UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we deal with the uniqueness problems on meromorphic functions concerning differential polynomials that share a small meromorphic function. Moreover, we improve some former results.

1 Introduction and Results

In this paper, we assume all the functions are non-constant meromorphic functions in the complex plane $C$. We shall use the standard notations of Nevanlinna theory of meromorphic functions such as $T(r, f), m(r, f), N(r, f), N(r, f), S(r, f), \text{etc.}$.

It is well known that if $f$ and $g$ share four distinct values CM, then $f$ is a Möbius transformation of $g$. Recently, corresponding to one famous question of Hayman [1], many uniqueness theorems for some certain types of differential polynomials sharing one value were obtained (See [2, 3, 4, 5]).

In 2001, M. Fang and W. Hong proved:

Theorem A [3]. Let $f$ and $g$ be two transcendental entire functions, $n \geq 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.

Afterwards, W. Lin and H. X. Yi improved Theorem A and obtained the following results:

Theorem B [4]. Let $f$ and $g$ be two transcendental entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.

Theorem C [4]. Let $f$ and $g$ be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then either $f(z) \equiv g(z)$ or $f = \{(n+2)h(1-h^{n+1})\}/\{(n+1)(1-h^{n+2})\}$ and $g = \{(n+2)(1-h^{n+1})\}/\{(n+1)(1-h^{n+2})\}$, where $h$ is a nonconstant meromorphic function.

Recently, W. Lin and H. X. Yi extended Theorem B and Theorem C concerning to fix-points (See [5]).

In this paper, some uniqueness problems of meromorphic functions are investigated, which are improvement and complementary results for the above theorems.

Throughout this paper, we use the following notations:

Let $E(f) = \{z|f(z) = 0\}$, where a zero of $f$ with multiplicity $m$ is counted $m$ times. If $E(f - \alpha) = E(g - \alpha)$, then we say that $f$ and $g$ share $\alpha$ CM, especially, we say that $f(z)$ and $g(z)$ have the same fixed-points if $\alpha(z) = z$. Let $E_k(f) = \{z\}$ zeros of $f(z)$ with multiplicity at most $k$, where a zero with multiplicity $m \leq k$ is counted $m$ times. Obviously, if $E(f) = E(g)$, then $E_k(f) = E_k(g)$, for $k = 1, 2, \cdots$.

Let $f$ be a meromorphic function. We denote by $\nu_k(f, r)$ the number of poles of $f$ with multiplicity at most $k$ in $|z| < r$ counting its multiplicities. We denote by $n_{(k,f)}$
the number of poles of \( f \) with multiplicity at least \( k \) in \( |z| < r \) counting its multiplicities. We denote by \( n_2(r, f) \) the number of poles of \( f \) in \( |z| < r \), where a simple pole is counted once and a multiple pole is counted two times. We denote by \( \mathfrak{m}(r, f) \) as the counting function of poles of \( f \) counted with ignoring multiplicities. \( N_k(r, f), N_k(r, f), N_2(r, f), N_k(r, f), N_k(r, f), N_k(r, f), N_k(r, f) \) and so on are defined in the usual way, respectively.

Let \( f, g \) and \( \alpha \) be meromorphic functions. Let \( \Psi_f(z) = f^{n+1}(z)(f^{m}(z) + a) + \alpha(z) \), where \( a \) is a constant. We note that

\[
\Psi_f'(z) = (n + m + 1)\{f^n(z)(f^m(z) + a_1)f'(z) + \alpha_1(z)\},
\]

where \( a_1 = (n + 1)a/(n + m + 1) \) and \( \alpha_1(z) = \alpha'(z)/(n + m + 1) \).

**Theorem 1.** Let \( f \) and \( g \) be two transcendental entire functions, \( \alpha \) be a meromorphic function such that \( T(r, \alpha_1) = o(T(r, f) + T(r, g)) \) and \( \alpha_1 \neq 0, \infty \). Let \( \Psi_f(z) \) be as above, and \( a \) be a nonzero constant. Suppose that \( m, n, k \) are positive integers such that \( (k - 1)n > 7 + 3m + k(5 + m) \). If \( E_k(\Psi_f') = E_k(\Psi_g') \), then \( f(z) \equiv g(z) \).

**Remark 1.** Under the condition of Theorem 1, letting \( k \to \infty \), we obtain that \( f(z) \equiv g(z) \) if \( E(\Psi_f') = E(\Psi_g) \) and \( n > 5 + m \). Obviously, Theorem 1 improves Theorem A and Theorem B.

**Theorem 2.** Let \( f \) and \( g \) be two transcendental meromorphic functions, \( \alpha_1 \) be a meromorphic function such that \( T(r, \alpha_1) = o(T(r, f) + T(r, g)) \) and \( \alpha_1 \neq 0, \infty \). Let \( a \) be a nonzero constant. Suppose that \( m, n, k \) are positive integers such that \( (k - 1)n > 14 + 3m + k(10 + m) \). If \( E_k(\Psi_f') = E_k(\Psi_g') \), then

(i) if \( m \geq 2 \), then \( f(z) \equiv g(z) \);

(ii) if \( m = 1 \), either \( f(z) \equiv g(z) \) or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where

\[
R(\omega_1, \omega_2) = (n + 1)(\omega_1^{n+2} - \omega_2^{n+2}) - (n + 2)(\omega_1^{n+1} - \omega_2^{n+1}).
\]

**Remark 2.** Under the condition of Theorem 2, letting \( k \to \infty \), we obtain that the result of Theorem 2 is still valid if \( E(\Psi_f') = E(\Psi_g) \) and \( n > 10 + m \). Obviously, Theorem 2 improves Theorem C.

**Theorem 3.** Let \( f \) and \( g \) be two transcendental meromorphic functions, \( \alpha \) be a meromorphic function such that \( T(r, \alpha) = o(T(r, f) + T(r, g)) \) and \( \alpha \neq 0, \infty \). Let \( a \) be a nonzero constant. Suppose that \( \Theta(\infty, f) + \Theta(\infty, g) > (2/5)\{(10 + m - n + 2(n + m + 2)/(k + 1)\} \) holds for positive integers \( m, n, k \) such that \( k \geq 2 \) and \( n \geq 10 + m \). If \( E_k(\Psi_f') = E_k(\Psi_g') \), then

(i) if \( m \geq 2 \), \( f(z) \equiv g(z) \);

(ii) if \( m = 1 \), either \( f(z) \equiv g(z) \), or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where

\[
R(\omega_1, \omega_2) = (n + 1)(\omega_1^{n+2} - \omega_2^{n+2}) - (n + 2)(\omega_1^{n+1} - \omega_2^{n+1}).
\]

As the consequence of Theorem 2 and Theorem 3, letting \( k \to \infty \), we have

**Remark 3.** Let \( f \) and \( g \) be two transcendental meromorphic functions, \( \alpha \) be a meromorphic function such that \( T(r, \alpha) = o(T(r, f) + T(r, g)) \) and \( \alpha \neq 0, \infty \). Let \( a \) be a nonzero constant, and \( m, n \) \((m \geq 2, n \geq 10 + m)\) be positive integers. If \( E(\Psi_f') = E(\Psi_g) \) and \( \Theta(\infty, f) > 0 \), then \( f(z) \equiv g(z) \).
Corollary. Let $f$ and $g$ be two transcendental meromorphic functions, $\alpha$ be a meromorphic function such that $T(r, \alpha) = o(T(r, f) + T(r, g))$ and $\alpha \not\equiv 0, \infty$. Let $a$ be a nonzero constant. Suppose that $\Theta(\infty, f) > 2/(n + 1)$ holds. If $E(f'') = E(f')$ holds for positive integers $m$ and $n \geq 10 + m$, then $f(z) \equiv g(z)$.

**Remark 4.** In the case $m = 1$, the following example shows that the condition of $\Theta(\infty, f) > 2/(n + 1)$ is necessary.

**Example.** Let

$$f = \frac{(n + 2)h(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}, \quad g = \frac{(n + 2)(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)},$$

where $u = \exp\{(2\pi i)/(n+2)\}$ and $h = (u^2e^z - u)/(e^z - 1)$. It is easy to find $E(f') = E(f'')$ and $\Theta(\infty, f) = 2/(n + 1)$, but $f(z) \not\equiv g(z)$.

### 2 Lemmas

For proving the theorems, we need the following lemmas.

**Lemma 1 [6].** Let $f(z)$ be a nonconstant meromorphic function, and

$$R(f) = \sum_{k=0}^n a_k f^k / \sum_{j=0}^m b_j f^j$$

be an irreducible rational function in $f$ with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max \{n, m\}$.

**Lemma 2.** Let $f$ and $g$ be two nonconstant meromorphic functions, and $\alpha$ be a meromorphic function such that $T(r, \alpha) = o(T(r, f) + T(r, g))$ and $\alpha \not\equiv 0, \infty$. Let $a$ be a nonzero constant, and $n$ and $m$ be positive integers. Set

$$F = f^n(f^m - a_1)f', \quad G = g^n(g^m - a_1)g'.$$

If $E_k(F - \alpha) = E_k(G - \alpha)$ and $(n - 6)k - m > 4$, then $S(r, f)$ and $S(r, g)$ are equivalent, that is, if $A(r) = S(r, f)$, then $A(r) = S(r, g)$, and also if $A(r) = S(r, g)$, then $A(r) = S(r, f)$.

**Proof.** By Lemma 1, we have

$$(n + m)T(r, f) = T(r, f^n(f^m + a)) + S(r, f) \leq T(r, F) + T(r, f') + S(r, f).$$

Therefore we have

$$T(r, F) \geq (n + m - 2)T(r, f) + S(r, f).$$

By the second fundamental theorem, we have

$$T(r, F) \leq N(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - \alpha}\right) + S(r, F) \leq N(r, F) + N\left(r, \frac{1}{F}\right) + N_k\left(r, \frac{1}{F - \alpha}\right) + N_{k+1}\left(r, \frac{1}{F - \alpha}\right) + S(r, f) \leq N(r, F) + N\left(r, \frac{1}{F}\right) + N_k\left(r, \frac{1}{G - \alpha}\right) + \frac{1}{k+1}N\left(r, \frac{1}{F - \alpha}\right) + S(r, f) \leq (4 + m)T(r, f) + \frac{1}{k+1}T(r, F) + T(r, G) + S(r, f).$$
Noting that $T(r, G) \leq T(r, g^n(g^m - 1)) + T(r, g^\prime) \leq (n + m + 2)T(r, g) + S(r, g)$, we deduce that
\[
\left(\frac{k(n + m - 2)}{k + 1} - 4 - m\right)T(r, f) \leq (n + m + 2)T(r, g) + S(r, f) + S(r, g).
\]
We note that $k(n + m - 2)/(k + 1) - 4 - m > 0$ and the conditions for $f$ and $g$ are symmetric. Thus $S(r, f)$ and $S(r, g)$ are equivalent.

**Lemma 3.** Let $F$ and $G$ be two nonconstant meromorphic functions such that $E_k(F - 1) = E_k(G - 1)$, and let
\[
H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right).
\]
If $H \neq 0$, then
\[
\frac{1}{2}\left\{T(r, F) + T(r, G)\right\} \leq N_2(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2(r, G) + N_2\left(r, \frac{1}{G}\right) + N_2\left(r, \frac{1}{F - 1}\right) + N_2\left(r, \frac{1}{G - 1}\right) + S(r),
\]
where $T(r) = \max\{T(r, F), T(r, G)\}$, $S(r) = o(T(r))$ ($r \to \infty, r \notin E$) and $E$ is a set of finite linear measure.

**Proof.** By the second fundamental theorem, we have
\[
T(r, F) \leq \mathcal{N}(r, F) + \mathcal{N}\left(r, \frac{1}{F}\right) - N_0\left(r, \frac{1}{F'}\right) + S(r, F)
\]
and
\[
T(r, G) \leq \mathcal{N}(r, G) + \mathcal{N}\left(r, \frac{1}{G}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, G),
\]
where $N_0(r, 1/F')$ is the counting function of the zeros of $F'$ in $|z| < r$ that is not the zeros of $F - 1$ and $F$. In the same way, we can define $N_0(r, 1/G')$. Thus we have
\[
T(r, F) + T(r, G) \leq \mathcal{N}(r, F) + \mathcal{N}(r, G) + \mathcal{N}\left(r, \frac{1}{F}\right) + \mathcal{N}\left(r, \frac{1}{G}\right) + \mathcal{N}\left(r, \frac{1}{F - 1}\right) + \mathcal{N}\left(r, \frac{1}{G - 1}\right) - N_0\left(r, \frac{1}{F'}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r).
\]
(1)
We also have
\[
\mathcal{N}\left(r, \frac{1}{F - 1}\right) + \mathcal{N}\left(r, \frac{1}{G - 1}\right) \leq \mathcal{N}_k\left(r, \frac{1}{F - 1}\right) + \mathcal{N}_k\left(r, \frac{1}{G - 1}\right) + \mathcal{N}_{k+1}\left(r, \frac{1}{F - 1}\right) + \mathcal{N}_{k+1}\left(r, \frac{1}{G - 1}\right) \leq \frac{1}{2}\left\{N_1\left(r, \frac{1}{F - 1}\right) + N_1\left(r, \frac{1}{G - 1}\right) + N_k\left(r, \frac{1}{F - 1}\right) + N_k\left(r, \frac{1}{G - 1}\right)\right\} + \mathcal{N}_{k+1}\left(r, \frac{1}{F - 1}\right) + \mathcal{N}_{k+1}\left(r, \frac{1}{G - 1}\right).
\]
Let $z_0$ be a simple pole of $F$. By a simple calculation, we know that $z_0$ is not a pole of $F''/F' - 2F'/F - 1$. Let $z_1$ be a zero of $F - 1$ with multiplicity $t$, where $1 \leq t \leq k$.

We know also that $z_1$ is not a pole of $H$, especially, $z_1$ is a simple zero of $H$ if $k = 1$. In fact, by a simple calculation, we can prove that any common simple 1-point of $F$ and $G$ is a zero of
\[
H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right).
\]
Let $z_0$ be a common simple 1-point of $F$ and $G$. If we expand $F$ in a neighborhood of $z_0$ as

$$F = 1 + A(z - z_0) + B(z - z_0)^2 + D(z - z_0)^3 + O((z - z_0)^4), \quad (A \neq 0).$$

Then we have

$$\frac{F''}{F'} - \frac{2F'}{F - 1} = \frac{-1}{z - z_0} + O(z - z_0).$$

Similarly, we have

$$\frac{G''}{G'} - \frac{2G'}{G - 1} = \frac{-1}{z - z_0} + O(z - z_0).$$

Thus we obtain that

$$H(z) = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right) = O(z - z_0),$$

that is, $z_0$ is a zero of $H$.

Similarly, by a simple calculation, we can also prove that any simple pole of $F$ is not a pole of $F'/F - 2F'/(F - 1)$, and any simple pole of $G$ is not a pole of $G'/G - 2G'/(G - 1)$.

Thus we have

$$N_1(r, \frac{1}{F - 1}) \leq N_1(r, \frac{1}{H}) \leq T(r, H) + S(r, F) = N(r, H) + S(r, F)$$

$$\leq \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}(r, \frac{1}{F - 1}) + \overline{N}(r, \frac{1}{G - 1})$$

$$+ N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, F). \quad (2)$$

Similarly we have

$$N_1(r, \frac{1}{G - 1}) \leq \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}(r, \frac{1}{F - 1}) + \overline{N}(r, \frac{1}{G - 1})$$

$$+ N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, G). \quad (3)$$

By (1), (2) and (3), we have the desired inequality.

**Lemma 4[7].** Let $H$ be defined as in Lemma 3. If $H \equiv 0$ and

$$\lim_{r \to \infty} \sup_{r \in I} \overline{N}(r, 1/F) + \overline{N}(r, 1/G) + \overline{N}(r, F) + \overline{N}(r, G)$$

$$\frac{T(r)}{	ext{lim sup}_{r \to \infty}} < 1,$$

where $I$ is a set of infinite linear measure, then $FG \equiv 1$ or $F \equiv G$.

**3 Proof of Theorems**

(I) **Proof of Theorem 2.**

Let

$$F = \frac{f^n(f^m + a_1)f'}{\alpha_1(z)}, \quad G = \frac{g^n(g^m + a_1)g'}{\alpha_1(z)}, \quad (4)$$

$$F_1 = \frac{1}{n + m + 1}f^{n+m+1} + \frac{a_1}{n + 1}f^{n+1}, \quad G_1 = \frac{1}{n + m + 1}g^{n+m+1} + \frac{a_1}{n + 1}g^{n+1}, \quad (5)$$

and

$$H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right).$$
Here \(a_1 = (n+1)a/(n+m+1)\) and \(\alpha_1 = (-\alpha')/(n+m+1)\). Then \(E_{k_1}(F-1) = E_{k_1}(G-1)\). By Lemma 1 and Lemma 2, we have \(S(r, f) = S(r, g)\) \((= S(r), \text{ say})\) and

\[
T(r, F) = (n + m + 1)T(r, f) + S(r, f), \quad T(r, G) = (n + m + 1)T(r, g) + S(r, g). \quad (6)
\]

Since \(F' = \alpha_1(z)F\) and \(G' = \alpha_1(z)G\), we deduce that

\[
T(r, F_1) + T(r, G_1) \leq T(r, F) + N\left( r, \frac{1}{F_1} \right) - N\left( r, \frac{1}{F} \right) + T(r, G) + N\left( r, \frac{1}{G_1} \right) - N\left( r, \frac{1}{G} \right) + S(r)
\]

\[
= T(r, F) + (n + 1)N\left( r, \frac{1}{F} \right) + N\left( r, \frac{1}{f^{m+a}} \right) - nN\left( r, \frac{1}{F} \right) - N\left( r, \frac{1}{f^{m+a}} \right) - N\left( r, \frac{1}{f'} \right) + T(r, G) + (n + 1)N\left( r, \frac{1}{G} \right) + N\left( r, \frac{1}{g^{m+a}} \right) - nN\left( r, \frac{1}{G} \right) - N\left( r, \frac{1}{g^{m+a}} \right) + S(r)
\]

\[
= T(r, F) + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f^{m+a}} \right) - N\left( r, \frac{1}{f^{m+a}} \right) - N\left( r, \frac{1}{g} \right) + T(r, G) + N\left( r, \frac{1}{g} \right) + N\left( r, \frac{1}{g^{m+a}} \right) - N\left( r, \frac{1}{g^{m+a}} \right) + S(r).
\]

(7)

If \(H \neq 0\), by Lemma 3, we have

\[
T(r, F) + T(r, G) \leq 2\left\{ N_2(r, F) + N_2\left( r, \frac{1}{F} \right) + N_2(r, G) + N_2\left( r, \frac{1}{G} \right) + \overline{N}(k+1)\left( r, \frac{1}{F-1} \right) + \overline{N}(k+1)\left( r, \frac{1}{G-1} \right) \right\} + S(r). \quad (8)
\]

It follows from (4) that

\[
N_2(r, F) + N_2\left( r, \frac{1}{F} \right) + N_2(r, G) + N_2\left( r, \frac{1}{G} \right)
\]

\[
\leq 2\left\{ \overline{N}(r, f) + N\left( r, \frac{1}{f} \right) \right\} + N\left( r, \frac{1}{f^{m+a_1}} \right) + N\left( r, \frac{1}{f'} \right) + 2\left\{ \overline{N}(r, g) + N\left( r, \frac{1}{g} \right) \right\} + N\left( r, \frac{1}{g^{m+a_1}} \right) + N\left( r, \frac{1}{g'} \right) + S(r). \quad (9)
\]
Thus we have, 

\[(n + m + 1) \left\{ T(r, f) + T(r, g) \right\} = \left\{ T(r, F_1) + T(r, G_1) \right\} + S(r) \]

\[\leq T(r, F') + T(r, G) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\]

\[+ N\left(r, \frac{1}{f^{m + a}}\right) + N\left(r, \frac{1}{g^{m + a}}\right) - N\left(r, \frac{1}{f^{m + a_1}}\right) - N\left(r, \frac{1}{g^{m + a_1}}\right) - N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{g}\right) + S(r)\]

\[\leq 4\left\{ N(r, f) + N(r, g) \right\} + 5\left\{ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) \right\}\]

\[+ 2\left\{ N\left(r, \frac{1}{f^{m + a}}\right) + N\left(r, \frac{1}{g^{m + a}}\right) \right\} + 2\left\{ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) \right\}\]

\[+ 2\left\{ \overline{N}(k + 1)\left(r, \frac{1}{f^{m + a}}\right) + \overline{N}(k + 1)\left(r, \frac{1}{G - 1}\right) \right\} + N\left(r, \frac{1}{f^{m + a}}\right) + N\left(r, \frac{1}{g^{m + a}}\right) - N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{g}\right) + S(r)\]

\[\leq 4\left\{ N(r, f) + N(r, g) \right\} + 5\left\{ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) \right\}\]

\[+ \left\{ T(r, F') + T(r, G') \right\} + N\left(r, \frac{1}{f^{m + a}}\right) + N\left(r, \frac{1}{g^{m + a}}\right)\]

\[+ \frac{2}{k + 1}\left\{ N\left(r, \frac{1}{F - 1}\right) + N\left(r, \frac{1}{G - 1}\right) \right\} + N\left(r, \frac{1}{f^{m + a}}\right) + N\left(r, \frac{1}{g^{m + a}}\right) + S(r)\]

\[\leq 4\left\{ N(r, f) + N(r, g) \right\} + 5\left\{ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) \right\}\]

\[+ 2\left\{ T(r, f) + T(r, g) \right\} + \frac{2}{k + 1}\left\{ T(r, F) + T(r, G) \right\}\]

\[+ 2m\left\{ T(r, f) + T(r, g) \right\} + S(r)\]

\[\leq \left\{ 11 + 2m + \frac{2(m + n + 2)}{k + 1} \right\} \left\{ T(r, f) + T(r, g) \right\} + S(r).\]

Hence we have,

\[(n + m + 1) \left\{ T(r, f) + T(r, g) \right\} \leq \left\{ 11 + 2m + \frac{2(m + n + 2)}{k + 1} \right\} \left\{ T(r, f) + T(r, g) \right\} + S(r).\]

Thus we have \( n + m + 1 \leq 11 + 2m + \{2(m + n + 2)/(k + 1)\}, \) which contradicts \((k - 1)n > 14 + 3m + (10 + m)k.\) Therefore we have \( H \equiv 0, \) that is,

\[\frac{F''}{F'} - \frac{2F'}{F - 1} \equiv \frac{G''}{G'} - \frac{2G'}{G - 1}.\]
Hence we see

\[
\frac{1}{G-1} = \frac{A}{F-1} + B,
\]

where \( A \neq 0 \) and \( B \) are constants. Thus \( E(F-1) = E(G-1) \), and

\[
T(r,F) = T(r,G) + S(r).
\]

Since

\[
\begin{align*}
N(r,F) & = N\left(r,\frac{1}{F}\right) + N(r,G) + N\left(r,\frac{1}{G}\right) \\
& \leq N(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^m+a_1}\right) \\
& \quad + N(r,g) + N\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g^m+a_1}\right) + S(r) \\
& \leq (m+4)\{T(r,f) + T(r,g)\} + S(r) \\
& \leq \frac{2(m+4)}{n+m-2}T(r) + S(r),
\end{align*}
\]

we have

\[
\limsup_{r \to \infty} \frac{N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + N(r,F) + N(r,G)}{T(r)} < 1,
\]

by Lemma 4 we get \( FG \equiv 1 \) or \( F \equiv G \).

We next discuss the following two cases.

Case 1. Suppose that \( FG \equiv 1 \), that is,

\[
f^n(f^m+a_1)f'g^n(g^m+a_1)g' \equiv \alpha^2(z).
\]

(a) Let \( z_0 \) be a zero of \( f \) of order \( p \) such that \( \alpha(z_0) \neq 0, \infty \). From (11) we know that \( z_0 \) is a pole of \( g \). Suppose that \( z_0 \) is a pole of \( g \) of order \( p \). From (11) we obtain that

(i) If \( p = 1 \), then \( n = np + mq + q + 1 \). This is a contradiction.

(ii) If \( p > 1 \), then \( np + p - 1 = np + mq + q + 1 \). This implies \( (n+1)(p-q) = mq + 2 > 0 \). Hence \( p \geq q + 1 \). Thus we have \( np + p - 1 < (n + m + 1)(p - 1) + 1 \). Therefore we see \( p \geq (n + m - 1)/m \).

(b) Let \( z_1 \) be a zero of \( f^m+a_1 \) of order \( p_1 \) such that \( \alpha(z_1) \neq 0, \infty \). From (11) we know that \( z_1 \) is a pole of \( g \). From (11) we obtain that

(i) If \( p_1 = 1 \), then \( 1 = nq_1 + mq_1 + q_1 + 1 \). This is a contradiction.

(ii) If \( p_1 > 1 \), then \( p_1 + p_1 - 1 = nq_1 + mq_1 + q_1 + 1 \). Thus \( p_1 \geq (n + m + 3)/2 \).

(c) Let \( z_2 \) be a zero of \( f' \) of order \( p_2 \) such that \( \alpha(z_2) \neq 0, \infty \) that is not a zero of \( f(f^m+a_1) \). From (11) we know that \( z_2 \) is a pole of \( g \). Suppose that \( z_2 \) is a pole of \( g \) of order \( q_2 \). From (11) we obtain that \( p_2 = nq_2 + mq_2 + q_2 + 1 \). Thus \( p_2 \geq n + m + 2 \).

Moreover, in the same method as above, we have the similar results for the zeros of \( g(g^m+a_1)g' \). On the other hand, we suppose that \( z_3 \) is a pole of \( f \) such that \( \alpha(z_3) \neq 0, \infty \).
From (11) we obtain that $z_3$ is a zero of $g(g^m + a_1)g'$. Thus we have

$$
\mathcal{N}(r, f) \leq \mathcal{N}(r, \frac{1}{g}) + \mathcal{N}(r, \frac{1}{g^m + a_1}) + \mathcal{N}_*(r, \frac{1}{g'}) \\
\leq \frac{m}{n + m - 1} \mathcal{N}(r, \frac{1}{g}) + \frac{2}{n + m + 3} \mathcal{N}(r, \frac{1}{g^m + a_1}) + \frac{1}{n + m + 2} \mathcal{N}(r, \frac{1}{g'}) \\
\leq \left( \frac{m}{n + m - 1} + \frac{2m}{n + m + 3} + \frac{2}{n + m + 2} \right) T(r, g) + S(r)
$$

where $n_*(r, g)$ is defined the number of zeros of $g'$ that is not zero of $g(g^m + a_1)$ in $|z| \leq r$, a zero point with multiplicity $m$ is counted $m$ times in the set. $N_*(r, 1/g)$ is defined in the terms of $n_*(r, 1/g)$ in the usual manner.

Hence

$$
mT(r, f) < \mathcal{N}(r, f) + \sum_{j=1}^{m} \mathcal{N}(r, \frac{1}{f - c_j}) + \mathcal{N}(r, \frac{1}{f}) + S(r) \\
\leq \left( \frac{m}{n + m - 1} + \frac{2m}{n + m + 3} + \frac{2}{n + m + 2} \right) T(r, g) + \sum_{j=1}^{m} \mathcal{N}(r, \frac{1}{f - c_j}) + S(r) \\
= \left( \frac{m}{n + m - 1} + \frac{2m}{n + m + 3} + \frac{2}{n + m + 2} \right) T(r, g) + \sum_{j=1}^{m} \mathcal{N}(r, \frac{1}{f - c_j}) + S(r)
$$

where $f^m - a_1 = (f - c_1)(f - c_2) \cdots (f - c_m)$. Similarly we have

$$
mT(r, g) < \left( \frac{m}{n + m - 1} + \frac{2m}{n + m + 3} + \frac{2}{n + m + 2} \right) T(r, f) + \sum_{j=1}^{m} \mathcal{N}(r, \frac{1}{g - c_j}) + S(r)
$$

Thus we have

$$
m(T(r, f) + T(r, g)) \leq \left( \frac{2m}{n + m - 1} + \frac{4m}{n + m + 3} + \frac{2}{n + m + 2} \right) (T(r, f) + T(r, g)) + S(r).
$$

Hence we have

$$
m < \frac{2m}{n + m - 1} + \frac{4m}{n + m + 3} + \frac{2}{n + m + 2},
$$

which contradicts with $n > m + 10$.

Case 2. Suppose that $F \equiv G$, then

$$
F_1 \equiv G_1 + C,
$$

where $C$ is a constant and

$$
F_1 = \frac{1}{n + m + 1} f^{n+m+1} + \frac{a_1}{n+1} f^{n+1}, \quad G_1 = \frac{1}{n + m + 1} g^{n+m+1} + \frac{a_1}{n+1} g^{n+1}.
$$

By Lemma 1 we have

$$
T(r, F_1) = (n + m + 1)T(r, f) + S(r), \quad T(r, G_1) = (n + m + 1)T(r, g) + S(r).
$$
It follows that
\[ T(r, f) = T(r, g) + S(r). \]  
(13)

Suppose that \( C \neq 0 \). By (13) we have
\[
(n + m + 1) T(r, g) = T(r, G_1) \\
< N\left(r, \frac{1}{G_1}\right) + N\left(r, \frac{1}{G_1 + C}\right) + N(r, G_1) + S(r) \\
\leq N(r, g) + N\left(r, \frac{1}{g^m + a}\right) + N\left(r, \frac{1}{f^m + a}\right) \\
+ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + S(r) \\
\leq (2m + 3) T(r, g) + S(r).
\]

Thus \( n + m + 1 \leq 2m + 3 \), which contradicts with \( n > m + 10 \). Therefore \( F_1 \equiv G_1 \), that is,
\[
f^{n+1}(f^m + a) \equiv g^{n+1}(g^m + a).
\]  
(14)

Thus \( f \) and \( g \) share \( \infty \) CM. Let \( h = f/g \). If \( h \neq 1 \), we have
\[ g^m = -\frac{a(h^{n+1} - 1)}{h^{n+m+1} - 1}. \]

If \( m \geq 2 \), we have
\[
(n - 1) T(r, h) \leq \sum_{j=1}^{n+1} N\left(r, \frac{1}{h - d_j}\right) + S(r, h) \\
\leq \frac{n + 1}{m} T(r, h) + S(r, h),
\]
where \( h^{n+m+1} - 1 = (h - 1)(h - d_1) \cdots (h - d_{n+m}) \). In fact, since each zero point of \( h - d_i \) has multiplicity at least \( m \), \( N(r, 1/(h - d_i)) \leq (1/m) N(r, 1/(h - d_i)) \leq (1/m) T(r, h) \). Thus \( (n - 1) \leq (n + 1)/m \), which contradicts with \( n > m + 10 \). Therefore \( h \equiv 1 \). Then \( f \equiv g \).

If \( m = 1 \), by (14), \( f \) and \( g \) satisfy the algebraic relation \( R(f, g) \equiv 0 \), where \( R(\omega_1, \omega_2) = (n + 1)(\omega_1^{n+2} - \omega_2^{n+2}) - (n + 2)(\omega_1^{n+1} - \omega_2^{n+1}) \). This completes the proof of Theorem 2.

(II) Proof of Theorem 1 and Theorem 3

By making use of Lemma 3 and a similar method to the proof of Theorem 2, we easily obtain the proof of Theorem 1 and Theorem 3.

References


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