POLYNOMIAL HULLS OF GRAPHS ON THE TORUS IN $\mathbb{C}^2$

TOSHIYA JIMBO

Received May 24, 2005; revised July 15, 2005

Abstract. We describe the polynomial hulls of graphs on the torus which are defined by the complex conjugate functions of polynomials in $\mathbb{C}^2$.

1. Introduction. Let $X$ be a compact subset in $\mathbb{C}^N$ and $\tilde{X}$ the polynomial hull of $X$. We denote by $C(X)$ the Banach algebra of all continuous functions on $X$ with sup-norm $\| \cdot \|_X$ and by $P(X)$ the closure in $C(X)$ of the polynomials in the coordinates.

Let $p(z, w)$ be an arbitrary polynomial in $\mathbb{C}^2$ and $f$ the restriction of the complex conjugate of $p$ to the unit torus $\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$. Let $G(f)$ denote the graph in $\mathbb{C}^3$ of $f$ on $\mathbb{T}^2$, i.e.,

$$G(f) = \{(z, w, f(z, w)) \in \mathbb{C}^3 : (z, w) \in \mathbb{T}^2\}.$$ 

H. Alexander[1]) and P. Ahern - W. Rudin ([2]) studied the structure of polynomial hulls of graphs on the unit sphere in $\mathbb{C}^n$. In this paper we consider the structure of polynomial hulls of graphs on $\mathbb{T}^2$ which are defined by the complex conjugates of polynomials in $\mathbb{C}^2$.

Assume that the degrees of $p(z, w) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} z^i w^j$ in $z$ and $w$ respectively are $m$ and $n$. We consider a polynomial $k(z, w) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} z^{m-i} w^{n-j}$ and rational function $h(z, w) = z^{-m} w^{-n} k(z, w)$. We have, for $(z, w) \in \mathbb{T}^2$,

$$\sum_{i=0}^m \sum_{j=0}^n a_{ij} \frac{1}{z^i w^j} = \frac{1}{z^m w^n} k(z, w) = h(z, w)$$

We set

$$\Delta(z, w) = \begin{vmatrix} \frac{\partial p}{\partial z}(z, w) & \frac{\partial p}{\partial w}(z, w) \\ \frac{\partial h}{\partial z}(z, w) & \frac{\partial h}{\partial w}(z, w) \end{vmatrix}.$$ 

We can write as a product

$$\Delta(z, w) = \frac{1}{z^{m+1} w^{n+1}} \prod_{i=1}^t q_i(z, w)^{n_i}$$

where $q_i(z, w)$ are irreducible polynomials. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, $\mathbb{T}$ its boundary and $\mathbb{D}^2$ the open unit polydisk in $\mathbb{C}^2$. For each $q_i(z, w)$ put

$$Z(q_i) = \{(z, w) \in \mathbb{C}^2 : q_i(z, w) = 0\},$$

$$Q_i = Z(q_i) \cap \mathbb{T}^2, \quad R_i = Z(q_i) \cap \overline{\mathbb{D}}^2.$$ 

We put $L = (\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{D}})$ and

$$V = \{(z, w) \in \mathbb{D}^2 \setminus (\mathbb{T}^2 \cup L) : p(z, w) = h(z, w)\}.$$ 

2000 Mathematics Subject Classification. Primary 32E20.

Key words and phrases. Polynomial hulls.
Let \([z, w, f; \mathbb{T}^2]\) be the uniform algebra generated by the coordinate functions \(z, w\) and \(f\) on \(\mathbb{T}^2\). Our result is that the polynomial hull of the graph \(G(f)\) can be determined as follows.

**Theorem.** Assume that \(\Delta(z, w) \neq 0\) on \(\mathbb{D}^2 \setminus L\). We put
\[
J = \{i \in \{1, 2, \ldots, t\} : \emptyset \neq Q_i \neq \bar{Q}_i, \bar{Q}_i \setminus (\mathbb{T}^2 \cup L) \subset V\}.
\]

(a) If \(J \neq \emptyset\), then we have \(\hat{G}(f) = \bigcup_{i \in J} \{(z, w, p(z, w)) : (z, w) \in \bar{Q}_i\} \cup G(f)\).

In this case \(p(z, w) = c_i\) (constant) on \(\bar{Q}_i\).

(b) If \(J = \emptyset\), then we have
\[
\hat{G}(f) = G(f), \text{ and } [z, w, f; \mathbb{T}^2] = C(\mathbb{T}^2).
\]

**2. Facts and lemmas.** Let \(M\) be a \(C^\infty\) real submanifold of an open set \(U\) in \(\mathbb{C}^N\). For a point \(\eta \in M\) we denote by \(T_{\eta}M\) the real tangent space of \(M\) at \(\eta\). \(M\) is called totally real at \(\eta\) if \(T_{\eta}M\) contains no non-trivial complex subspaces. \(M\) is called totally real if \(M\) is totally real at every point of \(M\). For a subset \(S\) of \(\mathbb{C}^2\) and a continuous function \(g\) on \(S\), we denote by \(G(g; S)\) the graph of \(g\) on \(S\), i.e.,
\[
G(g; S) = \{(z, w, g(z, w)) \in \mathbb{C}^3 : (z, w) \in S\}.
\]

When \(M\) is a totally real submanifold of \(U\) in \(\mathbb{C}^2\) and \(g\) is a \(C^\infty\) function in \(U\), it is known that the graph \(G(g; M)\) is totally real. For the graph \(G(f) = G(p; \mathbb{T}^2)\) we have that \(\hat{G}(f)\) is connected and so it does not contain any isolated points, since the polynomial hull of a compact connected set is connected. We need several facts and lemmas to decide the polynomial hull of \(\hat{G}(f)\).

**Theorem 2.1.** ([4], [7]). Let \(M\) be a \(C^\infty\) totally real submanifold of \(U\) in \(\mathbb{C}^N\).

(a) If \(X\) is a compact polynomially convex subset of \(M\), then \(P(X) = C(X)\).

(b) For a point \(\eta \in M\) there exists a small ball \(B_0\) centered at \(\eta\) such that \(\bar{B_0} \cap M\) is polynomially convex.

**Lemma 2.2.** ([5]). If \((z^0, w^0)\) is a point in \(V\) with \(\Delta(z^0, w^0) \neq 0\), then there is an open ball \(B_0\) centered at \((z^0, w^0)\) such that \(B_0 \cap V\) is totally real in \(B_0\).

**Lemma 2.3.** ([5]). Let \(X\) be a compact connected subset of \(\mathbb{C}^N\) and \(U\) an open subset of \(\mathbb{C}^N\) with \(U \cap X = \emptyset\). If \(\hat{X} \cap U\) is contained in a totally real submanifold \(M\) of \(U\), then we have \(\hat{X} \cap U = \emptyset\).

The proof of next lemma is obtained by the same way ([2]) in the case of the unit ball.

**Lemma 2.4.** Let \(g\) be a continuous function on \(\mathbb{T}^2\). If \((z^0, w^0) \in \mathbb{T}^2\) and \((z^0, w^0, \zeta^0) \in \hat{G}(g; \mathbb{T}^2)\), then \(\zeta^0 = g(z^0, w^0)\).

Next lemma is a special case of Lemma 1 in [6]. By using the results of uniform algebras it is also proved as follows.
Lemma 2.5. Let $g_1$ and $g_2$ be holomorphic functions on $\overline{\mathbb{D}^2}$ and $f = (\tilde{g}_1 + g_2)|_{T^2}$. Then we have

$$\widehat{G(f)} \subset G(\tilde{g}_1 + g_2; \overline{\mathbb{D}^2}).$$

Proof. Let $A = [z, w, \tilde{g}_1 + g_2; T^2] = [z, w, \tilde{g}_1; T^2]$ and $M_A$ the maximal ideal space of $A$. We denote by $X$ the joint spectrum of $z, w, \tilde{g}_1 + g_2$. Since a point evaluation of $T^2$ belongs $M_A$, $G(f)$ is contained in $X$, and so $G(f) \subset X$ (cf. [3]). For a point $(z_0, w_0, \zeta_0)$ in $\widehat{G(f)}$ there is a $\varphi \in M_A$ such that $z_0 = \varphi(z)$, $w_0 = \varphi(w)$ and $\zeta_0 = \varphi(\tilde{g}_1 + g_2)$. Then $|z_0| = |\varphi(z)| \leq \|z\|_{T^2} = 1$ and similarly $|w_0| \leq 1$. By using the polynomial approximation of $g_i$ we have that $\varphi(\tilde{g}_i) = g_i(z_0, w_0)$, $i = 1, 2$. Let $\mu$ be the representing measure on $T^2$ for $\varphi$. Then

$$\varphi(\tilde{g}_1) = \int_{T^2} \tilde{g}_1 d\mu = \int_{T^2} g_1 d\mu = \varphi(g_1).$$

Thus we have that $\varphi(\tilde{g}_1 + g_2) = \varphi(\tilde{g}_1) + \varphi(g_2) = \tilde{g}_1(z_0, w_0) + g_2(z_0, w_0)$ and $(z_0, w_0, \zeta_0)$ is contained in $G(\tilde{g}_1 + g_2; \overline{\mathbb{D}^2})$. 

\[\square\]

3. Proof of Theorem. @ We write $I = \{1, 2, \cdots, t\}$,

$$E_i = \{(z, w) \in R_i \setminus (T^2 \cup L): \frac{\partial q_i}{\partial z}(z, w) = 0, \text{ or } \frac{\partial q_i}{\partial w}(z, w) = 0\},$$

$$F_i = \bigcup_{j \in I \setminus \{i\}} (R_i \cap R_j) \setminus (T^2 \cup L),$$

$$R_i^* = R_i \setminus (T^2 \cup L \cup E_i \cup F_i),$$

$$\Sigma = \bigcup_{i \in I} R_i \setminus (T^2 \cup L).$$

It is known that the sets $E_i$ and $R_i \cap R_j$ ($i \neq j$) are finite at most, respectively, and $Z(q_i) \setminus (E_i \cup F_i)$ is a connected set in $\mathbb{C}^2$.

Step I. $\widehat{G(f)} \setminus G(\bar{\rho}; T^2 \cup L) \subset G(\bar{\rho}; \Sigma \cap V)$.

Proof. Let $\zeta$ be the third coordinate of $\mathbb{C}^3$. By Lemma 2.5 we have that

$$\widehat{G(f)} \subset \{(z, w, \zeta): (z, w) \in \overline{\mathbb{D}^2}, \zeta = p(z, w)\},$$

and by the definition of $k(z, w)$

$$\widehat{G(f)} \subset \{(z, w, \zeta): (z, w) \in \overline{\mathbb{D}^2}, |\zeta| \leq \|p\|_{T^2}, z^m w^n \zeta - k(z, w) = 0\}.$$

Hence we have $\widehat{G(f)} \setminus G(\bar{\rho}; T^2 \cup L) \subset G(\bar{\rho}; V)$. If a point $(z^0, w^0) \in V \setminus \Sigma$, then $\Delta(z^0, w^0) \neq 0$. By Lemma 2.2 there is a ball $B_0$ centered at $(z^0, w^0)$ such that $B_0 \cap (T^2 \cup L) = \emptyset$ and $B_0 \cap V$ is a totally real submanifold of $B_0$. Thus the graph $G(\bar{\rho}; B_0 \cap V)$ is also totally real and $(B_0 \times \mathbb{C}) \cap G(f) = \emptyset$. It follows from Lemma 2.3 that

$$G(\bar{\rho}; B_0 \cap V) \cap \widehat{G(f)} = \emptyset,$$

and so

$$G(\bar{\rho}; V \setminus \Sigma) \cap \widehat{G(f)} = \emptyset,$$

which proves Step I.

Note. It is sufficient to investigate $G(\bar{\rho}; V \setminus \Sigma)$, since the graph $\widehat{G(f)}$ is connected and $\widehat{G(f)} \subset G(\bar{\rho}; T^2) \cup G(\bar{\rho}; V \setminus \Sigma) \cup G(\bar{\rho}; L)$.
Assume that for some $i \in I$, $V \cap R_i^* \neq \emptyset$. For a point $(z^0, w^0)$ in $V \cap R_i^*$, there exist a neighborhood $U_0$ of $(z^0, w^0)$ in $R_i^*$ and holomorphic functions $\varphi(\lambda)$ and $\psi(\lambda)$ on $\mathbb{D}$ such that $(z^0, w^0) = (\varphi(0), \psi(0))$ and
$$U_0 = \{((\varphi(\lambda), \psi(\lambda)) : \lambda \in \mathbb{D}\}.$$  

Step II. The case that $\varphi(\lambda)$ and $\psi(\lambda)$ satisfy the condition
$$\frac{p(\varphi(\lambda), \psi(\lambda))}{h(\varphi(\lambda), \psi(\lambda))} \equiv 0 \text{ on } \mathbb{D}. \quad (1)$$
In this case, $q_i(z, w)$ is a common factor of $p(z, w) - p(z^0, w^0)$ and $k(z, w) - z^m w^n p(z^0, w^0)$, and so
$$R_i \setminus (T^2 \cup L) \subset V. \quad (2)$$

Proof. We obtain the power series on $\mathbb{D}$
$$p(\varphi(\lambda), \psi(\lambda)) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots,$$
$$h(\varphi(\lambda), \psi(\lambda)) = b_0 + b_1 \lambda + b_2 \lambda^2 + \cdots.$$ 
It follows from the assumption that for every polynomial $q(\lambda)$
$$0 = \int_{|\lambda|=1} \{p(\varphi(\lambda), \psi(\lambda)) - h((\varphi(\lambda), \psi(\lambda)))\} q(\lambda) d\lambda$$
$$= \int_{|\lambda|=1} \{\bar{a}_0 + \bar{a}_1 \lambda + \bar{a}_2 \lambda^2 + \cdots - b_0\} q(\lambda) d\lambda.$$ 
Thus $\bar{a}_1 = \bar{a}_2 = \cdots = 0$, $\bar{a}_0 = \bar{p}(z^0, w^0) = b_0$ and $\bar{a}_0 - h(\varphi(\lambda), \psi(\lambda)) \equiv 0$ on $\mathbb{D}$. Since $a_0$ depends on $q_i$, we put $c_i = a_0$. Then we can write that
$$k(z, w) - \bar{c}_i z^m w^n = q_i(z, w) k_i(z, w),$$
$$\overline{p(z, w)} - \bar{c}_i = q_i(z, w) \bar{p}_i(z, w)$$
for some polynomials $p_i(z, w)$ and $k_i(z, w)$. Thus (2) follows.

Step III. The case that (1) does not holds, i.e.,
$$\frac{p(\varphi(\lambda), \psi(\lambda))}{h(\varphi(\lambda), \psi(\lambda))} \not\equiv 0 \text{ on } \mathbb{D}. \quad (3)$$
In this case, we have
$$\widehat{G(f)} \setminus G(\bar{p}; T^2 \cup L) \subset G(\bar{p}; \Sigma_i \cap V) \quad (4)$$
where $\Sigma_i = \bigcup_{j \in I \setminus \{i\}} R_j \setminus (T^2 \cup L)$.

To show this we consider the condition (3) from two viewpoints of (5), (6) of Step IV and V.

Step IV. If
$$\frac{p(\varphi(\lambda), \psi(\lambda))}{h(\varphi(\lambda), \psi(\lambda))} \equiv 0 \text{ on } \mathbb{D}, \quad (5)$$
then we have $G(\bar{p}; (V \setminus R_i) \setminus \Sigma_i) \cap \widehat{G(f)} = \emptyset$.

Proof. Since $q_i(z, w)$ is an irreducible polynomial, it is a factor of $p(z, w) - c_i$. Thus $p(z, w) - c_i \equiv 0$ on $R_i$ and $\overline{c_i} - h(z, w) \not\equiv 0$ on $\overline{\mathbb{D}} \setminus (T^2 \cup L)$. Thus the set
$$V \cap R_i = \{(z, w) \in \overline{\mathbb{D}} \setminus (T^2 \cup L) : \overline{c_i} - h(z, w) = 0, q_i(z, w) = 0\}$$
is finite. Thus $G(\bar{p}; V \cap R_i)$ is the set of isolated points. Since $\widehat{G(f)}$ does not contain any isolated points, we have $G(\bar{p}; V \cap R_i \setminus \Sigma_i) \cap \widehat{G(f)} = \emptyset$, which proves (5).
Step V. Now let \((z^0, w^0) \in R^*_i\). Assume that
\[
\overline{p(\varphi(\lambda), \psi(\lambda))} - \overline{p(z^0, w^0)} \neq 0 \text{ on } \mathbb{D}.
\] (6)
We can assume that \(\varphi(\lambda) = z_0 + \lambda \text{ in } \rho \mathbb{D}\) for some positive \(\rho \mathbb{D}\). We put
\[
W_0 = \{(\varphi(\lambda), \psi(\lambda)) : \lambda \in \rho \mathbb{D}\}
\]
and
\[
W_0^* = \{(z_0 + \lambda, \psi(\lambda)) : \lambda \in \rho \mathbb{D}, \frac{\partial p}{\partial z}(\varphi(\lambda), \psi(\lambda)) + \frac{\partial p}{\partial w}(\varphi(\lambda), \psi(\lambda)) \frac{d\psi(\lambda)}{d\lambda} \neq 0\}.
\]
Step VI. If (6) holds, then \(G(\bar{p}; W_0^*)\) is totally real, and so
\[
G(\bar{p}; R_i \setminus (T^2 \cup L \cup \Sigma_i)) \cap \hat{G}(f) = \emptyset.
\] (7)
Proof. We put \(\lambda = x + iy\) and \(p = u + iv\) \((x, y, u, v \text{ real})\). The real tangent vectors at
\((z_0 + \lambda, \psi(\lambda), p(z_0 + \lambda, \psi(\lambda)))\) to \(G(\bar{p}; W_0)\) for \(\begin{bmatrix} \partial & \partial \end{bmatrix}_{\bar{\phi}} \) as follows.
\[
v_1 = (1, 0, \frac{\partial \text{Re}(\phi)}{\partial x}, \frac{\partial \text{Im}(\phi)}{\partial x}, \frac{\partial u}{\partial x}, -\frac{\partial v}{\partial x}),
\]
\[
v_2 = (0, 1, \frac{\partial \text{Re}(\phi)}{\partial y}, \frac{\partial \text{Im}(\phi)}{\partial y}, \frac{\partial u}{\partial y}, -\frac{\partial v}{\partial y}).
\]
The rank of the matrix defined by components of \(v_1, v_2, iv_1, iv_2\) is 4, since
\[
\begin{vmatrix}
1 & 0 & u_x & -v_x \\
0 & 1 & u_y & -v_y \\
0 & 1 & v_x & u_x \\
-1 & 0 & v_y & u_y
\end{vmatrix}
= -4(u_x^2 + v_x^2) = -4 \left| \frac{dp}{d\lambda} \right|^2.
\]
Thus \(G(\bar{p}; W_0^*)\) is a totally real manifold. It follows from Lemma 2.3 that
\[
G(\bar{p}; W_0^* \setminus \Sigma_i) \cap G(\hat{f}) = \emptyset.
\]
Since \(W_0 \setminus W_0^*\) is a set of isolated points, by connectivity of \(\hat{G}(f)\) we have
\[
G(\bar{p}; W_0 \setminus (W_0^* \cup \Sigma_i)) \cap G(\hat{f}) = \emptyset.
\]
When points \((z_0, w_0)\) run in \(R^*_i\), the corresponding neighborhoods \(U_0\) cover \(R^*_i\). Thus
\[
G(\bar{p}; R^*_i \setminus (\Sigma_i \cup T^2 \cup L) \cap \hat{G}(f) = \emptyset. \text{ Since the set } G(\bar{p}; R_i \setminus (R^*_i \cup T^2 \cup L) \text{ is finite, we have }
\]
\[
G(\bar{p}; R_i \setminus (R^*_i \cup \Sigma_i \cup T^2 \cup L)) \cap \hat{G}(f) = \emptyset, \text{ and the assertion (7) is proved. From (5) and (7) we obtain (4) of Step III.}
\]
By the above facts we obtain the following:
Step VII. If we put
\[
I_0 = \{i \in \{1, 2, \cdots, t\} : \emptyset \neq R_i \setminus (T^2 \cup L) \subset V\},
\]
then
\[
\hat{G}(f) \setminus G(\bar{p}; T^2 \cup L) \subset G(\bar{p}; \cup_{i \in I_0} R_i \cap V).
\]
For \(i \in I_0\), we consider the following cases:

(i). \(Q_i = \emptyset, R_i \neq \emptyset\). 
(ii). \(\emptyset \neq Q_i = \hat{Q}_i \neq R_i\).
(iii). \(\emptyset \neq Q_i \neq \hat{Q}_i = R_i\). 
(iv). \(\emptyset \neq Q_i \neq \hat{Q}_i \neq R_i\).
Step VIII. Assume that (ii) holds for \( i \in I_0 \), then

\[
G(\bar{p}; R_i \setminus (T^2 \cup L \cup \Lambda_i)) \cap \widehat{G(f)} = \emptyset,
\]

where \( \Lambda_i = \bigcup_{j \in I_0 \setminus \{i\}} R_j \).

Proof. We denote \( m_i \) by the maximal order of an irreducible factor \( q_i(z, w) \) in \( p(z, w) \), and we define a polynomial \( p_1(z, w) \) by

\[
p(z, w) - c_i = p_1(z, w)q_i(z, w)^{m_i}.
\]

By using \( p_1(z, w) \) we put \( K = \{(z, w) \in T^2 : p_1(z, w) = 0\} \). For a point \((z^0, w^0) \in R_i \setminus (K \cup T^2 \cup L)\), we put

\[
p_2(z, w) = \frac{1}{p_1(z^0, w^0)}p_1(z, w).
\]

Since \( Q_i \) and \( \{(z^0, w^0)\} \) are disjoint polynomially convex sets, there exist a polynomial \( p_0(z, w) \), a neighborhood \( U \) of \( Q_i \), and a neighborhood \( W \) of \( K \) in \( T^2 \) such that

\[
p_0(z^0, w^0) = 1, \quad \|p_0(z, w)p_2(z, w)\| < \frac{1}{2} \quad \text{on} \quad U,
\]

\[
|p_0(z, w)p_2(z, w)| < \frac{1}{2} \quad \text{on} \quad W.
\]

If we put \( M = \|p - c_i\|_{T^2} \), \( K_1 = \{(z, w) \in T^2 : p(z, w) - c_i = 0\} \), and put

\[
g_1(z, w, \zeta) = 1 - \frac{1}{2M^2}(\zeta - \bar{c}_i)(p(z, w) - c_i),
\]

then we have

\[
g_1(z, w, \zeta) = 1 \quad \text{on} \quad G(\bar{p}; K_1).
\]

Since \( |g_1| < 1 \) on \( G(\bar{p}; T^2 \setminus (U \cup W)) \), there exists a positive integer \( k \) such that

\[
|p_2(z, w)p_0(z, w)g_1(z, w, \zeta)^k| < \frac{1}{2} \quad \text{on} \quad G(\bar{p}; T^2 \setminus (U \cup W)).
\]

If we put \( g(z, w, \zeta) = p_2(z, w)p_0(z, w)g_1(z, w, \zeta)^k \), then

\[
|g(z, w, \zeta)| < \frac{1}{2} \quad \text{on} \quad G(f), \quad \text{and} \quad g(z^0, w^0, p(z^0, w^0)) = 1.
\]

Thus \((z^0, w^0, p(z^0, w^0)) \notin \widehat{G(f)} \) and so \( G(\bar{p}; R_i \setminus (K \cup T^2 \cup L)) \cap \widehat{G(f)} = \emptyset \). Since a set \((R_i \setminus K) \setminus (T^2 \cup L)\) is finite, by connectivity of \( G(f) \) we have

\[
G(\bar{p}; R_i \setminus (\Lambda_i \cup T^2 \cup L)) \cap \widehat{G(f)} = \emptyset,
\]

which proves (8).

In the case (i), if we choose a point \((z^*, w^*) \in T^2 \setminus \Lambda_i \), and put \( Q_i = \{(z^*, w^*)\} \), then we similarly obtain the proof of (i).

Step IX. Assume the (iii) holds, then

\[
G(\bar{p}; R_i) \subset \widehat{G(f)}.
\]

Proof. Since \( G(\bar{p}; Q_i) \subset G(f) = G(\bar{p}; T^2) \) and \( G(\bar{p}; Q_i) \subset \{(z, w, \zeta) \in C^3 : \zeta = c_i\} \), then we obtain (9).

Step X. Assume that (iv) holds. Then we have

\[
G(\bar{p}; R_i \setminus (L \cup T^2 \cup \hat{Q}_i)) \cap \widehat{G(f)} = \emptyset.
\]

Proof. Let \((z^0, w^0) \) be a point of \( R_i \setminus (L \cup T^2 \cup \hat{Q}_i) \). If \( Q_i \) in (ii) is replaced by \( \hat{Q}_i \), we similarly have (10).
5. Examples.

Example 5.1. If \( p(z, w) = \{(z + 1) - (w + 1)^2\}\{(z + 1)w^2 - z(w + 1)^2\} \) and \( f = \bar{p}|_{T^2} \), then \( h(z, w) = \frac{1}{z^2w^4}p(z, w) \) and
\[
\Delta(z, w) = \frac{2p(z, w)}{z^3w^5}g(z, w)
\]
where
\[
g(z, w) = wp_z(z, w) - 2zp_z(z, w)
\]
\[
= 2[(w + 1)z^2 + w^2(2w + 3)z - w^3(2w + 3)].
\]
The polynomial \( g(z, w) \) is irreducible. The sets defined by the section 1 are as follows:
\[
Q_1 = \{(z, w) \in T^2 : z - w^2 - 2w = 0\} = \{(-1, -1)\} = \hat{Q}_1.
\]
\[
R_1 = \{(z, w) \in \mathbb{D}^2 : z - w^2 - 2w = 0\}.
\]
\[
Q_2 = \{(z, w) \in T^2 : w^2 - z - 2zw = 0\} = \{(1, -1)\} = \hat{Q}_2.
\]
\[
R_2 = \{(z, w) \in \mathbb{D}^2 : w^2 - z - 2zw = 0\}.
\]
Then we have that \( R_j \setminus (T^2 \cup L) \subset V \) and \( \emptyset \neq Q_j = \hat{Q}_j \neq R_j, \ j = 1, 2 \), Since \( g(z, w) \) and \( p(z, w) - c \) for every \( c \in C \) are relatively prime polynomials. Thus \( R_3 \setminus (T^2 \cup L) \) is not contained in \( V \). Since \( I_0 = \{1, 2\} \) and \( J = \emptyset \), by the theorem we have
\[
\widehat{G(f)} = G(f).
\]

Example 5.2. If \( p(z, w) = (z + w)(w + 2)(2w + 1) \) and \( f = \bar{p}|_{T^2} \), then we have that \( h(z, w) = \frac{1}{z^3w^4}(z + w)(w + 2)(2w + 1) \) and
\[
\Delta(z, w) = \frac{2}{z^3w^4}(z + w)(w + 2)(2w + 1)g(z, w)
\]
where \( g(z, w) = -z(w^2 + 5w + 3) + w(3w^2 + 5w + 1) \). Since the polynomial \( g(z, w) \) is irreducible, the sets \( \{(z, w) \in \mathbb{D}^2 \setminus (T^2 \cup L) : z + w = 0\} \) and \( \{(z, w) \in \mathbb{D}^2 \setminus (T^2 \cup L) : 2w + 1 = 0\} \) are contained in \( V \), it follows from the theorem that
\[
\widehat{G(f)} = G(f) \cup \{(z, w, 0) \in \mathbb{D}^2 : z + w = 0\}.
\]

Example 5.3. ([5]). Let \( p(z, w) \) be a homogeneous polynomial:
\[
P(z, w) = cz^mw^n(z^k + a_1z^{k-1}w + a_2z^{k-2}w^2 + \cdots + a_kw^k)(a_k \neq 0)
\]
\[
= c(z - \lambda_1w)(z - \lambda_2w)\cdots(z - \lambda_kw)zw^n
\]
where \( k \) is a positive integer, \( m \) and \( n \) are nonnegative integers, and \( c, \lambda_1, \lambda_2, \cdots, \lambda_k \) are some constants with \( c\lambda_1\lambda_2\cdots\lambda_k \neq 0 \). We put
\[
J = \{j \in \{1, 2, \cdots, k\} : |\lambda_j| = 1\}.
\]

1. If \( J \neq \emptyset \), then \( \widehat{G(f)} = \bigcup_{j \in J} \{(z, w, 0) : z - \lambda_jw = 0, w \in D\} \cup G(f) \).
2. If \( J = \emptyset \), then \( G(f) = G(f) \), and moreover \( [z, w, f; T^2] = C(T^2) \).

\[\text{POLYNOMIAL HULLS OF GRAPHS ON THE TORUS 355}\]
Example 5.4. If \( p(z, w) = (z^2 - 1)w + z \) and \( f = \bar{p}|_{T^2} \), then \( h(z, w) = \frac{(1-z^2)+zw}{z^2w} \) and
\[
\Delta(z, w) = 1 - z^3w^2(z^2 - 1)g(z, w)
\]
where \( g(z, w) = zw^2 + 2(z^2 + 1)w + z \). We have that \( z - 1 \) is a factor of \( p(z, w) - 1 \) and \( z + 1 \) is a factor of \( p(z, w) + 1 \) and \( g(z, w) \) is an irreducible polynomial. Thus
\[
\hat{G}(f) = G(f) \cup \{(1, w, 1) : w \in \mathbb{D}\} \cup \{(-1, w, -1) : w \in \mathbb{D}\}.
\]

Acknowledgements
The author wishes to thank Professor Akira Sakai for variable suggestions for improving the manuscript.

References

Department of Mathematics, Nara University of Education, Takabatake, Nara 630-8528, Japan
E-mail address: jimbo@nara-edu.ac.jp