ON \textit{t}-LEVEL SUBALGEBRAS OF BCK-ALGEBRAS

KUNG HO KIM

Received September 6, 2008

Abstract. Using \textit{t}-norm \( T \), we introduce the notion of \textit{t}-level subalgebra, and some related properties are investigated.

1. Introduction. Y. Imai and Iseki [2, 3] introduced two classes of abstract: BCK-algebras and BCI-algebras and was extensively investigated by several researchers. L. A. Zander [8] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, O. G. Xi [5] applied this concept to BCK-algebras, and he introduced the notion of fuzzy subalgebras(ideals) of the BCK-algebras with respect to minimum. In this paper, using \textit{t}-norm \( T \), we introduce the notion of \textit{t}-level subalgebra, and some related properties are investigated.

2. Preliminaries. In what follows we use \( X \) to denote a BCK-algebra unless otherwise specified.

A BCK-algebra is an algebra \((X, \ast, 0)\) of type \((2, 0)\) satisfying the following axioms

\begin{enumerate}
  \item \((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0,\)
  \item \(x \ast (x \ast y) \ast y = 0,\)
  \item \(x \ast x = 0,\)
  \item \(x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y,\)
  \item \(x \ast 0 = 0 \text{ imply } x = 0,\)
\end{enumerate}

for all \(x, y \in X\). A partial ordering \(\leq\) on \(X\) can be defined by \(x \leq y\) if and only if \(x \ast y = 0\).

Let \(I\) be a nonempty subset of a BCK-algebra \(X\). Then \(I\) is called a subalgebra of \(X\) if \(x \ast y \in I, \) for all \(x, y \in I\).

Definition 2.1 ([7]). By a \textit{t}-norm \( T \), we mean a function \( T : [0, 1] \times [0, 1] \to [0, 1] \) satisfying the following conditions:

\begin{enumerate}
  \item \(T(x, 1) = x,\)
  \item \(T(x, y) \leq T(x, z) \text{ if } y \leq z,\)
  \item \(T(x, y) = T(y, x),\)
  \item \(T(x, T(y, z)) = T(T(x, y), z),\)
\end{enumerate}

for all \(x, y, z \in [0, 1]\).

For a \textit{t}-norm \( T \) on \([0, 1]\), denote by \( \Delta_T \) the set of element \( \alpha \in [0, 1] \) such that \( T(\alpha, \alpha) = \alpha, \) i.e., \( \Delta_T := \{ \alpha \in [0, 1] \mid T(\alpha, \alpha) = \alpha \}. \)

\textit{2000 Mathematics Subject Classification.} 06F35, 03G25, 03E72.

\textit{Key words and phrases.} BACK-algebra, \textit{t}-norm, \textit{T}-fuzzy subalgebra, idempotent \textit{T}-fuzzy subalgebra, \textit{t}-level subset, \textit{t}-level subalgebra.
Proposition 2.2. Every t-norm $T$ has a useful property:

$$T(\alpha, \beta) \leq \min(\alpha, \beta)$$

for all $\alpha, \beta \in [0, 1]$.

A fuzzy set in $X$ is a function $A : X \rightarrow [0, 1]$. A fuzzy set $A$ in $X$ is called a fuzzy subalgebra of $X$ if $A(x \ast y) \geq \min\{A(x), A(y)\}$ for all $x, y \in X$.

3. t-LEVEL SUBALGEBRAS OF BCK-ALGEBRAS.

Definition 3.1. A function $A : X \rightarrow [0, 1]$ is called a $T$-fuzzy subalgebra of $X$ with respect to a t-norm $T$ (briefly, a $T$-fuzzy subalgebra of $X$) if $A(x \ast y) \geq T(A(x), A(y))$ for all $x, y \in X$.

It is easy to show that every fuzzy subalgebra is a $T$-fuzzy subalgebra of $X$ with $T(\alpha, \beta) = \alpha \land \beta$ for each $\alpha, \beta \in [0, 1]$.

Definition 3.2. Let $T$ be a t-norm. A fuzzy set $A$ in $X$ is said to satisfy idempotent property if $\text{Im}(A) \subseteq \Delta_T$.

Proposition 3.3. Let $T$ be a t-norm on $[0, 1]$. If $A$ is an idempotent $T$-fuzzy subalgebra of $X$, then we have $A(0) \geq A(x)$ for all $x \in X$.

Proof. For every $x \in X$, we have

$$A(0) = A(x \ast x) \geq T(A(x), A(x)) = A(x).$$

This completes the proof. \hfill $\square$

Proposition 3.4. Let $T$ be a t-norm on $[0, 1]$. If $A$ is an idempotent $T$-fuzzy subalgebra of $X$, then the set

$$A^\omega = \{x \in X \mid A(x) \geq A(\omega)\}$$

is a subalgebra of $X$.

Proof. Let $x, y \in A^\omega$. Then $A(x) \geq A(\omega)$ and $A(y) \geq A(\omega)$. Since $A$ is an idempotent $T$-fuzzy subalgebra of $X$, it follows that

$$A(x \ast y) \geq T(A(x), A(y)) \geq T(A(x), A(\omega)) \geq T(A(\omega), A(\omega)) = A(\omega).$$

Thus, we have $A(x \ast y) \geq A(\omega)$, that is, $x \ast y \in A^\omega$. This completes the proof. \hfill $\square$

Corollary 3.5. Let $T$ be a t-norm. If $A$ is an idempotent $T$-fuzzy subalgebra of $X$, then the set

$$A_X = \{x \in X \mid A(x) = A(0)\}$$

is a subalgebra of a BCK-algebra $X$.

Proof. From the Proposition 3.3, $A_X = \{x \in X \mid A(x) = A(0)\} = \{x \in X \mid A(x) \geq A(0)\}$, hence $A_X$ is a subalgebra of $X$ from Proposition 3.4. \hfill $\square$

Let $\chi_I$ denote the characteristic function of a non-empty subset $I$ of $X$.

Theorem 3.6. Let $I \subseteq X$. Then $I$ is a subalgebra of a BCK-algebra $X$ if and only if $\chi_I$ is a $T$-fuzzy subalgebra of $X$. 
Proof. Let $I$ be a subalgebra of $X$. Then it is easy to show that $\chi_I$ is an $T$-fuzzy subalgebra of $X$. In fact, let $x, y \in I$. Then $x * y \in I$. Hence

$$
\chi_I(x * y) = 1 = T(\chi_I(x), \chi_I(y))
$$

If $x \in I$, $y \notin I$ (or $x \notin I$ and $y \in I$), then we have $\chi_I(x) = 1$ or $\chi_I(y) = 0$. This means that

$$
\chi_I(x * y) \geq T(\chi_I(x), \chi_I(y)) = 0.
$$

Conversely, suppose that $\chi_I$ is a $T$-fuzzy subalgebra of $X$. Now let $x, y \in I$. Then $\chi_I(x * y) \geq T(\chi_I(x), \chi_I(y)) = 1$, and so $\chi_I(x * y) = 1$, that is, $x * y \in I$. This proves the theorem. \(\square\)

Lemma 3.7 ([2]). Let $T$ be a $t$-norm. Then

$$
T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))
$$

for all $\alpha, \beta, \gamma, \delta \in [0, 1]$.

Proposition 3.8. If $A$ and $B$ are $T$-fuzzy subalgebras of $X$, then $A \land B : X \to [0, 1]$ defined by

$$(A \land B)(x) = T(A(x), B(x))$$

for all $x \in X$ is a $T$-fuzzy subalgebra of $X$.

Proof. Let $x, y \in X$. Then we have

\[
(A \land B)(x * y) = T(A(x * y), B(x * y)) \\
\geq T(T(A(x), A(y)), T(B(x), B(y))) \\
= T(T(A(x), B(x)), T(A(y), B(y))) \\
= T((A \land B)(x), (A \land B)(y)).
\]

This completes the proof. \(\square\)

Definition 3.9. A fuzzy subalgebra $A$ of a BCK-algebra $X$ is said to be normal if $A(0) = 1$.

Theorem 3.10. Let $A$ be a $T$-fuzzy subalgebra of a BCK-algebra $X$ and let $A^\circ$ be a fuzzy set in $X$ defined by $A^\circ(x) = A(x) + 1 - A(0)$ for all $x \in X$. Then $A^\circ$ is a normal $T$-fuzzy subalgebra of a BCK-algebra $X$ containing $A$.

Proof. For $x, y \in X$, we have

\[
A^\circ(x * y) = A(x * y) + 1 - A(0) \\
\geq T(A(x), A(y)) + 1 - A(0) \\
= T(A(x) + 1 - A(0), A(y) + 1 - A(0)) \\
= T(A^\circ(x), A^\circ(y)).
\]

Hence $A^\circ$ is a $T$-fuzzy subalgebra of $X$. Clearly, $A^\circ(0) = 1$ and $A \subseteq A^\circ$. \(\square\)

Definition 3.11. Let $A$ be a fuzzy subset of a set $X$, $T$ a $t$-norm and $\alpha \in [0, 1]$. Then we define a $t$-level subset of a fuzzy subset $A$ as

$$
A_T^\alpha = \{x \in X \mid T(A(x), \alpha) \geq \alpha\}.
$$

Theorem 3.12. Let $X$ be a BCK-algebra and $A$ a $T$-fuzzy subalgebra of $X$. Then $t$-level subset $A_T^\alpha$ is an subalgebra of $X$ where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$. 
Proof. $A^T_\alpha = \{ x \in M \mid T(A(x), \alpha) \geq \alpha \}$ is clearly nonempty. Let $x, y \in A^T_\alpha$. Then we have $T(A(x), \alpha) \geq \alpha$ and $T(A(y), \alpha) \geq \alpha$. Since $A$ is a $T$-fuzzy subalgebra of $X$, $A(x \ast y) \geq T(A(x), A(y))$ is satisfied. This means that
\[ T(A(x \ast y), \alpha) \geq T(T(A(x), A(y)), \alpha) = T(A(x), T(A(y), \alpha)) \geq T(A(x), \alpha) \geq \alpha. \]
Hence $x \ast y \in A^T_\alpha$. Therefore $A^T_\alpha$ is a subalgebra of $X$.

**Theorem 3.13.** Let $X$ be a BCK-algebra and $A$ a fuzzy subalgebra of $X$. Then every t-level subset $A^T_\alpha$ is a subalgebra of $X$ where $T(A(0), \alpha) \geq \alpha$ for all $\alpha \in [0, 1]$.

Proof. $A^T_\alpha = \{ x \in X \mid T(A(x), \alpha) \geq \alpha \}$ is clearly nonempty. Let $x, y \in A^T_\alpha$. Then we have $T(A(x), \alpha) \geq \alpha$ and $T(A(y), \alpha) \geq \alpha$. Since $A$ is a fuzzy subalgebra of $X$, $A(x \ast y) \geq \min\{A(x), A(y)\}$ is satisfied. This means that $T(A(x \ast y), \alpha) \geq T(\min\{A(x), A(y)\}, \alpha)$. If $\min\{A(x), A(y)\} = A(x)$ or $\min\{A(x), A(y)\} = A(y)$, in two cases, we have
\[ T(\min\{A(x), A(y)\}, \alpha) \geq \alpha \]
since $x, y \in A^T_\alpha$. Therefore, $T(A(x \ast y), \alpha) \geq \alpha$. Thus we get $x \ast y \in A^T_\alpha$. Hence $A^T_\alpha$ is a subalgebra of $X$.

**Theorem 3.14.** Let $X$ be a BCK-algebra and $A$ a fuzzy set of $X$ such that $A^T_\alpha$ is a subalgebra of $X$ where $T(A(x), \alpha) \geq \alpha$ for all $\alpha \in [0, 1]$. Then $A$ is a $T$-fuzzy subalgebra of $X$.

Proof. Let $x, y \in X, T(A(x), \alpha_1) = \alpha_1$ and $T(A(y), \alpha_2) = \alpha_2$. Then $x \in A^T_{\alpha_1}$ and $y \in A^T_{\alpha_2}$. Let $\alpha_1 < \alpha_2$. Then it follows that $T(A(x), \alpha_1) < T(A(y), \alpha_2)$ and $A^T_{\alpha_1} \subseteq A^T_{\alpha_2}$. So, $y \in A^T_{\alpha_1}$. Thus $x, y \in A^T_{\alpha_1}$ and since $A^T_{\alpha_1}$ is a subalgebra of $X$, by hypothesis, $x \ast y \in A^T_{\alpha_1}$. Therefore we have
\[ T(A(x \ast y), \alpha_1) \geq \alpha_1 = T(A(x), \alpha_1) \geq T(A(x), T(A(y), \alpha_1)) = T(T(A(x), A(y)), \alpha_1). \]
Thus we get $T(A(x \ast y), \alpha_1) \geq T(T(A(x), A(y)), \alpha_1)$. As a $t$-norm is monotone with respect to each variable and symmetric, we have $A(x \ast y) \geq T(A(x), A(y))$. Thus $A$ is a $T$-fuzzy subalgebra of $X$.

**Definition 3.15.** For each $i = 1, 2, 3, \ldots, n$, let $A_i$ be a $T$-fuzzy subalgebra in a BCK-algebra $X_i$. Let $t$ be a $t$-norm. Then the $T$-product of $A_i$ ($i = 1, 2, \ldots, n$) is the function $A_1 \times A_2 \times A_3 \times \cdots \times A_n : X_1 \times X_2 \times X_3 \times \cdots \times X_n \rightarrow [0, 1]$ defined
\[ (A_1 \times A_2 \times A_3 \times \cdots \times A_n)(x_1, x_2, x_3, \ldots, x_n) = T(A_1(x_1), A_2(x_2), A_3(x_3), \ldots, A_n(x_n)) \]
for $x_i \in X_i (i = 1, 2, \ldots, n)$.

**Theorem 3.16** ([1]). Let $A$ and $B$ be t-level subsets of the sets $G$ and $H$, respectively, and let $\alpha \in [0, 1]$. Then $A \times B$ is also t-level subset of $G \times H$.

**Definition 3.17.** Let $X$ be a BCK-algebra and $A$ a $T$-fuzzy subalgebra of $X$. The subalgebra $A^T_\alpha$ is called a $t$-level subalgebra of $X$ where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.

**Theorem 3.18.** Let $X_1$ and $X_2$ be two BCK-algebras, and $A, B$ $T$-fuzzy subalgebras of $X_1$ and $X_2$, respectively. Then the t-level subset $(A \times B)^T_\alpha$, for $\alpha \in [0, 1]$, is a subalgebra of $X_1 \times X_2$.

Proof. We know that $(A \times B)^T_\alpha = \{(x, y) \mid T((A \times B)(x, y), \alpha) \geq \alpha\}$. Since
\[ T((A \times B)(0_{X_1}, 0_{X_2}), \alpha) = T(T(A(0_{X_1}), B(0_{X_2})), \alpha) = T(A(0_{X_1}), T(B(0_{X_2}), \alpha)) \geq T(A(0_{X_1}), \alpha) \geq \alpha, \]
Theorem 3.20. \((A \times B)_T^\alpha\) is nonempty. Let \((x_1, y_1), (x_2, y_2) \in (A \times B)_T^\alpha\). Then \(T((A \times B)(x_1, y_1), \alpha) \geq \alpha\) and \(T((A \times B)(x_2, y_2), \alpha) \geq \alpha\). Since \(A \times B\) is a \(T\)-fuzzy subalgebra of \(X_1 \times X_2\), we have

\[
(A \times B)((x_1, y_1) \ast (x_2, y_2)) = (A \times B)(x_1 \ast x_2, y_1 \ast y_2) \geq T(A(x_1 \ast x_2), B(y_1 \ast y_2)).
\]

Since \(A\) and \(B\) are \(T\)-fuzzy subalgebras, we get

\[
T((A \times B)(x_1 \ast x_2, y_1 \ast y_2), \alpha) \geq T(T(A(x_1 \ast x_2), B(y_1 \ast y_2), \alpha)) = T(T(A(x_1 \ast x_2), T(B(y_1 \ast y_2), \alpha)) \geq T(A(x_1 \ast x_2), \alpha) \geq \alpha.
\]

Hence \((x_1, y_1) \ast (x_2, y_2) \in (A \times B)_T^\alpha\). Therefore \((A \times B)_T^\alpha\) is a subalgebra of \(X_1 \times X_2\). \(\square\)

Theorem 3.19 ([1]). Let \(A\) and \(B\) be fuzzy sets of the sets \(G\) and \(H\), respectively and \(T\) a \(t\)-norm and \(\alpha \in [0, 1]\). Then \(A_T^\alpha \times B_T^\alpha = (A \times B)_T^\alpha\).

Theorem 3.20. Let \(A_1, A_2, A_3, \ldots, A_n\) be fuzzy subalgebras under a minimum operation in \(BCK\)-algebras \(X_1, X_2, X_3, \ldots, X_n\), respectively and \(\alpha \in [0, 1]\). Then

\[
(A_1 \times A_2 \times \cdots \times A_n)_T^\alpha = A_1T^\alpha \times A_2T^\alpha \times \cdots \times A_nT^\alpha.
\]

Proof. Let \((a_1, a_2, a_3, \ldots, a_n) \in (A_1 \times A_2 \times \cdots \times A_n)_T^\alpha\). Then we have

\[
T(\min((A_1 \times A_2 \times \cdots \times A_n)(a_1, a_2, a_3, \ldots, a_n), \alpha) = T(\min(A_1(a_1), A_2(a_2), \ldots, A_n(a_n)), \alpha).
\]

For all \(i = 1, 2, \ldots, n\), \(\min(A_1(a_1), A_2(a_2), \ldots, A_n(a_n)) = A_i(a_i)\). This gives us

\[
T(\min(A_1(a_1), A_2(a_2), \ldots, A_n(a_n)), \alpha) = T(A_i(a_i), \alpha) \geq \alpha.
\]

Thus we have \(a_i \in A_iT^\alpha\). That is, \((a_1, a_2, a_3, \ldots, a_n) \in A_1T^\alpha \times A_2T^\alpha \times \cdots \times A_nT^\alpha\). Similarly, \((a_1, a_2, a_3, \ldots, a_n) \in A_1T^\alpha \times A_2T^\alpha \times \cdots \times A_nT^\alpha\). Then, for all \(i = 1, 2, \ldots, n\), we have \(a_i \in A_iT^\alpha\). That is, \(T(A_i(a_i), \alpha) \geq \alpha\). Since \(\min(A_1(a_1), A_2(a_2), \ldots, A_n(a_n)) = A_i(a_i)\) and \(T(A_i(a_i), \alpha) \geq \alpha\), we have

\[
T((A_1 \times A_2 \times \cdots \times A_n)(a_1, a_2, a_3, \ldots, a_n), \alpha) = T(\min(A_1(a_1), A_2(a_2), \ldots, A_n(a_n)), \alpha)
\]

\[
= T(A_i(a_i), \alpha)
\]

\[
\geq \alpha.
\]

Thus we have \((a_1, a_2, a_3, \ldots, a_n) \in (A_1 \times A_2 \times \cdots \times A_n)_T^\alpha\). \(\square\)

References

KUNG HO KIM

Department of Mathematics,
Chungk National University, Chungk,
Chungking 380-702, Korea
E-mail: ghkim@cjnu.ac.kr