ON IMPLICATIVE BCI-ALGEBRAS

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Abstract. In this paper, we give an axiom system of implicative BCI-algebras, investigate some properties of the branches of an implicative BCI-algebra, which are similar to those of implicative BCK-algebras, and show that for every initial section of an implicative BCI-algebra, it with respect to the BCI-ordering forms a Boolean algebra.

As is well known, commutative BCK-algebras, positive implicative BCK-algebras and implicative BCK-algebras are three classes of the most important BCK-algebras. In order to get the similar classes in BCI-algebras, J. Meng and X. L. Xin in [9], [11] and [10] introduced commutative BCI-algebras, positive implicative BCI-algebras and implicative BCI-algebras respectively, and investigated their fundamental properties similar to those of the corresponding algebras in BCK-algebras. And the author in [1], [2] and [3] gave some further properties of theirs.

The ideas of this paper are originated from [1]. Like [1], we will mainly use lattices and branches as well as initial sections to explore implicative BCI-algebras in this paper. And we will obtain a number of interesting results similar to those of implicative BCK-algebras.

0 Preliminaries For the notations and elementary properties of BCK and BCI-algebras, we refer the reader to [5], [4] and [8]. And we will use some familiar notions and properties of lattices without explanation.

Recall that according to the H. S. Li’s axiom system (see [7]), a BCI-algebra \((X; \ast, 0)\) means that it is an algebra of type \((2, 0)\), satisfying the following conditions: for any \(x, y, z \in X\),

\[
\begin{align*}
\text{BCI-1} & \quad ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0, \\
\text{BCI-2} & \quad x \ast 0 = x, \\
\text{BCI-3} & \quad x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y.
\end{align*}
\]

It is known that given a BCI-algebra \(X\), the following identities are valid:

\[
\begin{align*}
(0.1) & \quad (x \ast y) \ast z = (x \ast z) \ast y, \\
(0.2) & \quad x \ast y = x \ast (x \ast (x \ast y)), \\
(0.3) & \quad 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y), \\
(0.4) & \quad (x \ast y) \ast x = 0 \ast x.
\end{align*}
\]

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And $X$ with respect to its $BCI$-ordering $\leq$ forms a partially ordered set $(X; \leq)$ satisfying the following quasi-identities:

\begin{align}
(0.5) & \quad (x * y) * (x * z) \leq z * y, \\
(0.6) & \quad (x * z) * (y * z) \leq x * y, \\
(0.7) & \quad (x * (x * y)) * (x * (x * z)) \leq y * z,
\end{align}

where the binary relation $\leq$ on $X$ is defined as follows: $x \leq y$ if and only if $x * y = 0$. Moreover, the following assertions hold: for any $x, y, z \in X$,

\begin{align}
(0.8) & \quad x \leq y \text{ implies } z * y \leq z * x, \\
(0.9) & \quad x \leq y \text{ implies } x * z \leq y * z.
\end{align}

A minimal element $a$ of $X$ means that $a$ is an element in $X$ such that $x \leq a$ (i.e., $x * a = 0$) implies $x = a$ for any $x \in X$. Given a minimal element $a$ of $X$, the set $\{x \in X \mid x \geq a\}$ is called a branch of $X$, denoted by $V(a)$.

Given an element $c$ in $X$, the set $\{x \in X \mid x \leq c\}$ is called an initial section of $X$, denoted by $A(c)$.

**Theorem 0.1 ([8], §1.3).** Assume that $P$ is the set of all minimal elements of a BCI-algebra $X$. Then the collection $\{V(a) \mid a \in P\}$ of branches of $X$ forms a partition of $X$, that is, $X = \bigcup_{a \in P} V(a)$ and $V(a) \cap V(b) = \emptyset$ if $a \neq b$ for any $a, b \in P$. Moreover, the following fold: for any $x, y \in V(a)$,

\begin{align}
(0.10) & \quad 0 * (0 * x) = a, \\
(0.11) & \quad 0 * (x * y) = 0.
\end{align}

**Definition ([9], [11] and [10]).** A BCI-algebra $X$ is called commutative if

\[ x \leq y \text{ implies } x = y * (y * x) \quad \text{for all } x, y \in X; \]

it is called positive implicative if

\[ (x * (x * y)) * (y * x) = x * (x * (y * (y * x))) \quad \text{for all } x, y \in X; \]

it is called implicative if

\[ x * (x * y) = (y * (y * x)) * (x * y) \quad \text{for all } x, y \in X. \]

**Theorem 0.2 ([8], §2.4).** A BCI-algebra $X$ is commutative if and only if for any branch $V(a)$ of $X$, $x \in V(a)$ and $y \in V(a)$ imply

\[ x * (x * y) = y * (y * x). \]

Moreover, $(V(a); \leq)$ forms a lower semilattice such that for any $x, y \in V(a)$,

\begin{align}
(0.14) & \quad x \wedge y = y * (y * x), \\
(0.15) & \quad x * y = x * (x \wedge y).
\end{align}

**Theorem 0.3 ([1], Theorem 3.2).** If $A(c)$ is an initial section of a commutative BCI-algebra $X$, then $(A(c); \leq)$ is a distributive lattice with

\[ x \wedge y = y * (y * x) \quad \text{and} \quad x \vee y = c * ((c * x) \wedge (c * y)). \]
Theorem 0.4 ([3], Corollary 3). A BCI-algebra $X$ is positive implicative if and only if
\[(0.16)\quad x \ast y = ((x \ast y) \ast y) \ast (0 \ast y) \quad \text{for any } x, y \in X.\]
Thus
\[(0.17)\quad x \ast y = (x \ast y) \ast y \quad \text{if } y \geq 0.\]

Theorem 0.5 ([11], Theorem 6). A BCI-algebra $X$ is implicative if and only if it is commutative and positive implicative.

1 An axiom system of implicative BCI-algebras
Let’s begin our discussion with giving an axiom system of implicative BCI-algebras.

Theorem 1.1. An algebra $(X; \ast, 0)$ of type $(2,0)$ is an implicative BCI-algebra if and only if it satisfies the following identities:

1. $x \ast 0 = x$;
2. $x \ast x = 0$;
3. $(x \ast y) \ast z = (x \ast z) \ast y$;
4. $(x \ast z) \ast (x \ast y) = ((y \ast z) \ast (y \ast x)) \ast (x \ast y)$.

Proof. Necessity. (1) is just BCI-2. Repeatedly applying BCI-2, we have
\[x \ast x = ((x \ast 0) \ast (x \ast 0)) \ast (0 \ast 0).\]
Then BCI-1 implies $x \ast x = 0$, (2) holding. By (0.1), (3) is true. By the definition of the implicativity of $X$, we have
\[x \ast (x \ast y) = (y \ast (y \ast x)) \ast (x \ast y).\]
Right $\ast$ multiplying both sides of the last identity by $z$, we derive
\[(x \ast (x \ast y)) \ast z = ((y \ast (y \ast x)) \ast (x \ast y)) \ast z.\]
Then (0.1) gives $(x \ast z) \ast (x \ast y) = ((y \ast z) \ast (y \ast x)) \ast (x \ast y)$, showing (4).

Sufficiency. BCI-2 is just (1). Putting $z = 0$ in (4) and using (1), we have
\[(1.1)\quad x \ast (x \ast y) = (y \ast (y \ast x)) \ast (x \ast y),\]
which is the implicativity of $X$. It is easily seen from (1.1) and (1) that BCI-3 is true. It remains to show BCI-1. In fact, by (4), we have
\[x \ast (x \ast z) = ((z \ast y) \ast (z \ast x)) \ast (x \ast z).\]
Right $\ast$ multiplying both sides of the last identity by $z \ast y$, we obtain
\[(1.2)\quad ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = (((z \ast y) \ast (z \ast x)) \ast (x \ast z)) \ast (z \ast y).\]

By (3), the right side of (1.2) coincides with
\[(1.3)\quad (((z \ast y) \ast (z \ast y)) \ast (z \ast x)) \ast (x \ast z).\]

By (2), $(z \ast y) \ast (z \ast y) = 0 = z \ast z$, then (1.3) is identical with
\[(1.4)\quad ((z \ast z) \ast (z \ast x)) \ast (x \ast z).\]
Using (3) once again, (1.4) is the same as

\[(1.5) \quad ((z * (z * x)) * (x * z)) * z.\]

By (1.1), (1.5) is identical with \((x * (x * z)) * z\), that is, \((x * z) * (x * z)\) by (3). Now, since \((x * z) * (x * z) = 0\) by (2), we see that (1.2) is equivalent to

\[((x * y) * (x * z)) * (z * y) = 0,
\]

showing BCI-1. The proof is complete.

2 On branches of implicative BCI-algebras

We now consider the branches of an implicative BCI-algebra. It is known very well that the identity \(x * (y * x) = x\) is just the implicativity of BCK-algebras. It is interesting that the same identity holds in a branch of an implicative BCI-algebra.

**Proposition 2.1.** Let \(X\) be a BCI-algebra. If \(X\) is implicative, then for any branch \(V(a)\) of \(X\), \(x \in V(a)\) and \(y \in V(a)\) imply \(x * (y * x) = x\).

**Proof.** Since \(x, y \in V(a)\), we have \(0 * (x * y) = 0\) by (0.11). Then (0.4) gives

\[(2.1) \quad (x * (y * x)) * x = 0 * (y * x) = 0.
\]

On the other hand, replacing \(y\) by \(y * x\) in (0.12), we have

\[(2.2) \quad x * (x * (y * x)) = ((y * x) * ((y * x) * x)) * (x * (y * x)).
\]

Also, since every implicative BCI-algebra is positive implicative, by (0.16), we derive

\[(2.3) \quad y * x = ((y * x) * x) * (0 * x).
\]

Right \(\ast\) multiplying both sides of (2.3) by \((y * x) \ast x\), it follows

\[(2.4) \quad (y * x) * ((y * x) * x) = (((y * x) * x) * (0 * x)) * ((y * x) * x).
\]

By (0.4), the right side of (2.4) is equal to \(0 * (0 * x)\). Then

\[(2.5) \quad (y * x) * ((y * x) * x) = 0 * (0 * x).
\]

Right \(\ast\) multiplying both sides of (2.5) by \(x * (y * x)\), it yields

\[\quad ((y * x) * ((y * x) * x)) * (x * (y * x)) = (0 * (0 * x)) * (x * (y * x)).
\]

Comparison with (2.2) gives

\[x * (x * (y * x)) = (0 * (0 * x)) * (x * (y * x)),
\]

which means from (0.1) that

\[(2.6) \quad x * (x * (y * x)) = (0 * (x * (y * x))) * (0 * x).
\]

Moreover, since \(x, y \in V(a)\), by (0.3) and (0.11) as well as BCI-2, we obtain

\[0 * (x * (y * x)) = (0 * x) * (0 * (y * x)) = (0 * x) * 0 = 0 = 0 * x.
\]

Now, substituting \(0 * x\) for \(0 * (x * (y * x))\) in (2.6), and noticing \(0 * x) * (0 * x) = 0\), the following holds:

\[(2.7) \quad x * (x * (y * x)) = (0 * x) * (0 * x) = 0.
\]

Combining (2.1) with (2.7) and using BCI-3, it follows \(x * (y * x) = x\).
It is a pity that unlike Theorem 0.2, the converse of Proposition 2.1 is not true as shown in the following counter example.

**Example 2.1.** The set \( X = \{0, 1, 2, 3\} \) together with the operation * on \( X \) given by the Cayley table

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forms a BCI-algebra (see [6], the author H. Jiang denotes it by \( I_{4-2-1} \)). It is not difficult to see that the whole minimal elements of \( X \) are 0 and 2, and the branches \( V(0) = \{0, 1\} \) and \( V(2) = \{2, 3\} \). Now, it is easy to verify that for any branch \( V(a) \) of \( X \), \( x \in V(a) \) and \( y \in V(a) \) imply \( x * (y * x) = x \). However, \( X \) is not implicative. That is because

\[
3 * (3 * 1) = 1 \neq 0 = (1 * (1 * 3)) * (3 * 1).
\]

Nevertheless, we have still the next interesting fact.

**Proposition 2.2.** Let \( X \) be a BCI-algebra. If for any branch \( V(a) \) of \( X \), \( x \in V(a) \) and \( y \in V(a) \) imply \( x * (y * x) = x \), then \( X \) is commutative.

**Proof.** Let \( x \) and \( y \) be any elements in \( X \) such that \( x \leq y \) (i.e., \( x * y = 0 \)). By Theorem 0.1, there exists a minimal element \( a \) of \( X \) such that \( x \in V(a) \). Since \( a \leq x \) and \( x \leq y \), we obtain \( a \leq y \), that is, \( y \geq a \). Then \( y \in V(a) \). So our hypothesis gives \( x * (y * x) = x \). Hence \((0.6)\) implies

\[
x * (y * (y * x)) = (x * (y * x)) * (y * (y * x)) \leq x * y = 0.
\]

In other words, \( x \leq y * (y * x) \). The opposite inequality is naturally true. Therefore \( x = y * (y * x) \), and \( X \) is commutative. \( \square \)

As an implicative BCI-algebra \( X \) must be commutative, according to Theorem 0.2, every branch \( V(a) \) of \( X \) forms a lower semilattice \( (V(a); \leq) \), thus the greatest lower bound of any two elements in \( V(a) \) exists. And we have the following analogy.

**Proposition 2.3.** Let \( X \) be an implicative BCI-algebra and \( V(a) \) be a branch of \( X \). Then for any \( x, y, z \in V(a) \),

1. \((x * y) \land (y * x) = 0;\)
2. \((x \land y) * z = (x * z) \land (y * z);\)
3. the least upper bound \((z * x) \lor (z * y)\) of \( z * x \) and \( z * y \) exists and

\[
z * (x \land y) = (z * x) \lor (z * y).
\]

**Proof.** (1) Since \( x, y \in V(a) \), by (0.1) and Proposition 2.1, we have

\[
(y * x) * (x * y) = (y * (x * y)) * x = y * x.
\]

Then \((0.14)\) gives

\[
(x * y) \land (y * x) = (y * x) * ((y * x) * (x * y)) = (y * x) * (y * x) = 0.
\]
Also, by (0.6) and (0.15), we obtain
\[ z^* t \leq z , x \] (2.9)

Since

Now, comparing (2.10) with (2.9), we derive

Let

the following analogy.

Let

the least upper bound

That is,

It is not difficult to see that two elements in a branch of an implicative BCI-algebra have generally not their least upper bound. If the least upper bound exists, we also have the following analogy.

Proposition 2.4. Let \( x \) and \( y \) be any elements in a branch \( V(a) \) of a BCI-algebra \( X \). If the least upper bound \( x \lor y \) of \( x \) and \( y \) exists, then the following hold:

1. \((x \lor y) \ast x = y \ast x \) and \((x \lor y) \ast y = x \ast y \);
2. the least upper bound \((x \ast z) \lor (y \ast z)\) of \( x \ast z \) and \( y \ast z \) exists and

\[ (x \lor y) \ast z = (x \ast z) \lor (y \ast z) \]

for any \( z \in V(a) \);
3. \( z \ast (x \lor y) = (z \ast x) \lor (z \ast y) \) for any \( z \in V(a) \).
Corollary 2.5. Let \( x \) and \( y \) be any elements in a branch \( V(a) \) of an implicative BCI-algebra \( X \). If \( x \lor y \) exists, then \((x * y) \lor (y * x)\) exists and
\[
(x \lor y) \cdot (x \land y) = (x * y) \lor (y * x).
\]
3 On initial sections of implicative BCI-algebras

Finally let’s consider the initial sections of an implicative BCI-algebra. It is known that if $X$ is a BCK-algebra and $A(c)$ is an initial section of $X$, then $(A(c); \leq)$ forms a Boolean algebra (refer to [5], Theorem 12). It is interesting that the same conclusion is true if $X$ is an implicative BCI-algebra.

**Theorem 3.1.** Let $A(c)$ be an initial section of an implicative BCI-algebra $X$. Then $(A(c); \leq)$ is a Boolean algebra with $x \land y = y \ast (y \ast x)$, $x \lor y = c \ast ((c \ast x) \land (c \ast y))$ and $x' = (c \ast x) \ast (0 \ast x)$ for any $x, y \in A(c)$.

**Proof.** As any implicative BCI-algebra is commutative, by Theorem 0.3, $(A(c); \leq)$ is a distributive lattice with $x \land y = y \ast (y \ast x)$ and $x \lor y = c \ast ((c \ast x) \land (c \ast y))$ for any $x, y \in A(c)$. Also, $c$ is clearly the unit element of the lattice $A(c)$. Moreover, it is easy to verify from Theorem 0.1 that there exists some branch $V(a)$ of $X$ such that $A(c) \subseteq V(a)$.

Because $a$ is the least element of the branch $V(a)$, it is the zero element of the lattice $(A(c); \leq)$. It remains to show that $A(c)$ is a complemented lattice with $(c \ast x) \ast (0 \ast x)$ as the complement $x'$ of $x$ for any $x \in A(c)$. Let $u$ denote $(c \ast x) \ast (0 \ast x)$. Then we need to show is just the following facts:

1. $u \in A(c)$;  
2. $x \land u = a$;  
3. $x \lor u = c$.

In fact, by (0.6) and BCI-2, we have $(c \ast x) \ast (0 \ast x) \leq c \ast 0 = c$, that is, $u \leq c$. Then $u \in A(c)$, (i) holding. To show (ii) and (iii), let’s first assert that $u \ast x = c \ast x$. In fact, since $X$ is positive implicative, by (0.1) and (0.16), the following holds:

$((c \ast x) \ast (0 \ast x)) \ast x = ((c \ast x) \ast x) \ast (0 \ast x) = c \ast x$.

That is, $u \ast x = c \ast x$, as asserted. Now, we have

$x \land u = u \ast (u \ast x) = u \ast (c \ast x)$.

Because of $x \in V(a)$, by (0.4) and (0.10), we obtain

$u \ast (c \ast x) = ((c \ast x) \ast (0 \ast x)) \ast (c \ast x) = 0 \ast (0 \ast x) = a$.

Therefore $x \land u = a$, showing (ii). Because $X$ is commutative and $u \leq c$, we derive $c \ast (c \ast u) = u$. Then $(c \ast (c \ast u)) \ast x = u \ast x$, that is, $(c \ast x) \ast (c \ast u) = u \ast x$ by (0.1). So, the fact that $u \ast x = c \ast x$ gives $(c \ast x) \ast (c \ast u) = c \ast x$. Let $\ast$ multiplying both sides of the last equality by $c \ast x$, it follows

$(c \ast x) \ast ((c \ast x) \ast (c \ast u)) = (c \ast x) \ast (c \ast x)$.

That is, $(c \ast u) \land (c \ast x) = 0$, in other words, $(c \ast x) \land (c \ast u) = 0$. Therefore

$c \land ((c \ast x) \land (c \ast u)) = c \land 0 = c$.

Note that $x \lor u = c \ast ((c \ast x) \land (c \ast u))$, it yields $x \lor u = c$, proving (iii). \[\square\]

A BCI-algebra $X$ is called **locally bounded** if every branch $V(a)$ of $X$ is bounded, i.e., there is $m_a \in V(a)$ such that $x \leq m_a$ for all $x \in V(a)$.

**Corollary 3.2** ([10], Theorem 5). Assume that $X$ is a locally bounded implicative BCI-algebra. Then for every branch $V(a)$ of $X$, it with respect to the BCI-ordering $\leq$ forms a Boolean algebra $(V(a); \leq)$.
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REFERENCES


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