NEW DERIVATION OF CONSERVED QUANTITIES FOR HIGHER ORDER DIFFERENTIAL SYSTEM

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ABSTRACT. Through the application a suitable version of Noether’s theorem to the composite variational principle, a new method was investigated for the derivation of conserved quantities for second order differential system. In this article, the variables provided for higher order differential system are arranged to convert the system into the second order one. And then the method is applied effectively to construct conserved quantities of the higher order differential system.

Introduction. Noether’s theorem (Noether [7]) has been extensively initiated for the derivation of conserved quantities based on the symmetries in the Lagrangian or the Hamiltonian structures. However, without using the structures (which may fail to exist), Caviglia ([2], [3]) determined the new operative procedure for the quantities via the application of a suitable version of Noether’s theorem to the composite variational principle. The procedure was analyzed by Mimura and Nôno [6] with various viewpoints for the derivation of the quantities of a given second order differential system. Following Sarlet and Cantrijn [8], Mimura, Ikeda and Fujiwara [5] introduced some geometric notions associating with the equation field of the differential system to construct the quantities. The local version of the result in [6] was reformulated in [5] with the geometric notions.

In this paper, we give a further derivation of conserved quantities of higher order differential system by virtue of the method in [6], while some results in [6] were translated by Crăsmăreanu [4] for higher order one with the extended notion of adjoint for a linear operator. Arranging the original variables in higher order differential system, the system is converted into a second order one to apply the method (Remark 2 of Theorem 1 in [6]). The result in the consideration is carried into the case of higher order differential system, and one arrives at the theorem 1. Some postulations are imposed on unknown functions which give rise to the conserved quantities. Then the theorem 1 deduces to the theorem 2. In the context of the deduction, it can be observed that the theorem 2 is equivalent to the result of Crăsmăreanu [4]. Moreover the theorem 2 yields the theorem 3 by imposing an arbitrary degree of homogeneity on the higher order differential system. As illustrations for two types of third order differential equations, conserved quantities are constructed through which the solutions of the equations are determined completely.

1 Conserved quantity for higher order differential system. We set our starting point for a given k-th order differential system

\[ F^A(t, x, \dot{x}, \cdots, x^{(k)}) = 0 \quad (A = 1, \cdots, n; \ k \geq 2), \]

where \( x = (x^\sigma(t)), \dot{x} = (\dot{x}^\sigma(t)) = (dx^\sigma/dt), x^{(k)} = (d^k x^\sigma/dt^k), (\sigma = 1, \cdots, m; \ m \geq n) \). A conserved quantity of the system (1) is a quantity \( \Omega(t, x, \dot{x}, \cdots, x^{(k-1)}) \) satisfying \( d\Omega/dt = 0 \)

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on solutions to (1), where \(d/dt\) denotes the total differentiation with respect to \(t\):

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \ddot{x} \frac{\partial}{\partial x} + \cdots + x^{(k+1)} \frac{\partial}{\partial x^{(k)}},
\]

where \(\dot{x} \partial/\partial x = \sum_{\sigma=1}^{m} \dot{x}^{\sigma} \partial/\partial x^{\sigma}, \ddot{x} \partial/\ddot{x} = \sum_{\sigma=1}^{m} \ddot{x}^{\sigma} \partial/\ddot{x}^{\sigma}\) and so on. Particularly for a second order differential system, new derivation of conserved quantity has been given in ([6], Theorem 1 and its Remark 2). So by putting (in what follows the bracket \([ \ ]\) denotes Gauss's symbol)

\[
\begin{align*}
\frac{d^{2p} x^{\sigma}}{dt^{2p}} &= y^{\sigma}_{p} \quad (\sigma = 1, \cdots, m; \ p = 1, \cdots, \left[ \frac{k}{2} \right]),
\end{align*}
\]

the \(k\)-th order system (1) is converted into a second order one:

\[
\begin{align*}
F^{A}(t, y_{0}, y_{1}, \cdots, y_{\left[ \frac{k}{2} \right]}, \dot{y}_{0}, \dot{y}_{1}, \cdots, \dot{y}_{\left[ \frac{k}{2} \right]-1}) &= 0,
\end{align*}
\]

where \(y_{0} = (y_{0}^{\sigma}(t)) = (x^{\sigma}(t)), y_{p} = (y_{p}^{\sigma}(t)) \ (p = 1, \cdots, \left[ \frac{k}{2} \right]), y_{0} = (\dot{y}_{0}^{\sigma}(t)) = (\dot{x}^{\sigma}(t)), \dot{y}_{r} = (\dot{y}_{r}^{\sigma}(t)) \ (r = 1, \cdots, \left[ \frac{k-1}{2} \right]).\) Then by regarding the variables \(y = (y_{0}, y_{1}, \cdots, y_{\left[ \frac{k}{2} \right]-1})\) as \(q = (q^{\sigma}), (\kappa = 1, \cdots, \left( \left[ \frac{k}{2} \right] + 1 \right)n),\) the method in ([6], Remark 2 of Theorem 1) is applied to (3) to obtain the following theorem.

**Theorem 1** Let \(\mu = (\mu_{0}^{\sigma}, \mu_{1}^{\sigma}, \cdots, \mu_{\left[ \frac{k}{2} \right]}^{\sigma})\) and \(\xi = (\xi_{0}^{\sigma}, \xi_{1}^{\sigma}, \cdots, \xi_{\left[ \frac{k}{2} \right]}^{\sigma})\) be functions of \(t, y_{0}, y_{1}, \cdots, y_{\left[ \frac{k}{2} \right]-1}, \dot{y}_{0}, \dot{y}_{1}, \cdots, \dot{y}_{\left[ \frac{k}{2} \right]-1}\) satisfying the following system of equations on solutions to (3):

\[
\begin{align*}
\dot{\mu}_{\sigma} + \sum_{A=1}^{n} \mu_{A}^{\sigma} \frac{\partial F^{A}}{\partial y_{0}^{\sigma}} - d \left( \sum_{A=1}^{n} \mu_{A}^{\sigma} \frac{\partial F^{A}}{\partial y_{0}^{\sigma}} \right) &= 0 \quad (\sigma = 1, \cdots, m),
\end{align*}
\]

\[
\begin{align*}
\mu_{\left[ \frac{k}{2} \right]}^{\sigma} &= \sum_{A=1}^{n} \mu_{A}^{\sigma} \frac{\partial F^{A}}{\partial y_{\left[ \frac{k}{2} \right]}^{\sigma}} - d \left( \sum_{A=1}^{n} \mu_{A}^{\sigma} \frac{\partial F^{A}}{\partial y_{\left[ \frac{k}{2} \right]}^{\sigma}} \right) \quad (\sigma = 1, \cdots, m),
\end{align*}
\]

\[
\begin{align*}
\frac{n}{A=1} \sum_{\sigma=1}^{m} \mu_{A}^{\sigma} \left( \sum_{\ell=0}^{\left[ \frac{k}{2} \right]} \frac{\partial F^{A}}{\partial \dot{y}_{\ell}^{\sigma}} \xi_{\ell}^{\sigma} + \sum_{\ell=0}^{\left[ \frac{k-1}{2} \right]} \frac{\partial F^{A}}{\partial \dot{y}_{\ell}^{\sigma}} \ddot{\xi}_{\ell}^{\sigma} \right) + \sum_{\sigma=1}^{m} \sum_{\ell=1}^{\left[ \frac{k}{2} \right]} \mu_{\sigma}^{\ell} (\ddot{\xi}_{\ell-1}^{\sigma} - \xi_{\ell}^{\sigma}) = \frac{dK}{dt},
\end{align*}
\]

where \(K\) is a function of \(t, y_{0}, y_{1}, \cdots, y_{\left[ \frac{k}{2} \right]-1}, \dot{y}_{0}, \dot{y}_{1}, \cdots, \dot{y}_{\left[ \frac{k}{2} \right]-1}\); and if \(k \geq 4\), moreover

\[
\begin{align*}
\mu_{\sigma}^{\ell} &= \dot{\mu}_{\sigma}^{\ell+1} + \sum_{A=1}^{n} \mu_{A}^{\sigma} \frac{\partial F^{A}}{\partial y_{\ell}^{\sigma}} - d \left( \sum_{A=1}^{n} \mu_{A}^{\sigma} \frac{\partial F^{A}}{\partial y_{\ell}^{\sigma}} \right) \quad (\sigma = 1, \cdots, m; \ \ell = 1, \cdots, \left[ \frac{k}{2} \right] - 1).
\end{align*}
\]
Then, by using of the above $\mu$ and $\xi$, the following quantity $\Omega$ satisfying $\dot{\Omega} = 0$ on solutions to (3) is constructed:

$$
\Omega = \sum_{\sigma=1}^{m} \sum_{i=1}^{[\ell]} (\mu_{\sigma}^{\ell} \xi_{\ell-1}^{\sigma} - \mu_{\sigma}^{\ell} \xi_{\ell-1}^{\sigma}) + \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \sum_{i=0}^{[k-1]} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \xi_{\ell}^{\sigma} - K,
$$

which gives rise to a conserved quantity of (1) by denoting $\mu_{\sigma}^{\ell}$, $\xi_{\ell}^{\sigma}$, $F^A$ and $K$ as functions of the original variables $t$, $x$, $\dot{x}$, $\ldots$, $x^{(k-1)}$.

**Remark 1** Here note that $\left[\frac{k-1}{2}\right]$ is equal to $\left[\frac{k}{2}\right] - 1$ ($k$: even), or $\left[\frac{k}{2}\right]$ ($k$: odd). Accordingly, if $k$ is even, $F^A$ in (3) does not have the variables $y_{\ell}^{\sigma}$, i.e., $\partial F^A / \partial y_{\ell}^{\sigma} = 0$. So that the term $\sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}}$ may be deleted in the appearance of (5).

When $k \geq 4$, at first, (5) is substituted for $\dot{\mu}_{\sigma}^{\ell}$ in (7) with $\ell = \left[\frac{k}{2}\right] - 1$ to see

$$
\dot{\mu}_{\sigma}^{\left[\frac{k}{2}\right]-1} = \mu_{\sigma}^{\left[\frac{k}{2}\right]} + \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right)
$$

$$
= \frac{d^2}{dt^2} \left( \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right) \right)
$$

$$
+ \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right)
$$

$$
= \sum_{A=1}^{n} \sum_{i=0}^{1} \frac{d^{2(1-i)}}{dt^{2(1-i)}} \left( \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right) \right),
$$

where $\frac{d^{0}F}{dt^{0}}$ denotes that $d^{0}F/dt^{0} = F$ for an arbitrary function $F$. And then the appearance of $\mu_{\sigma}^{\left[\frac{k}{2}\right]-1}$ is substituted for $\dot{\mu}_{\sigma}^{\left[\frac{k}{2}\right]-1}$ in (7) with $\ell = \left[\frac{k}{2}\right] - 2$ to see

$$
\dot{\mu}_{\sigma}^{\left[\frac{k}{2}\right]-2} = \mu_{\sigma}^{\left[\frac{k}{2}\right]-1} + \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right)
$$

$$
= \sum_{A=1}^{n} \sum_{i=0}^{1} \frac{d^{2(1-i)+2}}{dt^{2(1-i)+2}} \left( \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right) \right)
$$

$$
+ \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \sum_{A=1}^{n} \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right)
$$

$$
= \sum_{A=1}^{n} \sum_{i=0}^{2} \frac{d^{2(2-i)}}{dt^{2(2-i)}} \left( \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} - \frac{d}{dt} \left( \mu_{A}^{0} \frac{\partial F^A}{\partial y_{\ell}^{\sigma}} \right) \right),
$$
Therefore, the functions

\begin{align}
\mu_\kappa^\ell &= \sum_{A=1}^{[\frac{k}{2}]-\ell} \sum_{i=0}^{[\frac{k}{2}] - \ell} \left( \frac{d^2(\frac{k}{2} - \ell - i)}{dt^2(\frac{k}{2} - \ell - i)} \left( \mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{\frac{k}{2} - i}} \right) - \frac{d^2(\frac{k}{2} - \ell - i) + 1}{dt^2(\frac{k}{2} - \ell - i) + 1} \left( \mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{\frac{k}{2} - i}} \right) \right) \\
(\sigma = 1, \ldots, m; \; \ell = 1, \ldots, \left[\frac{k}{2}\right]; \; k \geq 4).
\end{align}

For \( k \geq 4 \) and \( \ell = 1 \), (9) is written as

\begin{align}
\mu_\kappa^1 &= \sum_{A=1}^{[\frac{k}{2}]-1} \sum_{i=0}^{[\frac{k}{2}] - 1} \left( \frac{d^2(\frac{k}{2} - i - 1)}{dt^2(\frac{k}{2} - i - 1)} \left( \mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{\frac{k}{2} - i}} \right) - \frac{d^2(\frac{k}{2} - i) - 1}{dt^2(\frac{k}{2} - i) - 1} \left( \mu_A^0 \frac{\partial F^A}{\partial y^\sigma_{\frac{k}{2} - i}} \right) \right) \\
&= 0 \quad (\sigma = 1, \ldots, m; \; k \geq 2).
\end{align}

**Remark 2** By putting \( y_0^\sigma = x^\sigma \) and \( y_1^\sigma = \ddot{x}^\sigma \), a second order differential system (1) with \( k = 2 \):

\[ F^A(t, x, \dot{x}, \ddot{x}) = 0 \]

is converted into

\begin{align}
(3)' \quad \begin{cases}
F^A(t, y_0, y_1, y_0) = 0 \\
y_0^\sigma - y_1^\sigma = 0 \\
(A = 1, \ldots, n; \; \sigma = 1, \ldots, m).
\end{cases}
\end{align}

In this case, (11) reduces to

\begin{align}
(11)' \quad \sum_{A=1}^{n} \left( \mu_A^0 \frac{\partial F^A}{\partial y_0^\sigma} - \frac{d}{dt} \left( \mu_A^0 \frac{\partial F^A}{\partial y_0^\sigma} \right) + \frac{d^2}{dt^2} \left( \mu_A^0 \frac{\partial F^A}{\partial y_1^\sigma} \right) \right) = 0.
\end{align}

And (10) (or (5) with \( k = 2 \)) is substituted for (6) with \( k = 2 \) to see

\begin{align}
(12) \quad \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \mu_A^0 \left( \frac{\partial F^A}{\partial y_0^\sigma} \dot{x}_0^\sigma + \frac{\partial F^A}{\partial y_0^\sigma} \ddot{x}_0^\sigma + \frac{\partial F^A}{\partial y_1^\sigma} \dot{x}_1^\sigma \right) = \frac{dK}{dt}.
\end{align}

Therefore, the functions \( \mu_A^0(t, y_0, y_0) \) and \( \xi_0^\sigma(t, y_0, y_0) \) satisfying (11)' and (12) on solutions to (3)' yields the following quantity \( \Omega \) satisfying \( \Omega = 0 \) on solutions to (3)'

\begin{align}
(8)' \quad \Omega = \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \left( \mu_A^0 \frac{\partial F^A}{\partial y_0^\sigma} \xi_0^\sigma + \left( \mu_A^0 \frac{\partial F^A}{\partial y_0^\sigma} \frac{d}{dt} \left( \mu_A^0 \frac{\partial F^A}{\partial y_1^\sigma} \right) \right) \xi_0^\sigma \right) = K(t, y_0, y_0).
\end{align}

By denoting \( y_0^\sigma, \dot{y}_0^\sigma \) and \( y_1^\sigma \) as the original variables \( x^\sigma \), \( \dot{x}^\sigma \) and \( \ddot{x}^\sigma \) respectively, (8)' turns into the conserved quantities obtained in ([6], Remark 2 of Theorem 1):

\begin{align}
\Omega = \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \left( \mu_A^0 \frac{\partial F^A}{\partial x^\sigma} \xi_0^\sigma + \left( \mu_A^0 \frac{\partial F^A}{\partial x^\sigma} \frac{d}{dt} \left( \mu_A^0 \frac{\partial F^A}{\partial \ddot{x}^\sigma} \right) \right) \xi_0^\sigma \right) = K(t, x, \dot{x}).
\end{align}
For an arbitrary given \( \xi_\sigma^\ell \) in (6), particularly put \( \xi_\sigma^\ell = d^{2\ell} \xi_\sigma^0 / dt^{2\ell} \), i.e., \( \xi_\sigma^\ell = \xi_\sigma^{\ell-1} (\sigma = 1, \cdots, m; \ \ell = 1, \cdots, [\frac{1}{2}]) \). Then (6) reduces to

\[
\sum_{A=1}^n \sum_{\sigma=1}^m \mu_A^0 \left( \sum_{\ell=0}^{\frac{n}{2}} \frac{\partial F_A}{\partial y_\sigma^\ell} \frac{d^{2\ell} \xi_\sigma^0}{dt^{2\ell}} + \sum_{\ell=0}^{\frac{n}{2}+1} \frac{\partial F_A}{\partial y_\sigma^\ell} \frac{d^{2\ell+1} \xi_\sigma^0}{dt^{2\ell+1}} \right) = \frac{dK}{dt},
\]

which is just the equation (1.10b) in (Crasmareanu [4], Theorem). Therefore the result of Crasmareanu is concluded completely in the theorem 1:

**Theorem 2** Let \( \mu_A^0 \) and \( \xi_0^\ell \) be functions of \( t, y_0, y_1, \cdots, y_{\frac{n}{2}-1}, \dot{y}_0, \dot{y}_1, \cdots, \dot{y}_{\frac{n}{2}-1} \) satisfying the equations (11) and (13) on solutions to (3). Then the following quantity \( \Omega = 0 \) on solutions to (3) is constructed:

\[
\begin{align*}
\Omega &= \sum_{A=1}^n \sum_{\sigma=1}^m \sum_{\ell=1}^{\frac{n}{2}} \sum_{i=0}^{\frac{\ell}{2} - i} \frac{d^2(y_{\frac{n}{2}-\ell-i}^\ell)}{dt^2(y_{\frac{n}{2}-\ell-i}^\ell)} \left( \mu_A^0 \frac{\partial F_A}{\partial y_\sigma^\ell} \right) - \frac{d^2(y_{\frac{n}{2}-\ell-i}^\ell+1)}{dt^2(y_{\frac{n}{2}-\ell-i}^\ell+1)} \left( \mu_A^0 \frac{\partial F_A}{\partial y_\sigma^{\ell+1}} \right) \frac{d^{2\ell} \xi_\sigma^0}{dt^{2\ell}} \\
&\quad - \frac{d^2(y_{\frac{n}{2}-\ell-i}^\ell)}{dt^2(y_{\frac{n}{2}-\ell-i}^\ell)} \left( \mu_A^0 \frac{\partial F_A}{\partial y_\sigma^\ell} \right) - \frac{d^2(y_{\frac{n}{2}-\ell-i+1}^\ell+1)}{dt^2(y_{\frac{n}{2}-\ell-i+1}^\ell+1)} \left( \mu_A^0 \frac{\partial F_A}{\partial y_\sigma^{\ell+1}} \right) \frac{d^{2\ell} \xi_\sigma^0}{dt^{2\ell}} \\
&\quad + \sum_{A=1}^n \sum_{\sigma=1}^m \sum_{\ell=0}^{\frac{n}{2}-1} \mu_A^0 \frac{\partial F_A}{\partial y_\sigma^\ell} \frac{d^{2\ell} \xi_\sigma^0}{dt^{2\ell}} - K \quad (k \geq 2),
\end{align*}
\]

which gives rise to a conserved quantity of (1) by denoting \( \mu_A^0, \xi_0^\ell, F_A \) and \( K \) as functions of the original variables \( t, x, \dot{x}, \cdots, x^{(k-1)} \).

**Remark 3** If \( k \) is even, the term \( \partial F_A / \partial y_\sigma^\ell \) in (11), (13) and (14) may be deleted (see Remark 1).

**Remark 4** Here note that (13) is derived by putting \( \xi_\sigma^\ell = d^{2\ell} \xi_\sigma^0 / dt^{2\ell} \) (\( \sigma = 1, \cdots, n; \ \ell = 1, \cdots, [\frac{1}{2}] \)). But whenever \( k = 2 \), it can be directly obtained by substituting (10) for (6) without the postulation \( \xi_\sigma^\ell = \xi_0^\ell \).

Here we impose on \( F_A \) an arbitrary degree \( s \) of homogeneity with respect to \( y_\sigma^\ell \ (\sigma = 1, \cdots, n; \ \ell = 0, \cdots, [\frac{1}{2}] \) and \( \dot{y}_\sigma^\ell \ (\sigma = 1, \cdots, n; \ \ell = 0, \cdots, [\frac{1}{2}] \)) to have the identity:

\[
\sum_{\sigma=1}^m \sum_{\ell=0}^{\frac{n}{2}} \frac{\partial F_A}{\partial y_\sigma^\ell} y_\sigma^\ell + \sum_{\ell=0}^{\frac{n}{2}+1} \frac{\partial F_A}{\partial y_\sigma^\ell} \dot{y}_\sigma^\ell = sF_A,
\]

which vanishes on solutions to (3). In viewing of \( y_\sigma^p = y_{\sigma-1}^p \) in (3), since

\[
y_\sigma^p = \frac{d^4 y_{\sigma-2}^p}{dt^4} = \cdots = \frac{d^{2p} y_{\sigma}^0}{dt^{2p}},
\]

(15) guarantees that \( \xi_\sigma^0 = y_\sigma^0 \) satisfies (13) with \( K = 0 \) on solutions to (3). Therefore the theorem 2 reduces to
Theorem 3  Let $F^A$ in (3) be homogeneous function of degree $s$ with respect to $y_0, y_1, \cdots, y_{\lfloor \frac{k}{2} \rfloor}, \tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_{\lfloor \frac{k}{2} \rfloor}$. Then the function $\mu^0_A$ of $t, y_0, y_1, \cdots, y_{\lfloor \frac{k}{2} \rfloor}, \tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_{\lfloor \frac{k}{2} \rfloor}$ satisfying the equation (11) on solutions to (3) yields the following quantity $\Omega$ satisfying $\dot{\Omega} = 0$ on solutions to (3):

\[
\Omega = \sum_{A=1}^{n} \sum_{\ell=1}^{m} \left( \frac{d^2(\frac{k}{2})_{\ell-i}+1}{dt^2(\frac{k}{2})_{\ell-i}} \left( \mu^0_A \frac{\partial F^A}{\partial y_{\frac{k}{2}}^\sigma} \right) - \frac{d^2(\frac{k}{2})_{\ell-i}}{dt^2(\frac{k}{2})_{\ell-i}+1} \left( \mu^0_A \frac{\partial F^A}{\partial y_{\frac{k}{2}}^\sigma} \right) \right) \frac{d^2(t-1)y_0^\sigma}{dt^2-1} \]

\[
- \sum_{A=1}^{n} \sum_{\ell=1}^{m} \sum_{i=0}^{n-m} \left( \frac{d^2(\frac{k}{2})_{\ell-i}}{dt^2(\frac{k}{2})_{\ell-i}+1} \left( \mu^0_A \frac{\partial F^A}{\partial y_{\frac{k}{2}}^\sigma} \right) - \frac{d^2(\frac{k}{2})_{\ell-i+1}}{dt^2(\frac{k}{2})_{\ell-i+1}+1} \left( \mu^0_A \frac{\partial F^A}{\partial y_{\frac{k}{2}}^\sigma} \right) \right) \frac{d^2(t-1)y_0^\sigma}{dt^2-1} \]

\[
+ \sum_{A=1}^{n} \sum_{\ell=1}^{m} \sum_{i=0}^{n-m} \mu^0_A \frac{\partial F^A}{\partial y_{\frac{k}{2}}^\sigma} \frac{d^2y_0^\sigma}{dt^2} \quad (k \geq 2),
\]

which gives rise to a conserved quantity of (1) by denoting $\mu^0_A$ and $F^A$ as functions of the original variables $t, x, \tilde{x}, \cdots, x^{(k-1)}$.

2 A reduction to a third order differential system. Particularly consider third order differential system

\[
(1)'' \quad F^A(t, x, \dot{x}, \ddot{x}) = 0 \quad (A = 1, \cdots, n).
\]

By putting $x^\sigma = y_0^\sigma$ and $\ddot{x}^\sigma = y_1^\sigma$ $(\sigma = 1, \cdots, m)$, the system (1)'' can be converted into a second order one:

\[
(3)'' \quad \begin{cases} 
F^A(t, y_0, y_1, \tilde{y}_0, \tilde{y}_1) = 0 \\
y_0'' - y_1'' = 0
\end{cases} \quad (A = 1, \cdots, n; \sigma = 1, \cdots, m).
\]

Then the theorem 3 reduces to

**Corollary.** Let $F^A$ in (3)'' be homogeneous function of degree $s$ with respect to $y_0, y_1, \tilde{y}_0, \tilde{y}_1$. Then the function $\mu^0_A$ of $t, y_0, y_1, \tilde{y}_0$ satisfying the equation

\[
(11)'' \quad \sum_{A=1}^{n} \left( \frac{d^3}{dt^3} \left( \mu^0_A \frac{\partial F^A}{\partial y_0^\sigma} \right) - \frac{d^2}{dt^2} \left( \mu^0_A \frac{\partial F^A}{\partial y_1^\sigma} \right) + \frac{d}{dt} \left( \mu^0_A \frac{\partial F^A}{\partial y_0^\sigma} \right) - \mu^0_A \frac{\partial F^A}{\partial y_0^\sigma} \right) \]

on solutions to (3)'' yields the following quantity $\Omega$ satisfying $\dot{\Omega} = 0$ on solutions to (3)'' is constructed:

\[
\Omega = \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \left( \mu^0_A \frac{\partial F^A}{\partial y_0^\sigma} - \frac{d}{dt} \left( \mu^0_A \frac{\partial F^A}{\partial y_1^\sigma} \right) \right) y_0^\sigma - \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \left( \mu^0_A \frac{\partial F^A}{\partial y_0^\sigma} - \frac{d}{dt} \left( \mu^0_A \frac{\partial F^A}{\partial y_1^\sigma} \right) \right) y_0^\sigma \]

\[
+ \sum_{A=1}^{n} \sum_{\sigma=1}^{m} \mu^0_A \left( \frac{\partial F^A}{\partial y_0^\sigma} + \frac{\partial F^A}{\partial y_1^\sigma} \right),
\]

which gives rise to a conserved quantity of (1)" by denoting $\mu^0_A$ and $F^A$ as functions of the original variables $t, x, \dot{x}, \ddot{x}$. 

Example 1. First consider the following linear differential equation

\[ \dot{x} - \frac{a}{t^2} x = 0 \quad (a: \text{const.}, \ a > -\frac{1}{4}). \]

By putting \( x = y_0 \) and \( \ddot{x} = y_1 \), (17) is converted into a second order system

\[ \begin{cases} \dot{y}_1 - \frac{a}{t^2} y_0 = 0 \\ \ddot{y}_0 - y_1 = 0, \end{cases} \]

in which \( \dot{y}_1 - (a/t^2)\dot{y}_0 \) is homogeneous function of degree one with respect to \( \dot{y}_0 \) and \( \dot{y}_1 \).

The equation (11)′ reduces to

\[ \ddot{\mu}^0 - \frac{d}{dt} \left( \frac{a}{t^2} \mu^0 \right) = 0, \]

which is integrated as

\[ \ddot{\mu}^0 - \frac{a}{t^2} \mu^0 = C \quad (C: \text{const.}). \]

Here put \( \mu^0 = t^m \) (\( m: \text{const.} \)). Then the homogeneous equation of (20) is written as

\[ \ddot{\mu}^0 - \frac{a}{t^2} \mu^0 = (m^2 - m - a)t^{m-2} = 0, \]

whose solution is

\[ \mu^0 = C_1 t^{m_1} + C_2 t^{m_2} \quad (C_1, C_2: \text{const.}), \]

where \( m_1 \) and \( m_2 \) are the constants:

\[ m_1 = \frac{1 + \sqrt{1 + 4a}}{2}, \quad m_2 = \frac{1 - \sqrt{1 + 4a}}{2}. \]

Accordingly, since \( \mu^0 = t^2 \) is a solution of (19) (also (20)), the solution of (19) is determined as

\[ \mu^0 = C_1 t^{m_1} + C_2 t^{m_2} + C_3 t^2 \quad (C_1, C_2, C_3: \text{const.}), \]

which is substituted for (16)′ to obtain the conserved quantity:

\[ \Omega = C_1 (t^{m_1} y_1 - m_1 t^{m_1-1} \dot{y}_0) + C_2 (t^{m_2} y_1 - m_2 t^{m_2-1} \dot{y}_0) + C_3 (t^2 y_1 - 2t \dot{y}_0 + (2 - a)y_0). \]

Since \( C_1, C_2 \) and \( C_3 \) are arbitrary constants, (22) includes the following three conserved quantities:

\[ \Omega_1 = t^{m_1} \ddot{x} - m_1 t^{m_1-1} \dot{x}, \]
\[ \Omega_2 = t^{m_2} \ddot{x} - m_2 t^{m_2-1} \dot{x}, \]
\[ \Omega_3 = t^2 \ddot{x} - 2t \dot{x} + (2 - a)x, \]

which are independent if

\[
\begin{vmatrix}
  t^{m_1} & -m_1 & 0 \\
  t^{m_2} & -m_2 & 0 \\
  t^2 & -2t & 2 - a
\end{vmatrix} = (2 - a)(m_1 t^{m_2} - m_2 t^{m_1}) \neq 0.
\]
When $a \neq 2$, by eliminating $\ddot{x}$ and $\dot{x}$ in $\Omega_1$, $\Omega_2$ and $\Omega_3$, the solution of (17) is determined immediately as

\[ x = A_1 t^{2-m_1} + A_2 t^{2-m_2} + A_3, \]

where $A_1$, $A_2$ and $A_3$ are the constants:

\[ A_1 = \frac{2 - m_2}{(2 - a)(m_2 - m_1)} \Omega_1, \quad A_2 = \frac{2 - m_3}{(2 - a)(m_2 - m_1)} \Omega_2, \quad A_3 = \frac{1}{2-a} \Omega_3. \]

When $a = 2$, it follows that $\Omega_1 = \Omega_3$, which and $\Omega_2$ lead respectively to

\[ \Omega_1 = \Omega_3 = t^2 \ddot{x} - 2 t \dot{x}, \]

\[ \Omega_2 = \frac{\ddot{x}}{t} + \frac{\dot{x}}{t^2}. \]

In $\Omega_1$ and $\Omega_2$, $\ddot{x}$ is eliminated to have

\[ \dot{x} = \frac{a_2}{3} t^2 - \frac{a_1}{3} \frac{1}{t}, \]

which is integrated as

\[ x = B_1 \log |t| + B_2 t^3 + B_3, \]

where $B_1 = - \Omega_1/3$, $B_2 = \Omega_2/9$ and $B_3$ are arbitrary constants.

**Example 2.** Next consider the following linear differential equation

(23) \quad \ddot{x} + f(t) \dot{x} = 0.

By putting $x = y_0$ and $\ddot{x} = y_1$, (23) is converted into a second order system

(24) \quad \begin{cases} \dot{y}_1 + f(t) y_1 = 0 \\ \dot{y}_0 - y_1 = 0. \end{cases}

Then (11)$'$ reduces to

\[ \frac{d^3 \mu^0}{dt^3} - \frac{d^2}{dt^2} f(t) \mu^0 = 0, \quad \text{i.e.,} \quad \ddot{\mu}^0 - f(t) \mu^0 = C_1 t + C_2 \quad (C_1, C_2: \text{const.}), \]

whose solution

\[ \mu^0 = e^{\int f(t) dt} \left( C_1 \int t e^{-\int f(t) dt} dt + C_2 \int e^{\int f(t) dt} dt + C_3 \right) \quad (C_1, C_2, C_3: \text{const.}) \]

is substituted for (16)$'$ to construct the conserved quantity

\[ \Omega = C_1 \left( - t y_0 + y_0 + y_1 e^{\int f(t) dt} \int t e^{-\int f(t) dt} dt \right) \]

\[ + C_2 \left( - \dot{y}_0 + y_1 e^{\int f(t) dt} \int e^{-\int f(t) dt} dt \right) + C_3 y_1 e^{\int f(t) dt}. \]
Since $C_1$, $C_2$ and $C_3$ are arbitrary constants, (25) includes the following conserved quantities:

\[
\Omega_1 = -t\dot{x} + x + \bar{x}\int e^{-f(t)}dt - \int te^{-f(t)}dt,
\]
\[
\Omega_2 = -\dot{x} + \bar{x}\int e^{-f(t)}dt - \int te^{-f(t)}dt,
\]
\[
\Omega_3 = \bar{x}\int e^{-f(t)}dt,
\]
in which $\bar{x}$ and $\dot{x}$ is eliminated to determine the solution of (23):

\[
x = \Omega_1 - \Omega_2 t + \Omega_3 \left(t \int e^{-f(t)}dt - \int te^{-f(t)}dt \right),
\]
where $\Omega_1$, $\Omega_2$ and $\Omega_3$ are arbitrary constants.

References


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