ON SUNS, MOONS AND BEST APPROXIMATION IN M-SPACES

T.D.Narang*, Sangeeta, Shavetambry Tejpal†

Received June 9, 2006; revised July 28, 2006

Abstract. A metric space \((X, d)\) in which for every \(x, y \in X\) and for every \(t, 0 \leq t \leq 1\) there exists exactly one point \(z \in X\) such that \(d(x, z) = (1 - t)d(x, y)\) and \(d(z, y) = td(x, y)\) is called an M-space. In this paper we discuss suns and moons in M-spaces and characterize these via best approximation thereby extending corresponding known results in normed linear spaces to M-spaces.

Introduction The concept of a sun in Approximation Theory was first introduced in normed linear spaces by Klee [5] but the terminology 'sun' was proposed by Effimov and Steckin [3]. We recall that a set \(V\) is a sun iff whenever \(v_0 \in V\) is a best approximation to some element \(x \notin V\) then \(v_0\) is a best approximation to every element on the ray from \(v_0\) through \(x\). Since every convex set in a normed linear space has this property, a sun may be regarded as a generalization of a convex set. Vlasov [10], who developed the concept further, showed that in a smooth Banach space every proximinal sun is convex. In view of Vlasov's result, the most famous unsolved problem in Approximation Theory viz. whether or not every Chebyshev set in a Hilbert space is convex, may be stated equivalently as "Is every Chebyshev set in a Hilbert space a sun?" The concept of a moon, which is a generalization of sun, was introduced by Amir and Deutsch [1] and their special interest was in determining those normed linear spaces in which every moon is a sun. Knowing such spaces is quite useful as it is much easier to verify that a given set is a moon than verify it is a sun.

Our purpose in this paper is to discuss these concepts in M-spaces [4] (also called strongly convex spaces [9]) and extend some of the results proved in [1] and [6] to M-spaces.

To start with, we give a few notations and recall a few definitions.

Let \((X, d)\) be a metric space and \(x, y, z \in X\). We say that \(z\) is between \(x\) and \(y\) if \(d(x, z) + d(z, y) = d(x, y)\). For any two points \(x, y\) of \(X\), the set \(\{z \in X : d(x, z) + d(z, y) = d(x, y)\}\) is called the metric segment and is denoted by \(G(x, y)\).

A metric space \((X, d)\) is said to be convex [9] if for every \(x, y \in X\) and for every \(t, 0 \leq t \leq 1\) there exists at least one point \(z \in X\) such that

\[
d(x, z) = (1 - t)d(x, y)\quad \text{and}\quad d(z, y) = td(x, y)
\]

The space \((X, d)\) is said to be strongly convex [9] or an M-space [4] if such a \(z\) exists and is unique for each pair \(x, y\) of \(X\).

Thus for strongly convex metric spaces each \(t, 0 \leq t \leq 1\), determines a unique point of the segment \(G(x, y)\).

Let \(G(x, y, -)\) denote the largest line segment containing \(G(x, y)\) for which \(x\) is an extreme point i.e. the ray starting from \(x\) and passing through \(y\), \(G_1(x, y, -)\) denotes \(G(x, y, -) \setminus G(x, y)\) and \(K(v_0, x) \equiv \bigcup B(z, d(z, v_0))\), \(z \in G_1(v_0, x, -)\) where \(B(z, r)\) stands for an open ball with centre \(z\) and radius \(r\).

2000 Mathematics Subject Classification. 41A65, 51K05, 51K99.

Key words and phrases. M-space, best approximation, Chebyshev set, solar point, lunar point, sun, moon.
A subset $V$ of an M-space $(X,d)$ is said to be a cone with vertex $v_0$ if $G(v_0, y, -) \subseteq V$ whenever $y \in V$.

Let $V$ be a non-empty subset of a metric space $(X,d)$ and $x \in X$. An element $v_0 \in V$ is called a best approximation to $x$ if $d(x,v_0) = \text{dist}(x,V)$. We denote by $P_V(x)$, the set of all best approximants to $x$ in $V$. The set $V$ is said to be proximinal if $P_V(x) \neq \emptyset$ for each $x \in X$ and is said to be Chebyshev if $P_V(x)$ is exactly singleton for each $x \in X$.

For $v_0 \in V$, $P_V^{-1}(v_0) = \{x \in X : v_0 \in P_V(x)\}$. It is easy to prove (see [8]) that if $x \in P_V^{-1}(v_0)$ then $x_\lambda \in P_V^{-1}(v_0)$ for every $x_\lambda \in G[v_0,x]$ i.e. $v_0 \in P_V(x_\lambda)$. On the other hand, $v_0$ may not be in $P_V(x_\lambda)$ for $x_\lambda \in G_1(v_0,x,-)$. This motivates the following definition introduced in normed linear spaces by Effimov and Steckin [3]: If $V$ is a proximinal subset of an M-space $(X,d)$, a point $v_0 \in V$ is called a solar point (see Fig. 1 left diagram) if $x \in P_V^{-1}(v_0)$ implies $x_\lambda \in P_V^{-1}(v_0)$ for every $x_\lambda \in G_1(v_0,x,-)$. The set $V$ is called a sun (see Fig. 1 right diagram) if for each $x \in X \setminus V$, every $v_0 \in P_V(x)$ is a solar point of $V$ i.e. for all $v_0 \in P_V(x)$, $v_0 \in P_V(z)$ for all $z \in G_1(v_0,x,-)$.

Let $V$ be a subset of an M-space $(X,d)$. A point $v_0 \in V$ is called a lunar point if $x \in X$ and $K(v_0,x) \cap V \neq \emptyset$ imply $v_0 \in K(v_0,x) \cap V$. The set $V$ is called a moon if each of its point is lunar.

The set $V = \{(x,y) \in R^2 : x^2 + 4y^2 \geq 1\}$ in Euclidean 2-space $R^2$ is a moon (see [2], p.38). We shall see that each sun in an M-space is a moon. However, the converse is not true (see [1]).

For proximinal subsets of an M-space, we have:

**Theorem 1** A proximinal subset $V$ of an M-space $(X,d)$ is a sun if and only if for any $v_0 \in V$, the set $P_V^{-1}(v_0)$ is a cone with vertex $v_0$.

**Proof** Suppose $V$ is a sun and $x \in P_V^{-1}(v_0)$ i.e. $v_0 \in P_V(x)$. We want to show that $G(v_0, x, -) \subseteq P_V^{-1}(v_0)$. Since $v_0 \in P_V(x)$ and $V$ is a sun, $v_0 \in P_V(z)$ for all $z \in G_1(v_0,x,-)$ and consequently for all $z \in G(v_0, x, -)$ i.e. $z \in P_V^{-1}(v_0)$ for all $z \in G(v_0, x, -)$ i.e. $P_V^{-1}(v_0)$ is a cone with vertex $v_0$.

Conversely, let $x \in X \setminus V$ and $y \in P_V(x)$ i.e. $x \in P_V^{-1}(y)$ where $y \in V$. Since $P_V^{-1}(y)$ is a cone with vertex $y$, $G(y,x,-) \subseteq P_V^{-1}(y)$ i.e. $y \in P_V(z)$ for all $z \in G(y,x,-)$. Hence $V$ is a sun.

**Theorem 2** A proximinal subset $V$ of an M-space $(X,d)$ is a sun if and only if for any $v_0 \in V$ and $x \in P_V^{-1}(v_0)$, $K(v_0,x) \cap V = \emptyset$.

**Proof** Suppose $V$ is a sun. Let $v_0 \in V$ and $x \in P_V^{-1}(v_0)$. Since $v_0 \in P_V(x)$ and $V$ is a sun, $v_0 \in P_V(z)$ for all $z \in G(v_0,x,-)$. To show $K(v_0,x) \cap V = \emptyset$. Suppose $u \in K(v_0,x) \cap V$.
i.e. \( u \in B(z, d(z,v_0)) \) for some \( z \in G_1(v_0, x, -) \) i.e. \( d(z,u) \leq d(z,v_0) \) and so \( v_0 \notin P_V(z) \) as \( u \in V \), a contradiction. Therefore \( K(v_0, x) \cap V = \emptyset \).

For the converse part, suppose \( V \) is not a sun. Then there exists \( x \in X \setminus V \) and \( v_0 \in P_V(x) \) such that \( v_0 \notin P_V(z) \) for some \( z \in G(v_0, x, -) \). Then \( d(z,v_1) \leq d(z,v_0) \) where \( v_1 \in P_V(z) \) i.e. \( v_1 \in B(z, d(z,v_0)) \) for some \( z \in G(v_0, x, -) \). i.e \( v_1 \in K(v_0, x) \). Also \( v_1 \in V \) and therefore \( K(v_0,x) \cap V \neq \emptyset \), a contradiction. Hence \( V \) is a sun.

Note In normed linear spaces, Theorem 2 was proved by Amir and Deutsch [1] (see also [6], p. 467).

**Lemma 3** \( K(v_0,x) = K(v_0,y) \) for all \( y \in G[v_0, x] \), where \( x \in X \), \( V \subset X \) and \( v_0 \in P_V(x) \).

**Proof** \( K(v_0,x) = \bigcup B(z_1, d(z_1, v_0)) \), \( z_1 \in G_1(v_0, x, -) \), \( K(v_0, y) = \bigcup B(z_2, d(z_2, v_0)) \), \( z_2 \in G_1(v_0, y, -) \).

Let \( z \in K(v_0,x) \) then \( z \in B(z_1, d(z_1, v_0)) \) for at least one \( z_1 \in G_1(v_0, x, -) \). Now any \( z_1 \in G_1(v_0, x, -) \) is also a point on \( G_1(v_0, y, -) \) i.e. \( z_1 = z_2 \) for some \( z_2 \in G_1(v_0, y, -) \) i.e. \( z \in \bigcup B(z_2, d(z_2, v_0)) \), \( z_2 \in G_1(v_0, y, -) \). Therefore,

\[
K(v_0,x) \subseteq K(v_0,y)
\]

Let \( z \in K(v_0,y) \) i.e \( z \in B(z_2, d(z_2, v_0)) \) for at least one \( z_2 \in G_1(v_0, y, -) \). If \( z_2 \in G_1(v_0, x, -) \) then \( z \in K(v_0,x) \) and so \( K(v_0,y) \subseteq K(v_0,x) \). If \( z_2 \in G[y,x] \), consider \( z \in G_1(v_0, x, -) \). Then

\[
d(z,z') \leq d(z, z_2) + d(z_2, z') \\
\leq d(z_2, v_0) + d(z_2, z') \\
= d(z', v_0).
\]

Therefore \( z \in B(z', d(z', v_0)) \) and so \( z \in K(v_0,x) \). Consequently

\[
K(v_0,x) \subseteq K(v_0,y)
\]

(1) and (2) imply \( K(v_0,x) = K(v_0,y) \).

The following theorem shows that we may assume in the definition of lunar point that \( x \) has \( v_0 \) as a best approximation from \( V \).

**Theorem 4** Let \( V \) be a subset of an \( M \)-space \((X,d)\) and \( v_0 \in V \). Then the following are equivalent:

(i) \( v_0 \) is a lunar point

(ii) whenever \( v_0 \) is a best approximation to \( x \) with \( K(v_0, x) \cap V \neq \emptyset \) then \( v_0 \in \overline{K(v_0, x) \cap V} \).

**Proof** (i) \( \Rightarrow \) (ii) is trivial. (ii) \( \Rightarrow \) (i). Let \( x \in X \) and \( K(v_0, x) \cap V \neq \emptyset \). To show \( v_0 \in K(v_0, x) \cap V \). If \( v_0 \) is a best approximation to \( x \) then by (ii), \( v_0 \in \overline{K(v_0, x) \cap V} \). If \( v_0 \) is not a best approximation to \( x \) then two cases arise:

(a) \( v_0 \) is not a local best approximation to \( x \).

(b) \( v_0 \) is a local best approximation to \( x \).

Case (a): If \( v_0 \) is not a local best approximation to \( x \) i.e. for all \( \epsilon \geq 0 \) there exists \( v_\epsilon \in V \) such that \( d(v_\epsilon, v_0) \leq \epsilon \) and \( d(v_\epsilon, x) \leq d(v_0, x) \). Then \( v_\epsilon \in B(x, d(v_0, x)) \subset K(v_0, x) \). Therefore every neighbourhood of \( v_0 \) contains an element \( v_\epsilon \) of \( K(v_0, x) \cap V \) other than \( v_0 \). i.e. \( v_0 \) is a limit point of \( K(v_0, x) \cap V \) and so \( v_0 \in \overline{K(v_0, x) \cap V} \). Hence \( v_0 \) is a lunar point.

Case (b): If \( v_0 \) is a local best approximation to \( x \) i.e. \( v_0 \) is a best approximation to \( x \) from \( V \cap B(v_0, \epsilon) \) for some \( \epsilon \geq 0 \). Let \( z \in G[v_0, x] \) such that \( d(z, v_0) \leq \frac{\epsilon}{2} \) then by Lemma 3 \( K(v_0, z) = K(v_0, x) \) and \( v_0 \) is a best approximation to \( z \) from \( V \) [7]. So (ii) implies \( v_0 \in \overline{K(v_0, z) \cap V} = \overline{K(v_0, x) \cap V} \) and therefore \( v_0 \) is a lunar point.
**Remark** For normed linear spaces, above theorem was proved by Amir and Deutsch [1].

**Corollary 5** Every sun in an M-space is a moon.

**Proof** Let $V$ be a sun. Suppose $V$ is not a moon i.e. there exists $v_0 \in V$ which is not a lunar point i.e. $v_0$ is a best approximation to $x \in X$ with $K(v_0, x) \cap V \neq \emptyset$ but $v_0 \notin K(v_0, x) \cap V$.

As $V$ is a sun, Theorem 2 implies $K(v_0, x) \cap V = \emptyset$ whenever $v_0$ is a best approximation to $x \in X$.

Since these two statements are contradictory, the result follows.

**Acknowledgements** First author is thankful to University Grant Commission, India and third author is thankful to Council of Scientific and Industrial Research (CSIR), India for financial support.

**References**


DEPARTMENT OF MATHEMATICS, GURU NANAK DEV UNIVERSITY, AMRITSAR-143005 (INDIA)

* E-mail:tdnarang1948@yahoo.co.in
† E-mail:shwetambry@rediffmail.com