COMMON FIXED POINT THEOREMS IN SMALL SELF DISTANCE QUASI-SYMMETRIC DISLOCATED METRIC SPACE

F. M. Zeyada, M. R. A. Moubarak* and A. H. Soliman

ABSTRACT. In this paper we introduce common fixed point theorems in a new type of generalized metric space so called a small self distance quasi-symmetric dislocated metric space (ssd-q-s-d-metric space for short). Our results are generalizations of Theorem 2.1 [1] due to Mohamed Aamri and Driss El Moutawakil.

1 Introduction and Preliminaries

There have been a number of generalizations of metric space. One such generalization is symmetric space. M. Aamri and D. El Moutawakil [1] introduced the following theorem in symmetric space.

Theorem 2.1. Let \( d \) be a symmetric for \( X \) that satisfies (W.3) and (HE). Let \( A \) and \( B \) be two weakly compatible selfmappings of \((X, d)\) such that (1) \( d(Ax, Ay) \leq \phi(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\}) \) for all \((x, y) \in X^2\), (2) \( A \) and \( B \) satisfy the property (E.A), and (3) \( AX \subseteq BX \). If the range of \( A \) or \( B \) is a complete subspace of \( X \), then \( A \) and \( B \) have a unique common fixed point.

The aim of the present paper is to give generalizations of Theorem 2.1 [1] in a type of generalized metric space weaker than symmetric space so called small self distance quasi-symmetric dislocated metric space.

Let \( X \) be a nonempty set and let \( d \) be a distance function. The pair \((X, d)\) is called a distance space [3].

We need the following conditions:

\[
\begin{align*}
& (d_1) \forall x \in X, d(x, x) = 0, \\
& (d_2) \forall x, y \in X, d(x, y) = 0 \Rightarrow x = y, \\
& (d_3) \forall x, y \in X, d(x, y) = d(y, x), \\
& (d_4) \forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y), \\
& (d_5) \forall x, y \in X, d(x, x) \leq \min\{d(x, y), d(y, x)\}
\end{align*}
\]

for all \( x, y, z \in X \). If \( d \) satisfies conditions \((d_1) - (d_4)\), then \((X, d)\) is called a metric space. If it satisfies conditions \((d_2) - (d_4)\), then \((X, d)\) is called a dislocated metric space [3]. Also \((X, d)\) is called a symmetric space if satisfies \((d_1) - (d_3)\).

Definition 1.2 [2]. Let \( A \) and \( B \) be two selfmappings of a metric space \((X, d)\). We say that \( A \) and \( B \) satisfy the property (E.A) if there exists a sequence \((x_n)\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t
\]

for some \( t \in X \).

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* Corresponding author.
2 Main results Definition 2.1. A distance space \((X, d)\) is called a small self distance quasi-symmetric-dislocated metric space (ssd-q-s-d-metric space, for short) if \(d\) satisfies \((d_2)\)′ and \((d_3)\).

Example 2.1. Let \(X\) be a nonempty set and \(d : X \times X \to [0, \infty)\) defined by \(d(x, y) = \frac{1}{3}\) if \(x = y\) and \(d(x, y) = 1\) if \(x \neq y\). Then \((X, d)\) is a small self distance quasi-symmetric-dislocated metric space.

Definition 2.2. Let \((X, d)\) be a ssd quasi-symmetric dislocated metric space and let \(Y \subset X\). \(Y\) said to be \(l\)-closed (resp. \(r\)-closed) if \(d(x, Y) = 0\) (resp. \(d(Y, x) = 0\)), then \(x \in Y\).

Definition 2.3. Two selfmapping \(A\) and \(B\) of ssd-q-s-d-metric \(X\) are said to be weakly compatible if they commute at there coincidence points; i.e., if \(Bu = Au\) for some \(u \in X\), then \(BAu = ABu\).

Definition 2.4. Let \((X, d)\) a ssd-q-sd-metric space. Then \((X, d)\) satisfies \((\ell u; 3)\) if for every sequence \((x_n)\) in \(X\) and \(x, y \in X\), if \(\lim_{n \to \infty} d(x, x_n) = \lim_{n \to \infty} d(y, x_n) = 0\), then \(x = y\); and satisfies \((ru; 3)\) if for every sequence \((x_n)\) in \(X\) and \(x, y \in X\), \(\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x_n, y) = 0\), then \(x = y\).

Definition 2.5. Let \((X, d)\) be a ssd-q-s-d-metric space. Two self mappings \(A\) and \(B\) of \((X, d)\) are said to have the property \((\ell - E.A - H_E)\) if

(a) \(AX \subseteq BX\),
(b) there exists a sequence \((x_n)\) such that \(\lim_{n \to \infty} d(t, Ax_n) = \lim_{n \to \infty} d(t, Bx_n) = \lim_{n \to \infty} d(Bx_n, Ax_n) = 0\) for some \(t \in X\).

Also, \(A\) and \(B\) are said to have the property \((r - E.A - H_E)\) if

\(a')\) \(AX \subseteq BX\),
\(b')\) there exists a sequence \((x_n)\) such that \(\lim_{n \to \infty} d(Ax_n, t) = \lim_{n \to \infty} d(Bx_n, t) = \lim_{n \to \infty} d(Bx_n, Ax_n) = 0\) for some \(t \in X\).

In the sequel, we need a function \(\phi : R^+ \to R^+\) satisfying the condition \(0 < \phi(t) < t\) for each \(t > 0\).

Theorem 2.1. Let \((X, d)\) be a ssd-q-sd-metric space that satisfies \((\ell u; 3)\). Let \(A\) and \(B\) be two weakly compatible selfmappings of \((X, d)\) such that

1. \(d(Ax, Ay) \leq \phi(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\})\) \(\forall x, y \in X\);
2. \(A\) and \(B\) satisfies \((\ell - E.A - H_E)\). If \(AX\) or \(BX\) is \(l\)-closed. Then \(A\) and \(B\) have a unique common fixed point.

Proof. From (2), there exists a sequence \((x_n)\) in \(X\) such that \(\lim_{n \to \infty} d(t, Ax_n) = \lim_{n \to \infty} d(t, Bx_n) = \lim_{n \to \infty} d(Bx_n, Ax_n) = 0\). Since \(BX\) is \(l\)-closed or \(AX\) is \(l\)-closed, then \(t \in BX\) or \(t \in AX\). Thus there exists \(u \in X\) such that \(Bu = t\). Now, we prove that \(Au = Bu\). If \(Au \neq Bu\), then from \((\ell u; 3)\), \(\lim_{n \to \infty} d(Au, Ax_n) = \alpha > 0\). Thus for \(0 < \epsilon < \alpha\), there exists \(n_0 (\epsilon) \in N\) such that \(\forall n \geq n_0 (\epsilon), |d(Au, Ax_n) - \alpha| < \epsilon\), i.e., \(\alpha - \epsilon < d(Au, Ax_n) < \alpha + \epsilon\). Thus \(\forall n \geq n_0 (\epsilon)\),

\[d(Au, Ax_n) \leq \phi(\max\{d(Bu, Bx_n), d(Bu, Ax_n), d(Bx_n, Ax_n)\})\]
\[< \max\{d(Bu, Bx_n), d(Bu, Ax_n), d(Bx_n, Ax_n)\}\]

Letting \(n \to \infty\) we have \(\lim_{n \to \infty} d(Au, Ax_n) = 0\). So from \((\ell u; 3)\), \(Au = Bu\). The weak compatibility of \(A\) and \(B\) implies that \(ABu = BAu\) and then \(AAu = ABu = BAu = BBu\). Let us show that \(Au\) is a common fixed point of \(A\) and \(B\). Suppose that \(AAu \neq Au\), then
Proof. unique common fixed point
Let us show that compatibility of which is a contradiction. Therefore \( Au = AAu = BAu = BBu. \) Second if \( d(Au, AAu) \neq 0, \) then

\[
d(Au, AAu) \leq \phi(\max\{d(Bu, BAu), d(BAu, Bu)\}) = \phi(d(Au, AAu))
\]

which is a contradiction. Therefore \( Au = AAu = BAu. \) Hence \( Au \) is a common fixed point of \( A \) and \( B. \) Suppose \( u \) and \( v \) are two fixed points of \( A \) and \( B \) and \( u \neq v. \) Then \( d(u, v) > 0 \) or \( d(v, u) > 0. \) If \( d(u, v) > 0, \) then

\[
d(u, v) = d(Au, Av) \leq \phi(\max\{d(Bu, v), d(Bu, v)\}) = \phi(d(u, v)) < d(u, v),
\]

which is a contradiction. Also if \( d(v, u) > 0, \) one can deduce that \( d(v, u) < d(v, u) \) which is a contradiction. Therefore \( u = v. \)

**Theorem 2.2.** Let \((X, d)\) be a ssd-q-sd-metric space that satisfies \((r \, w.3)\). Let \( A \) and \( B \) be two weakly compatible selfmappings of \((X, d)\) such that

1. \( d(Ax, Ay) \leq \phi(\max\{d(Bx, By), (Ax, By), d(Bx, Ax)\}) \) \( \forall x, y \in X; \)
2. \( A \) and \( B \) satisfies \((r - E.A - H_E)\). If \( AX \) or \( BX \) is \( r- \) closed, then \( A \) and \( B \) have a unique common fixed point

**Proof.** From (2), there exists a sequence \((x_n)\) in \( X \) such that \( \lim_{n \to \infty} d(Ax_n, t) = \lim_{n \to \infty} d(Bx_n, t) = \lim_{n \to \infty} d(Ax_n, Ax_n) = 0. \) Since \( BX \) is \( r- \) closed or \( AX \) is \( r- \) closed, then \( t \in BX \) or \( t \in AX. \) Thus there exists \( u \in X \) such that \( Bu = t. \) Now, we prove that \( Au = Bu. \) If \( Au \neq Bu, \) then from \((ru3), \) \( \lim_{n \to \infty} d(Ax_n, Bu) = \alpha > 0. \) Thus for \( 0 < \epsilon < \alpha, \) there exists \( n_0(\epsilon) \in N \) such that \( \forall n \geq n_0(\epsilon), \) \( d(Ax_n, Bu) < \alpha - \epsilon < d(Ax_n, Bu) < \epsilon + \alpha. \) Thus \( \forall n \geq n_0(\epsilon), \)

\[
d(Ax_n, Au) \leq \phi(\max\{d(Bx_n, Bu), d(Ax_n, Bu), d(Bx_n, Ax_n)\})
\]

Letting \( n \to \infty \) we have \( \lim_{n \to \infty} d(Ax_n, Au) = 0. \) So from \((ru3), \) \( Au = Bu. \) The weak compatibility of \( A \) and \( B \) implies that \( ABu = BAu \) and then \( AAu = ABu = BAu = BBu. \) Let us show that \( Au \) is a common fixed of \( A \) and \( B. \) Suppose that \( AAu \neq Au, \) then

\[
d(AAu, Au) \leq \phi(\max\{d(Bu, Bu), d(Au, Bu), d(Bu, Au)\}) = \phi(d(Au, Au))
\]

which is a contradiction. Therefore \( Au = AAu = BAu = BBu. \) Second if \( d(Au, AAu) \neq 0, \) then

\[
d(Au, AAu) \leq \phi(\max\{d(Bu, BAu), d(Au, BAu), d(Bu, Au)\}) = \phi(d(Au, AAu))
\]

< \( d(Au, AAu), \)
which is a contradiction. Therefore \( Au = AAu = BAu \). Hence \( Au \) is a common fixed of \( A \) and \( B \). Suppose \( u \) and \( v \) are two fixed points of \( A \) and \( B \) and \( u \neq v \). Then \( d(u, v) > 0 \) or \( d(v, u) > 0 \). If \( d(u, v) > 0 \), then
\[
d(u, v) = d(Au, Av) \leq \phi(\max\{d(Bu, Bv), d(Au, Bv), d(Bu, Au)\}) = \phi(d(u, v)) < d(u, v),
\]
which is a contradiction. The same is obtained if \( d(v, u) > 0 \). Therefore \( u = v \).

**Conclusion.** Since any symmetric space is ssd-q-s-d-metric space and the conditions in Theorem 2.1 \([1]\) implies the conditions in Theorem 2.1 or in Theorem 2.2, then Theorem 2.1 \([1]\) is obtained as a corollary of Theorem 2.1 or Theorem 2.2.

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**References**


**Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut, Egypt**

*Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt*

E-mail address: a...h...soliman@yahoo.com