications, 2 (2014), 77-99.

[29] H. P. Sankappanavar, *A note on regular De Morgan semi-Heyting algebras*, Demonstratio Mathematica, 49, No 3, (2016), 252-265.

[30] H. P. Sankappanavar, *Regular dually Stone semi-Heyting algebras*. In Preparation.

[31] H. P. Sankappanavar, *De Morgan semi-Heyting and Heyting algebras*, New Trends in Algebra and Combinatorics. Proceeding of the 3rd International Congress in Algebra and Combinatorics (ICAC2017), held on 25-28 August 2017, Hong Kong, China, March 10, 2020.

[32] J, Varlet, *A regular variety of type* $\langle 2, 2, 1, 1, 0, 0\rangle$, Algebra Univ. 2 (1972), 218-223.

[33] H. Werner, **Discriminator algebras**, Studien zur Algebra und ihre Anwendungen, Band 6, Academie–Verlag, Berlin, 1978.

Department of Mathematics
State University of New York
New Paltz, NY 12561

sankapph@newpaltz.edu

OPEN MAPPING THEOREMS WITH FINITE FIBRES FOR C-SPACES

Chieko Komoda

Received March 20, 2018 ; revised May 28, 2018, June 17, 2018

Abstract

In this paper we study theorems for C-spaces and finite C-spaces on dimension-raising open mappings and dimension-lowering open mappings with finite fibres.

Keywords and phrases. A-weakly infinite-dimensional, S-weakly infinite-dimensional, C-spaces, finite C-spaces, mapping theorems.

2000 Mathematics Subject Classification. Primary 54F45.

1 Introduction

In this paper we assume that all spaces are normal and all mappings are continuous.

A space X is *A-weakly infinite-dimensional* or *Alexandroff weakly infinite-dimensional* if for every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X there exists a collection $\{L_i : i < \omega\}$ of closed subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$, and $\bigcap_{i<\omega} L_i = \emptyset$.

A space X is a *C-space* [1] if for every countable collection $\{\mathcal{G}_i : i < \omega\}$ of open covers of X there exists a countable collection $\{\mathcal{H}_i : i < \omega\}$ of collections of pairwise disjoint open subsets of X such that $\mathcal{H}_i$ is a refinement of $\mathcal{G}_i$ for every $i < \omega$ and $\bigcup_{i<\omega} \bigcup\{H : H \in \mathcal{H}_i\} = X$.

It is easily seen that every C-space is A-weakly infinite-dimensional. However, it is still unknown whether every compact A-weakly infinite-dimensional metrizable space is a C-space.

A space X is S-*weakly infinite-dimensional* or *Smirnov weakly infinite-dimensional* if for every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X there exists a collection $\{L_i : i < \omega\}$ of closed subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$, and $\bigcap_{i \leq n} L_i = \emptyset$ for some $n < \omega$. It directly follows from the definition that every S-weakly infinite dimensional space is A-weakly infinite dimensional, and every compact A-weakly infinite dimensional space is S-weakly infinite dimensional.

A space X is a *finite C-space* [2] if for every collection $\{\mathcal{G}_i : i < \omega\}$ of finite open covers of X there exists a collection $\{\mathcal{H}_i : i < \omega\}$ of collections of pairwise disjoint open subsets of X such that $\mathcal{H}_i$ is a refinement of $\mathcal{G}_i$ for every $i < \omega$ and $\bigcup_{i \leq n} \bigcup \{H : H \in \mathcal{H}_i\} = X$ for some $n < \omega$. It is well-known [2] that every finite C-space is S-weakly infinite-dimensional. There exists a C-space which is not a finite C-space (see [1, Example 2.15]). However, every compact C-space is a finite C-space.

For paracompact spaces, Gutev and Valov [5] proved the countable sum theorem for C-spaces. For countably paracompact and collectionwise normal spaces, the author proved the countable sum theorem for C-spaces (cf. [6, Corollary 3.2]). Addis and Gresham [1] proved that every finite-dimensional, paracompact space is a C-space. By the same proof, we can show that every finite-dimensional space is a finite C-spaces. The following two Lemmas will play a important role in the proof of our main theorems.

Lemma A *If there exists a closed subset K of a countably paracompact collectionwise normal space X satisfying the following conditions (1) and (2), then X is a C-space.*

(1) K is a C-space,

(2) for every closed subset F of X with $F \cap K = \emptyset$, F is a C-space.

Proof. Let $\{\mathcal{G}_i : i < \omega\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, be a collection of open covers of X. Since K is a countably paracompact C-space, by [6, Lemma 2.1], there exists a collection $\{\mathcal{U}_{2i} : i < \omega\}$, where $\mathcal{U}_{2i} = \{U_\lambda : \lambda \in \Lambda_{2i}\}$, of discrete collections of open subsets of K such that $U_\lambda \subset G_\lambda \cap K$ and $\bigcup_{i<\omega} \bigcup\{U_\lambda : \lambda \in \Lambda_{2i}\} = K$. Since X is collectionwise normal, by [6, Lemma 2.2], there exists a collection $\{\mathcal{H}_{2i} : i < \omega\}$, where $\mathcal{H}_{2i} = \{H_\lambda : \lambda \in \Lambda_{2i}\}$, of discrete collections of open subsets of X such that $H_\lambda \cap K = U_\lambda$ and $H_\lambda \subset G_\lambda$. Let us set $F = X - \bigcup_{i<\omega} \bigcup\{H : H \in \mathcal{H}_{2i}\}$. Similarly there exsits a collection $\{\mathcal{H}_{2i+1} : i < \omega\}$ of discrete collections of open subsets of X such that $\mathcal{H}_{2i+1}$ is a refinement of $\mathcal{G}_{2i+1}$ for every $i < \omega$ and $\bigcup_{i<\omega} \bigcup\{H : H \in \mathcal{H}_{2i+1}\} \supset F$. We get the required collection $\{\mathcal{H}_i : i < \omega\}$.

Lemma B *If there exists a closed subset K of a space X satisfying the following conditions (1) and (2), then X is a finite C-space.*

(1) K is a finite C-space,

(2) for every closed subset F of X with $F \cap K = \emptyset$, F is a finite C-space.

Proof. Let $\{\mathcal{G}_i : i < \omega\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, be a collection of finite open covers of X. Since K is a finite C-space, there exists a collection $\{\mathcal{U}_{2i} : i < \omega\}$, where $\mathcal{U}_{2i} = \{U_\lambda : \lambda \in \Lambda_{2i}\}$, of finite collections of pairwise disjoint open subsets of K such that $U_\lambda \subset G_\lambda \cap K$ and $\bigcup_{i=1}^{n} \bigcup\{U_\lambda : \lambda \in \Lambda_{2i}\} = K$ for some $n < \omega$. Since K is normal, there exists $\{\mathcal{F}_{2i} : i \leq n\}$, where $\mathcal{F}_{2i} = \{F_\lambda : \lambda \in \Lambda_{2i}\}$, of collections of closed subsets of K such that $F_\lambda \subset U_\lambda$ and $\bigcup_{i=1}^{n} \bigcup\{F_\lambda : \lambda \in \Lambda_{2i}\} = K$. There exists a collection $\{\mathcal{H}_{2i} : i < \omega\}$, where $\mathcal{H}_{2i} = \{H_\lambda : \lambda \in \Lambda_{2i}\}$, of finite collections of pairwise disjoint open subsets of X for every $i < \omega$ such that $F_\lambda \subset H_\lambda \subset G_\lambda$ and $\bigcup_{i=1}^{n} \bigcup\{H_\lambda : \lambda \in \Lambda_{2i}\} \supset K$. For every $i > n$ we let $\mathcal{H}_{2i} = \{\emptyset\}$. Let us set $F = X - \bigcup_{i \leq n} \bigcup\{H : H \in \mathcal{H}_{2i}\}$. For a space F repeating above procedure we obtain the required collection $\{\mathcal{H}_i : i < \omega\}$.

2 Dimension-raising mappings

Polkowski [8] proved the following theorem.

Theorem [8]. *If $f : X \longrightarrow Y$ is an open mapping of an A-weakly infinite-dimensional space X onto a countably paracompact space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then Y is A-weakly infinite-dimensional.*

We shall prove the following theorem. This is an analogy of the above Polkowski's theorem.

2.1. Theorem *If $f : X \longrightarrow Y$ is an open mapping of a C-space X onto a countably paracompact and collectionwise normal Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then Y is a C-space.*

To prove Theorem 2.1 we need the following theorem and lemma.

2.2. Theorem(cf.[4, Lemma 6.7]) *If $f : X \longrightarrow Y$ is a closed mapping of a countably paracompact C-space X onto a space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then Y is a C-space.*

2.3. Lemma([3, Lemma 6.3.12]) *If all fibres of an open mapping $f : X \longrightarrow Y$ defined on a space X are finite and have the same cardinality, then f is closed.*

2.4 Proof of theorem 2.1. Let $K_j = \{y \in Y : |f^{-1}(y)| = j\}$ for every $j \in \mathbb{N}$. It is easy to see that the union $\bigcup_{j \leq i} K_j$ is closed in Y for every $i \in \mathbb{N}$. Inductively, we show that the union $\bigcup_{j \leq i} K_j$ is a C-space for every $i \in \mathbb{N}$. To this end, it suffices to show that every closed subspace Z of Y contained in K_i is a C-space, cf. Lemma A. By Lemma 2.3, the restriction $f|_{f^{-1}(Z)} : f^{-1}(Z) \longrightarrow Z$ is perfect. As the inverse image of a countably paracompact space under a perfect mapping is countably paracompact, then $f^{-1}(Z)$ is countably paracompact. By Theorem 2.2, Z is a C-space. Thus the union $\bigcup_{j \leq i} K_j$ is a closed C-space for every $i \in \mathbb{N}$. By countable sum theorem, Y is a C-space.

The following theorem is a counterpart for finite C-spaces of Polkowski's result.

2.5. Theorem *If $f : X \longrightarrow Y$ is an open mapping of a weakly paracompact finite C-space X onto a space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then Y is a finite C-space.*

To prove Theorem 2.5 we need the following theorem and lemma.

2.6. Theorem([4, Theorem 6.4]) *If $f : X \longrightarrow Y$ is a mapping of a compact C-space X onto a space Y such that $|f^{-1}(y)| < \mathfrak{c}$ for every $y \in Y$, then Y is a C-space.*

For each space X and $n < \omega$ we let

$$G_n(X) = \bigcup\{U \subset X : U \text{ is open and } \dim \mathrm{Cl\,U} \leq \mathrm{n}\}$$

and

$$S(X) = X - \bigcup_{n<\omega} G_n(X).$$

Sklyarenko ([9, Theorem 3]) proved the following lemma in the case when X is S-weakly infinite dimensional.

2.7. Lemma *A weakly paracompact space X is a finite C-space if and only if $S(X)$ is a compact finite C-space and every closed subspace $F \subset X$ disjoint from $S(X)$ is finite dimensional.*

Proof. Assume that the space X is a finite C-space. We shall show that $S(X)$ is compact. Suppose $S(X)$ is not compact. Since $S(X)$ is weakly paracompact, $S(X)$ is not pseudocompact. Thus there exists a countable discrete closed subspace F of $S(X)$. Let us set $F = \{x_i : i < \omega\}$. We can take a discrete collection $\{U_i : i < \omega\}$ of open subsets of X with $x_i \in U_i$ for every $i < \omega$. Thus we have $\dim \mathrm{Cl}\, U_i > i$ for every $i < \omega$. Let us set $Y = \bigcup\{\mathrm{Cl}\, U_i : i < \omega\}$. Since $\bigcup\{\mathrm{Cl}\, U_i : i < \omega\}$ is homeomorphic to $\bigoplus\{\mathrm{Cl}\, U_i : i < \omega\}$, Y is not a S-weakly infinite dimensional subspace of X. Thus Y is not a finite C-space. The contradiction shows that $S(X)$ is compact. Let F be a closed subset of X disjoint from $S(X)$. First, we shall show that $F \subset G_n(X)$ for some $n < \omega$. Suppose that for every $n < \omega$, $F \not\subset G_n(X)$. Since $F \setminus G_n(X)$ is infinite for every $n < \omega$, inductively, we choose points $x_1, x_2, \cdots$ such that $x_n \in F \setminus (G_n(X) \cup \{x_1, x_2, \cdots, x_{n-1}\})$ for every $n < \omega$. The space $E = \{x_n : n < \omega\}$ is a closed discrete subspace of F. For a space $E = \{x_n : n < \omega\}$ repeating above procedure we obtain a contradiction. Thus $F \subset G_n(X)$ for some $n < \omega$. Since X is weakly paracompact, by the point finite sum theorem, $\dim F \leq n$. By Lemma B, the converse holds. Lemma 2.7 has been proved.

2.8 Proof of Theorem 2.5. By Lemma 2.7, $S(X)$ is compact. Applying Theorem 2.6 to $f|_{S(X)}$, $f(S(X))$ is a finite C-space. For each closed subspace $F \subset Y$ disjoint from $f(S(X))$, as $f^{-1}(F) \cap S(X) = \emptyset$, by Lemma 2.7, we take an integer n with $\dim f^{-1}(F) \leq n$. As the restriction $f|_{f^{-1}(F)} : f^{-1}(F) \longrightarrow F$ is open, by Nagami [7] (cf.

[3, 3.3.G]), $\dim F = \dim f^{-1}(F) \leq n$. Thus F is a finite C-space. By Lemma B, Y is a finite C-space.

3 Dimension-lowering mappings

The following theorem is a counterpart for C-spaces of Polkowski's result, which was proved in the case when A-weakly infinite-dimensional (see [8, Theorem 3.3 (ii)]).

3.1. Theorem *If $f : X \longrightarrow Y$ is an open mapping of a paracompact space X onto a C-space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then X is a C-space.*

To prove Theorem 3.1 we need the following lemma.

3.2. Lemma([7], cf [8, Lemma B]) *If $f : X \longrightarrow Y$ is an open mapping of a space X to a space Y and there exists an integer $n \geq 1$ such that $|f^{-1}(y)| = n$ for every $y \in Y$, then f is a local homeomorphism.*

3.3 Proof of Theorem 3.1. For every $n \in \mathbb{N}$ we set

$$Y_n = \{y \in Y : |f^{-1}(y)| = n\} \text{ and } X_n = f^{-1}(Y_n).$$

It is easy to see that the union $Y'_n = \bigcup_{k \leq n} Y_k$ is closed in Y for every $n \in \mathbb{N}$, therefore the union $X'_n = \bigcup_{k \leq n} X_k$ is also closed in X. Since X is the union of countable collection $\{X'_n : n \in \mathbb{N}\}$ of closed subsets of X, by the countable sum theorem for C-spaces, we only prove that X'_n is a C-space for every $n \in \mathbb{N}$. Let $f_n : X_n \longrightarrow Y_n$ be the mapping defined by $f_n(x) = f(x)$ for every $x \in X_n$.

Obviously, X'_1 is a C-space, because f_1 is a homeomorphism. Assume that X'_{n-1} is a C-space. To prove that X'_n is a C-space, it suffices to show that every closed subset Z of X'_n contained in X_n is a C-space.

By Lemma 3.2, the mapping f_n is a local homeomorphism. Thus for every $x \in X_n$ we can take a neighborhood U_x of x in X_n such that the restriction $f_n|_{U_x} : U_x \longrightarrow Y_n$ is an embedding. Since X_n is open in X'_n, U_x is open in X'_n. We may assume that U_x is an F_σ-set of X'_n. Let $U_x = \cup\{A(x,m) : m \in \mathbb{N}\}$, where $A(x,m)$ is closed in X'_n. For every $y \in Y_n$ let us set $f^{-1}(y) = \{x(y,1), x(y,2), \ldots, x(y,n)\}$. Then the intersection $\bigcap_{i=1}^{n} f(U_{x(y,i)})$ is a neighborhood of y in Y_n'. Take an open F_σ-set V_y of y in Y_n' such that $y \in V_y \subset \bigcap_{i=1}^{n} f(U_{x(y,i)})$. Let $V_y = \cup\{B(y,\ell) : \ell \in \mathbb{N}\}$, where $B(y,\ell)$ is closed in Y_n'. The set $W(y,i) = U_{x(y,i)} \cap f^{-1}(V_y)$ is homeomorphic to $f(W(y,i))$. We have

$$W(y,i) = \bigcup\{A(x(y,i),m) \cap f^{-1}(B(y,\ell)) : m, \ell \in \mathbb{N}\}.$$

We shall prove that $A(x(y,i),m) \cap f^{-1}(B(y,\ell))$ is a C-space. Since $f_n|_{U_x(y,i)}$ is an embedding, $A(x(y,i),m) \cap f^{-1}(B(y,\ell))$ is homeomorphic to $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$.

By Lemma 2.3, f_n is closed, therefore $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$ is closed in Y_n. Since $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell))) \subset B(y,\ell) \subset Y_n$, $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$ is closed in $B(y,\ell)$. As $B(y,\ell)$ is a C-space, $f_n(A(x(y,i),m) \cap f^{-1}(b(y,\ell)))$ is a C-space. Thus $A(x(y,i),m) \cap f^{-1}(b(y,\ell))$ is a C-space. By the countable sum theorem for C-spaces, $W(y,i)$ is a C-space. Since Z is paracompact, the open cover $\mathcal{W} = \{W(y,i) \cap Z : y \in Y_n, 1 \leq i \leq n\}$ of Z has a locally-finite closed refinement $\mathcal{F}$. Since every member of $\mathcal{F}$ is a C-space, by the locally finite sum theorem for C-spaces (cf. [6, Theorem 1.1(i)]), Z is a C-space. Theorem 3.1 has been proved.

3.4. Theorem *If $f : X \longrightarrow Y$ is a closed-and-open mapping of a space X onto a weakly paracompact finite C-space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then X is a finite C-space.*

Proof. Since for every $y \in Y$ $|f^{-1}(y)| < \omega$, the closed mapping $f : X \longrightarrow Y$ is perfect. As $S(Y)$ is compact, $f^{-1}(S(Y))$ is compact. By Theorem 3.1, $f^{-1}(S(Y))$ is a finite C-space. For each closed subset $F \subset X$ disjoint from $f^{-1}(S(Y))$, as $f(F) \cap S(Y) = \emptyset$, by Lemmma 2.7, we take integer n with $\dim f(F) \leq n$. As $f|_F : F \longrightarrow f(F)$ is closed, by [3, Theorem 3.3.10], $\dim F \leq \dim f(F) \leq n$. Thus F is a finite C-space, by Lemma B, X is a finite C-space.

References

[1] D. F. Addis and J. H. Gresham, *A class of infinite-dimensional spaces, Part I: Dimension theory and Alexandroff's Problem*, Fund. Math. 101(1978), 195-205.

[2] P. Borst, *Some remarks concerning C-spaces*, Top. Appl. 154(2007), 665-674.

[3] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, 1995.

[4] V. V. Fedorchuk, *Some classes of weakly infinite-dimensional spaces*, Journ. of Math. Sci. 155, No. 4 (2008), 523-570.

[5] V. Gutev and V. Valov, *Continuous selections and C-spaces*, Proc. Amer. Math. Soc. 130 (2002), 233-242.

[6] C. Komoda, *Sum theorems for C-spaces*, Sci. Math. Japonicae 59(2004), 71-77.

[7] K. Nagami, *Mappings of finite order and dimension theory*, Jap. Journ. of Math. 30(1960), 25-54.

[8] L. Polkowski, *Some theorems on invariance of infinite dimension under open and closed mappings*, Fund. Math. 119(1983), 11-34.

[9] E. G. Sklyarenko, *On dimensional properties of infinite dimensional spaces*, Amer. Math. Soc. Transl. Ser. 2, 21(1962), 35-50.

Communicated by *Yasunao Hattori*

Chieko Komoda

Department of Liberal Arts, Kurume National College of Technology

Kurume, Fukuoka, 830-8555, Japan

E-mail address: komoda@kurume-nct.ac.jp

CZECH REPUBLIC

(a) **Milan Vlach**
(b) Charles University, Faculty of Mathematics and Physics
Malostranske namesti 25, 118 00 Prague 1, Czech Republic
(b’) milan.vlach@mff.cuni.cz and mvlach@ksi.ms.mff.cuni.cz
(c) Game theory, Fair division, Optimization

FINLAND

(a) **Arto Salomaa**
(b) Jaanintie 34 A 26, 20540 Turku, Finland
(b') asalomaa@utu.fi
(c) Formal languages, automata, computability, cryptography

(a) **Jarkko Kari**
(b) Department of Mathematics and Statistics, FI-20014 University of Turku, Finland
(b') jkari@utu.fi
(c) automata theory, cellular automata, tilings, symbolic dynamics

GERMANY

(a)**Klaus Denecke**
(b) 14542 Werder (Havel), Germany, Zanderweg 3
(b') klausdenecke@hotmail.com
(c) General Algebra, Discrete Mathematics, Multiple-valued Logic, Ordered Sets and Lattices, Theory of Semigroups

GREECE

(a) **Maria Fragoulopoulou**
(b) Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, Greece
(b') fragoulop@math.uoa.gr
(c) Non-normed Topological Algebras, Topological Algebras with an Involution, Unbounded Operator Theory,
Tensor products of Topological Algebras and Topological *-Algebras.

HUNGARY

(a) **Gyula Maksa**
(b) Institute of Mathematics, University of Debrecen, H-4002 Debrecen, Pf. 400, Hungary
(b') maksa@science.unideb.hu
(c) Functional equations

(a) **Kálmán Györy**
(b) University of Debrecen, Institute of Mathematics, 4010 Debrecen, Hungary
(b') gyory@science.unideb.hu
(c) Number Theory (mainly Diophantine and Algebraic Number Theory)

(a) **Pál Dömösi**
(b) Institute of Mathematics and Informatics, Nyíregyháza University, Nyíregyháza, Sóstói út 31/B, H-4400, Hungary
(b') domosi.pal@nye.hu
(c) Theoretical Computer Science, Algebra

ISRAEL

(a) **Dany Leviatan**
(b) School of Mathematics, Tel Aviv University, 6139001 Tel Aviv, Israel
(b') leviatan@post.tau.ac.il
(c) Approximation Theory, Computer Added Geometric Design, Summability

ITALY

(a) **Angelo Favini**
(b) Dept. of Mathematics, Bologna Univ., Piazza di Porta S. Donato, 5, 40126, Bologna, Italy
(b') angero.favini@unibo.it
(c) Evolution equations and Control theory for abstract differential equations and PDE.

(a) **Antonio Di Crescenzo**
(b) Università di Salerno, Dipartimento di Matematica, Via Giovanni Paolo II, n.132, 84084 Fisciano (SA), Italy
(b') adicrescenzo@unisa.it
(c) Applied Probability, Stochastic Processes and Applications, Reliability Theory, Queueing Systems, Stochastic Models in Biology, Information Measures

(a) **Tonia Ricciardi**
(b) Department of Mathematics and its Applications, Federico II University, Via Cintia, 80126 Naples, Italy.
(b') tonia.ricciardi@unina.it
(c) Nonlinear elliptic partial differential equations

NETHERLANDS

(a) **Grzegorz Rozenberg**
(b) Leiden Institute of Advanced Computer Science (LIACS) Leiden University, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
(b') rozenber@liacs.nl
(c) Natural computing, Formal languages and automata theory

POLAND

(a) **Dariusz Zagrodny**
(b) Faculty of Mathematics and Natural Science, College of Science, Cardinal Stefan Wyszynski University, Dewajtis 5, 01-815 Warsaw, Poland
(b')dariusz.zagrodny@wmii.uni.lodz.pl
(c) Nonsmooth Analysis (this is my main field of research), Nonlinear Programming (convex, nonconvex), Maximal Monotonicity (with respect to duality)

(a) **Henryk Hudzik**
(b) Faculty of Economics and Information Technology, The State University of Applied Sciences in Płock, Nowe Trzepowo 55, 09-402 Płock, Poland
and
Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska Street 87, 61-614 Poznań, Poland.
(b') hudzik@amu.edu.pl
(c) Function spaces theory and abstract Banach spaces theory, Banach Lattices, Geometry of Banach Spaces, Composition and multiplication operators between Köthe spaces

(a) **Krzysztof Szajowski**
(b) Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology, Wybrzeze, Wyspianskiego 27, PL-50-370 Wroclaw, Poland
(b') Krzysztof.Szajowski@pwr.edu.pl
(c) Applied Probability, Game Theory, Operations Research, Mathematical Statistics

(a) **Piotr Multarzynski**
(b) Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland
(b') multarz@mini.pw.edu.pl
(c) Algebraic analysis (calculus of right invertible operators); Algebraic approach to differential geometry; Discrete counterparts of the classical concepts in analysis and differential geometry; q-calculus, Sikorski and Froelicher differential (or smooth) spaces; 12H10, 39A12, 39A70, 47B39, Groupoids - theory and applications in physics.

(a) **Tomasz Kubiak**
(b) Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
(b') tkubiak@amu.edu.pl
(c) Many-valued topology (fuzzy topology), Pointfree topology (frames and locales)

P.R.OF CHINA

(a) **Minghao Chen**
(b) Xidazhi Street 92,Harbin ,150001,China Department of Mathematics,Harbin Institute of Technology
(b') chenmh130264@aliyun.com;chenmh130264@hit.edu.cn
(c) Uncertain dynamical systems;Fuzzy differential equation; Fuzzy optimization;Fuzzy sets theory

ROMANIA

(a) **Adrian Petrusel**
(b) Babes-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, Department of Mathematics, Kogalniceanu street no. 1, 400084 Cluj-Napoca, Romania
(b') petrusel@math.ubbcluj.ro
(c) Nonlinear Analysis

(a) **Ioan A. Rus**
(b) Department of Mathematics, Babes-Bolyai University, Str. Kogalniceanu No. 1, 400084 Cluj-Napoca, Romania
(b') iarus@math.ubbcluj.ro
(c) Fixed Point Theory

(a) **Vasile Berinde**
(b) Department of Mathematics and Computer Science, Faculty of Sciences Technical University of Cluj-Napoca North University Center at Baia Mare, Victoriei Nr. 76, 430122 Baia Mare, Romania
(b') vberinde@cubm.utcluj.ro
(c) Fixed Point Theory, iterative approximation of fixed points

RUSSIA

(a) **Andrei Vesnin**
(b) Sobolev Institute of Mathematics, pr. ak. Koptyuga 4, Novosibirsk, 630090, Russia
(b') vesnin@math.nsc.ru
(c) Low-dimensional topology, Knot theory, Hyperbolic manifolds and orbifolds.

(a) **Semen S. Kutateladze**
(b) The Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Academician Koptyug's Avenue 4, Novosibirsk, 630090, RUSSIA
(b') sskut@member.ams.org and sskut@math.nsc.ru
(c) Functional Analysis, Operator Theory, Convex Geometry, Optimization and Programming, Nonstandard Analysis, Boolean Valued Models

UNITED STATES OF AMERICA

(a) **Andreas Blass**
(b) Mathematics Department, University of Michigan, Ann Arbor, MI 48109-1043, USA
(b') ablass@umich.edu
(c) Mathematical logic, set theory, category theory

(a) **John B Conway**
(b) Professor Emeritus, George Washington University,Phillip Hall 801 22nd St. NW
Washington, DC 20052, U.S.A
(b’) Conway@gwu.edu
(c) Functional Analysis and Operator Theory

(a) **Paul Cull**
(b) Computer Science, Kelley Engineering Center, Oregon State University, Corvallis, OR 97331, USA
(b') pc@cs.orst.edu
(c) Difference Equations and Dynamical Systems, Computer Science (Theory, Algorithms, Networks), Mathematical Biology (Population Models, Neural Nets)

(a) **W. Wistar Comfort**
(b) Department of Mathematics, Wesleyan University, Wesleyan Station, Middletown, CT USA 06459
(b') wcomfort@wesleyan.edu
(c) Topological theory of topological groups, General (set-theoretic) topology

JAPAN

(a) **Mariko Yasugi**
(b) non-public
(b’) yasugi@cc.kyoto-su.ac.jp
(c) Logic Oriented Mathematics

(a) **Haruo Maki**
(b) non-public
(b') makih@pop12.odn.ne.jp
(c) (Topological) digital n-spaces (n>0), Generalized closed sets (after Levine),
Operation theory in topology (in the sense of Kasahara and Ogata)

(a) **Kohzo Yamada**
(b) Faculty of Education, Shizuoka Univ., 836 Ohya, Shizuoka 422-8529, Japan
(b’) kohzo.yamada@shizuoka.ac.jp
(c) General Topology

(a) **Yasunao Hattori**
(b) Shimane Univ., Matsue, Shimane 690-8504, Japan
(b’) hattori@riko.shimane-u.ac.jp
(c) General Topology

(a) **Yoshikazu Yasui**
(b) Department of Modern Education, Faculty of Education, Kio University, 4-2-2, Umami-naka, Koryo-cho,
Kitakaturagi-gun, Nara, 635-0832, Japan
(b’) y.yasui@kio.ac.jp
(c) General Topology

(a) **Eiichi Nakai**
(b) Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
(b') eiichi.nakai.math@vc.ibaraki.ac.jp
(c) Real analysis, harmonic analysis, Fourier analysis, function spaces, singular and fractional integrals

(a) **Jun Kawabe**
(b) Division of Mathematics and Physics, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
(b') jkawabe@shinshu-u.ac.jp
(c) Measure and integration, Vector measure, Nonadditive measure

(a) **Shizu Nakanishi**
(b) non-public
(b’) shizu.nakanishi@nifty.ne.jp
(c) measures and integrations

(a) **Jun Ichi Fujii**
(b) Department of Educational Collaboration(Science, Mathematics and Information),Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
(b') fujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Masaru Nagisa**
(b) Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Yayoi-cho, Chiba, 263-8522, Japan
(b’) nagisa@math.s.chiba-u.ac.jp
(c) operator algebra, operator theory

(a) **Hiroyuki Osaka**
(b) Graduate School of Science and Engineering, Ritsumeikan University, 1-1-1 Noji-higashi, Kusatsu, Shiga 525-8577 Japan
(b') osaka@se.ritsumei.ac.jp
(c) Operator Theory and Operator Algebras

(a) **Masatoshi Fujii**
(b) non-public
(b’) mfujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Wataru Takahashi**
(b) Keio Research and Education Center for Natural Science,Keio University,Kouhoku-ko,Yokohama 223-8521, Japan
(b’) wataru@is.titech.ac.jp, wataru@a00.itscom.net
(c) Nonlinear Functional Analysis

(a) **Shigeo Akashi**
(b) Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science, 2641, Yamazaki, Noda-City, Chiba-Prefecture, 278-8510, Japan
(b’) akashi@is.noda.tus.ac.jp
(c) Information Theory, Entropy Analysis, Applied Mathematics, Functional Analysis

(a) **Yoshitsugu Kabeya**
(b) Department of Mathematical Sciences, Osaka Prefecture University, 1-1, Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531, Japan
(b’) kabeya@ms.osakafu-u.ac.jp
(c) Partial Differential Equations, Ordinary Differential Equations

(a) **Atsushi Yagi**
(b) Dept. of Applied Physics, Graduate School of Engineering, Osaka Univ., 2-1 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') yagi@ap.eng.osaka-u.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

(a) **Yoshimasa Nakamura**
(b) Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
(b)' ynaka@i.kyoto-u.ac.jp
(c) integrable systems, numerical linear algebra, special functions

(a) **Yasumasa Fujisaki**
(b) Department of Information and Physical Sciences, Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') fujisaki@ist.osaka-u.ac.jp
(c) Control Systems Theory

(a) **Naruhiko Aizawa**
(b) Department of Physical Science, Graduate School of Science, Osaka Prefecture University,Sakai,Osaka 599-8531,Japan
(b')aizawa@p.s.osakafu-u.ac.jp
(c) representation theory

(a) **Hisao Nagao**
(b) non-public
(b') nagao.hisao@aqua.plala.or.jp
(c) Multivariate Analysis, Sequential Analysis, Jackknife Statistics and Bootstrap Method

(a) **Masanobu Taniguchi**
(b) Dept. of Applied Mathematics, School of Fundamental Science & Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo,169-8555, Japan, Tel & Fax: 03-5286-8386
(b') taniguchi@waseda.jp
(c) Statistical Inference for Stochastic Processes

(a) **Masao Kondo**
(b) non-public
(b') kondo@sci.kagoshima-u.ac.jp
(c) Time Series Analysis

(a) **Masao Fukushima**
(b) Dept. of Systems and Mathematical Science, Faculty of Science and Engineering, Nanzan University, Nagoya, Aichi 466-8673, Japan
(b)' fuku@nanzan-u.ac.jp
(c) Mathematical Programming, Nonlinear Optimization

(a) **Ryusuke Hohzaki**
(b) Department of Computer Science, National Defense Academy, 1-10-20, Hashirimizu, Yokosuka, 239-8686, Japan
(b') hozaki@cc.nda.ac.jp
(c) Reviewable area: Operations Research, Search theory, Game theory

(a) **Hiroaki Ishii**
(b) Department of Mathematical Sciences, School of Science and Technology, Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669-1337, Japan
(b') ishiroaki@yahoo.co.jp
(c) Operations Research and Fuzzy Theory, especially Mathematical Programming (Stochastic Programming, Combinatorial Optimization, Fuzzy Programming), Scheduling Theory, Graph and Network Theory, Inventory control, Mathematical evaluation method

(a) **Junzo Watada**
(b) Universiti Teknologi PETRONAS Department of Computer & Information Sciences 32610 Seri Iskandar,Perak Darul Ridzuam,Malaysia Office Phone:
+60-5-368-7517 Mobile:+60-13-598-0208
Professor Emeritus,Waseda University,Japan
(b') junzow@osb.att.ne.jp
(c) Fuzzy systems, Management Engineering

(a) **Kensaku Kikuta**
(b) School of Business Administration, University of Hyogo,
8-2-1 Gakuen-nishi-machi, Nishi-ku, Kobe City 651-2197 JAPAN
(b') kikuta@biz.u-hyogo.ac.jp
(c) Game Theory, Operations Research,

(a) **Wuyi Yue**
(b) Dept. of Intelligence and Informatics, Faculty of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada-ku , Kobe 658-8501, JAPAN
(b') yue@konan-u.ac.jp
(c) Queueing Networks, Performance Analysis and Modeling, Communications Networks, Operations Research, Markov Processes, Probabilistic Methods, Systems Engineering

(a) **Hiroaki Sandoh**
(b) Faculty of Policy Studies Kwansei Gakuin University 2-1, Gakuen, Sanda-shi, Hyogo 669-1337 Japan
(b') sandoh@kwansei.ac.jp
(c) Operations Research and Management Science, Stochastic modeling

(a) Katsunori Ano
(b) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku Saitama-city, 337-8570, Japan
(b') k-ano@shibaura-it.ac.jp
(c) Optimal Stopping, Mathematical Finance, Applied Probability

(a) **Koyu Uematsu**
(b) Graduate School of Management and Information Sciense Faculty of Global Business ,Osaka International University 6-21-57 Tohdacho, Moriguchi-Shi, Osaka,570-8555,Japan
(b') uematsu@oiu.jp
(c) Stochastic Process and its Applications,Reliability Analysis,and Game Theory

(a) **Yoshiki Kinoshita**
(b) Dept. of Information Sciences , Faculty of Science, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa 259-1293, Japan
(b') yoshiki@kanagawa-u.ac.jp
(c) Software Science, Programming language semantics

(a) **Shunsuke Sato**
(b) non-public
(b')ss_22362@nifty.com
(c) Mathematical biology in general

(a)**Tadashi Takahashi**
(b)Department of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada, Kobe, Hyogo 658-8501, Japan
(b') takahasi@konan-u.ac.jp
(c)Mathematics Education

(a) **Benoit Collins**
(b) Department of Mathematics, Faculty of Science, Kyoto University
(b') collins@math.kyoto-u.ac.jp
(c) Random Matrix Theory, Free Probability, Quantum Information Theory
Quantum Groups (operator algebra side), Operator Algebra

(a) **Yoko Watamori**
(b) Department of Mathematics and Information Sciences, Graduate School of Science, Osaka Prefecture University, Sakai, Osaka 599-8531, Japan
(b') watamori@mi.s.osakafu-u.ac.jp
(c) Directional statistics, Multivariate Analysis

(a) **Koichi Osaki**
(b)Department of Mathematical Sciences,School of Science and Technology, Kwansei Gakuin University,
2-1 Gakuen, Sanda, 669-1337, Japan.
(b')osaki@kwansei.ac.jp
(c)Nonlinear partial differential equations, Infinite-dimensional dynamical systems
****

Managing Editor
Koyu Uematsu (Professor of Osaka International University)
International Society for Mathematical Sciences
1-5-12-202 Kaorigaoka-cho, Sakai-ku, Sakai-city, 590-0011,Japan
uematsu@jams.jp

Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html)and would like to take in charge of the paper for refereeing.
2. Title of the paper.
3. Authors' names.
4. Corresponding author's name, e-mail address and postal address (affiliation).
5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and sent it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.

b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.

c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.

(4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.

b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.

c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names

3'. 3. in Japanese for Japanese authors

4. Corresponding author's name and postal address (affiliation)

4'. 4. in Japanese for Japanese authors

5. ISMS membership number
6. E-mail address

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$225. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$225) ☐Print ＋ Online (US$225)
University (Institution)	
Department	
Postal Address where SCMJ should be sent	
E-mail address	
Person in charge	Name: Signature:
Payment Check one of the two.	☐Bank transfer ☐Credit Card (Visa, Master)
Name of Associate Membership	1.
	2.

Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in “**Join ISMS !**” on the inside of the back cover.

Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4’s address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in “**Join ISMS !**”)

Individual Membership Application Form

1. Name	
2. Birth date	
3. Academic background	
4. Affiliation	
5. 4’s address	
6. Doctorate	
7. Contact address	
8. E-mail address	
9. Special fields	
10. Membership category	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.
The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

****

Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.
Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.
(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:
(1) Post Office Transfer Account - 00930-3-73982 or
(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.
All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	¥8,000	US$80， Euro75	US$50, Euro47
1-year Students member	¥4,000	US$50， Euro47	US$30， Euro28
Life member*	Calculated as below*	US$750， Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73－age) × ¥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

CONTENTS

MOHSEN KIAN AND YUKI SEO :
NORM INEQUALITIES RELATED TO THE MATRIX GEOMETRIC MEAN OF NEGATIVE POWER 191

HIROSHI TATEISHI :
ON THE BANG-BANG PRINCIPLE FOR DIFFERENTIAL INCLUSIONS IN A REFLEXIVE SEPARABLE BNANACH SPACE 201

HIDETOSHI MARUBAYASHI, SRI WAHYUNI, INDAH EMILIA WIJAYANTI AND IWAN ERNANTO :
STRONGLY GRADED RINGS WHICH ARE MAXIMAL ORDERS 207

NARUHIKO AIZAWA:
VERMA MODULES OVER A $Z_2 \otimes Z_2$ GRADED SUPERALGEBRA AND INVARIANT DIFFERENTIAL EQUATIONS 211

JOEMAR C. ENDAM AND JENETTE S. BANTUG:
CAUCHY'S THEOREM FOR B-ALGEBRAS 221

MASARU NAGISA:
TRANSFORMS ON OPERATOR MONOTONE FUNCTIONS 229

HANAMANTAGOUDA P. SANKAPPANAVAR:
JI-DISTRIBUTIVE, DUALLY QUASI-DE MORGAN SEMI-HEYTING AND HEYTING ALGEBRAS 245

CHIEKO KOMODA :
OPEN MAPPING THEOREMS WITH FINITE FIBRES FOR C-SPACES 273

Notices from the ISMS

Board of Editors 1
Call for Papers for SCMJ 11
Call for ISMS Members 13