LUKASIEWICZ FINITELY LOCAL ALGEBRAS

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Abstract. In this paper finitely local L-algebras are introduced as a generalization of quasi-local L-algebras. The class of finitely local algebras includes the semilocal L-algebras. Some properties are studied and characterizations are given.

1 Introduction

A \( L \)-algebra is said to be local if it has a unique maximal ideal [5]. Local \( L \)-algebras are also characterized as the \( L \)-algebra where for each element \( x \) exists a positive integer \( n \) such that \( nx = 1 \) or \( nx' = 1 \) [1].

A generalization of these algebras are semilocal \( L \)-algebras which are defined and studied in [4]. They are characterized as the \( L \)-algebra with finitely many maximal ideals. Another generalization, that arises from the second characterization of the local \( L \)-algebras, are the quasi-local \( L \)-algebras in which for each element \( x \) exists a positive integer \( n \) such that \( nx \) or \( nx' \) is a boolean element [7].

These two generalizations, semilocal and quasi-local, are independent, i.e. there are semilocal algebras that are not quasi-local and vice versa.

In this paper we define a new class of \( L \)-algebras, called finitely local, containing both quasi-local and semilocal \( L \)-algebras.

2 Preliminaries

Following [9] we recall that a \( L \)-algebra \( <A, +, \cdot, 0, 1> \) (Lukasiewicz-algebra or MV-algebra [1], [2], [4]) is a sistem such that, \( \forall x, y \in A, \)

1) \( <A, +, 0> \) is an Abelian monoid
2) \( x + 1 = 1 \)
3) \( (x')' = x \)
4) \( 0' = 1 \)
5) \( x + x' = 1 \)
6) \( (x' + y')' + y = (x + y)' + x. \)

Setting as well

\( i) \quad x \cdot y = (x' + y')' \)
\( ii) \quad x \lor y = (x' + y')' + y \)
\( iii) \quad x \land y = (x' \lor y')' \)
\( iv) \quad x \leq y \text{ if and only if } x' + y = 1. \)

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The structure \(<A, \lor, \land, \leq, 0, 1>\) is a bounded distributive lattice. A \(L\)-algebra \(A\) is said to be a \(L\)-chain if the \(\leq\) order is linear. Every \(L\)-algebra is a subdirect product of \(L\)-chains [2].

Given a \(L\)-algebra \(A\), let \(B_A\) denote the set of its boolean (idempotent) elements, i.e. the set of all \(x \in A\) with \(2x = x\). The set \(B_A\) is a Boolean subalgebra of \(A\) [2] and, \(\forall x, y \in B_A\), \(x + y = x \lor y\) and \(x \cdot y = x \land y\).

A non-empty subset \(I \subseteq A\) is an ideal if it is closed under + and if \(x \in I\), \(y \in A\) with \(y \leq x\) imply \(y \in I\).

For \(a, b \in A\), \(a \leq b\), let \(A_{a,b} = \{x \in A : a \leq x \leq b\}\).

The system \(<A_{a,b}, \oplus, \land, \lor; a, b>\) is a \(L\)-algebra with respect to the following operations:

\[
x \oplus y = a + [(x' + a)(y' + a)]'(y' + a)'
\]

\[
x' = a + (y' + x)'
\]

Every \(L\)-algebra is isomorphic to \(A_{0,b'} \times A_{a,b}\), where \(b \in B_A\) with \(b \neq 0, 1\).

Recall that an element \(a \in A\) has finite order \(n\) if \(n\) is the least positive integer such that \(na = 1\) and we write \(\text{ord}(a) = n\). If no such \(n\) exists, we say that \(a\) has infinite order.

An element \(a \in A\) is said to be quasi-archimedean if \(na \in B_A\) for some integer \(n > 0\). If no such \(n\) exists the element \(a\) is said to be non-archimedean.

A \(L\)-algebra \(A\) is said to be

- \(n\)-local if, for each \(x \in A\), \(\text{ord}x \oplus \text{ord}x'\) is finite, i.e. if and only if \(A\) has a unique maximal ideal;
- \(\text{semi-local}\) if, for each \(a \in A\), \(a\) or \(a'\) is quasi-archimedean;
- \(\text{finitely local}\) if it has only finitely many maximal ideals (we refer the reader to [5], [7] and [4]).

3 Finitely local \(L\)-algebras

**Definition 1** A \(L\)-algebra \(A\) is called \(n\)-local, \(n \geq 2\), if the following properties hold:

1) For each non-archimedean \(x \in A\) there exist \(b \in B_A\), \(b \neq 0\), and a positive integer \(m\) such that \(mx \land b\) is non-archimedean and \(mx' \land b\) is boolean;

2) For any \(b_1, b_2, \ldots, b_n \in B_A\), for which \(b_i \land b_j = 0\) \(\forall i \neq j\), there exists \(k, 1 \leq k \leq n\), such that \(x < b_k\) is not true for every non-archimedean element \(x \in A\).

A is called 1-local if it is a local \(L\)-algebra.

Hence, we say that \(A\) is a finitely local \(L\)-algebra if it is \(n\)-local for some \(n \geq 1\).

We remark that if \(A\) is \(n\)-local, then it is \(m\)-local for any \(m \geq n\).

Throughout this paper, \(\text{Rad}(A)\) denotes the radical of \(A\), that is, the intersection of all maximal ideals of \(A\) and \(\pi\) denotes the canonical omomorphism of \(A\) on \(A_{\text{rad}(A)}\).

**Lemma 1** Let \(A\) be a \(L\)-algebra. Then \(B_A\) is isomorphic to \(B_{A_{\text{rad}(A)}}\).

**Proof.** It is suffices to check that if \([b]\) is a boolean element of \(A_{\text{rad}(A)}\), then there exists a unique \(y \in B_A\) such that \(\pi(y) = [b]\).

Let \(x\) be an element of \(A\) such that \(\pi(x) = [b]\) and put \(y = (2(2x))'\). Thus

\[
\pi(y) = \pi((2(2x))') = (2(2\pi(x))')' = \pi(x) = [b].
\]
Now we note that
\[
((2y)' + y)' = ((2(2x)' + (2x)')' + (2(2x)' + (2x))' = (2x)' + 2(2x)' = (2x)' + (2x)'
\]
Since, by (2) of Definition 1,
\[
\text{we obtain } ((2(2x)' + 2x)' + (2x)')' = \text{ is a semilocal } \mathbb{L} \text{-algebra.}
\]

We show that Proposition 3

\[
\text{Let } A \text{ be a semilocal } \mathbb{L} \text{-algebra. Then } A \text{ is a } \text{finitely local } \mathbb{L} \text{-algebra.}
\]

**Proof.** Let \( A \) be a direct product of finitely many local \( \mathbb{L} \)-algebras. Then, by Theorem 2.6 in [4], \( A \) is semilocal.

Conversely, suppose that \( A \) is a semilocal \( \mathbb{L} \)-algebra. Thus the \( \mathbb{L} \)-algebra \( A \) is a direct product of finitely many simple \( \mathbb{L} \)-chains (see [4]).

Let \( b_1, b_2, \ldots, b_n \) be the atoms of \( B_{\text{Rad}(A)} \). By Lemma 7, we can suppose that \( b_1, b_2, \ldots, b_n \) are atoms of \( B_A \). Then \( A \) is isomorphic to the direct product \( A_{b_1} \times A_{b_2} \times \cdots \times A_{b_n} \), where each \( A_{b_i} \simeq \frac{A}{\text{Rad}(A)} \), \( i = 1, 2, \ldots, n \), is semilocal. Since \( B_{A_{b_i}} = \{0, 1\} \), the \( \mathbb{L} \)-algebras \( A_{b_i} \) are local.

**Proposition 3** Let \( A \) be a semilocal \( \mathbb{L} \)-algebra. Then \( A \) is a finitely local \( \mathbb{L} \)-algebra.

**Proof.** Take \( b_1, b_2, \ldots, b_n \in B_A \) as in Proposition 1. Let \( x \) be a non-archimedean element of \( A \) and let \( B_x = \{ b_i : x \wedge b_i \text{ is non-archimedean} \} \).

We remark that \( B_x \) is not empty: otherwise, by Proposition 7 ii) in [7], the element \( \bigvee (x \wedge b_i) = x \) would be quasi-archimedean.

Thus, put \( b = \bigvee \{ b_i : b_i \in B_x \} \), the element \( x \wedge b \) is non-archimedean. But, since for each \( b_i, i = 1, 2, \ldots, n, x \wedge b_i \) or \( x' \wedge b_i \) is quasi-archimedean, the element \( x' \wedge b \) is quasi-archimedean.

Hence we conclude that \( A \) is a \( m \)-local \( \mathbb{L} \)-algebra with \( m \leq n \).

**Proposition 4** \( A \) is a quasi-local \( \mathbb{L} \)-algebra if and only if it is a 2-local \( \mathbb{L} \)-algebra.

**Proof.** If \( A \) is a quasi-local \( \mathbb{L} \)-algebra, then the claim follows from Proposition 8 in [7]. Suppose that \( A \) is a 2-local \( \mathbb{L} \)-algebra. Let \( x \) be a non-archimedean element of \( A \). By hypothesis, there exist \( b \in B_A, b \neq 0 \), and a positive integer \( m \) such that \( mx \wedge b \) is non-archimedean and \( mx' \wedge b \) is boolean.

Since, by (2) of Definition 1, \( mx \wedge b \) non-archimedean gives \( mx' \wedge b' \) quasi-archimedean, then the element \( mx' = (mx' \wedge b) \vee (mx' \wedge b') \) is quasi-archimedean.

Let \( I_A = \{ x \in A : x \wedge z \text{ is quasi-archimedean, } \forall z \in A \} \).

We show that

**Proposition 5** \( I_A \) is an ideal of \( A \).
Proof. Take \( x \in I_A \) and \( y \in A \) with \( y \leq x \). Since \( y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) \) is quasi-archimedean, the element \( y \) lies in \( I_A \).

Let \( x, y \in I_A \), then we can find an integer \( n \geq 1 \) such that \( nx, ny \in B_A \). Since \( x \wedge z \) and \( y \wedge z \) are quasi-archimedean, from \( n((x + y) \wedge z) = (nx + ny) \wedge nz = (nx \wedge nz) \vee (ny \wedge nz) = n((x \wedge z) \vee (y \wedge z)) \) follows that the element \( (x + y) \wedge z \) is also quasi-archimedean. Hence \( x + y \in I_A \).

\[ \square \]

**Proposition 6** Let \( A \) be a \( n \)-local \( L \)-algebra which is not \( (n-1) \)-local. Then there exist \( b_1, b_2, ..., b_{n-1} \in B_A \), with \( b_i \wedge b_j = 0 \) for \( i \neq j \), with the following properties

1) \( (\bigvee b_i) \in I_A \)

2) For each \( x \) non-archimedean, there is \( 0 < j < n \) such that \( x \wedge b_j \) is non-archimedean.

**Proof.** Since \( A \) is not \( (n-1) \)-local, there exist \( b_1, b_2, ..., b_{n-1} \in B_A \), \( b_i \wedge b_j = 0 \) for \( i \neq j \), such that for each \( b_i \) there is a non-archimedean element \( x \in A \) with \( x < b_i \). Since \( A \) is \( n \)-local, for each non-archimedean element \( x \in A \), \( x \notin (\bigvee b_i)' \). Then \( (\bigvee b_i)' \in I_A \). This implies that the element \( (x \wedge (\bigvee b_i)') \) is quasi-archimedean, for each \( x \) non-archimedean.

Then, since \( x = x \wedge (b_1 \vee b_2 \vee ... \vee b_{n-1} \vee (\bigvee b_i)') = (x \wedge b_1) \vee (x \wedge b_2) \vee ... \vee (x \wedge b_{n-1}) \vee (x \wedge (\bigvee b_i)') \), at least one element \( x \wedge b_j \) must be non-archimedean.

\[ \square \]

**Theorem 7** \( A \) is a finitely local \( L \)-algebra if and only if \( \frac{A}{I_A} \) is a semilocal \( L \)-algebra.

**Proof.** Suppose that \( \overline{A} = \frac{A}{I_A} \) is semilocal. First we show that if \( [x] \in B_{\overline{A}} \), then there exists \( b \in B_A \) such that \( [x] = [b] \). From \( [x] = 2[x] \) we have \( z = ((2x)' + x)' \in I_A \). Let \( n \) be the positive integer such that \( nz \in B_A \). Then \( x + nz = x + (n+1)z = 2x + nz = 2x + 2nz = 2(x + nz) \), that is \( x + nz \in B_A \) and \( [x + nz] = [x] \).

Now let \( x \) be a non-archimedean element of \( A \).

We remark that \( [x] \) is also non-archimedean: otherwise would exist a non-archimedean \( a \in A \) and \( b \in B_A \) such that \( [a] = [b] \). This implies that there is a boolean \( c \in I_A \) such that \( a + c = a \vee c = b \) which gives \((e' \wedge (a \vee c)) \vee (a \wedge c) = a \). Hence \( a \wedge c \) would be non-archimedean which is a contradiction being \( c \in I_A \).

Since \( \overline{A} \) is semilocal, we can take the atoms \( [b_1], [b_2], ..., [b_n] \) of \( B_{\overline{A}} \) (see proposition 1).

Then, from \( [x] = ([b_1] \wedge [x]) \vee ([b_2] \wedge [x]) \vee ... \vee ([b_n] \wedge [x]) \), follows that there is at least a \( [b_i] \wedge [x] \) which is non-archimedean. Hence \( b_i \wedge x \) is non-archimedean.

On the other hand, since \( \overline{A}_{b_i} \) is local, we have \( [b_i] \wedge [x'] = [b_i] \) which implies \( b_i \wedge x' \) quasi-archimedean. Now it is easy to conclude that \( A \) is \( n+1 \)-local.

Conversely, suppose \( A \) finitely local. Let \( b_1, b_2, ..., b_{n-1} \in B_A \) as in the above proposition. Let \( J_i \) be the ideal generated by

\[ H_i = \{ b \in B_A : b \wedge (x \wedge b_i) \text{ quasi-archimedean } \forall x \} \cup \{ x \wedge b_i \text{ non-archimedean} \}. \]

We show that \( J_i \) is a maximal ideal of \( A \) containing \( I_A \). Let \( \overline{J} \) be an ideal of \( A \) with \( J_i \subseteq \overline{J} \). Take \( a \in \overline{J} - J_i \). This element \( a \) is quasi-archimedean: otherwise, by proposition 5, we would have \( a < b_i \), for \( j \neq i \), which implies \( a \in J_i \), since \( b_j \in H_i \). Then there exists \( n \) such that \( na = b \in B_A \). Since \( b \notin H_i \), there is a non-archimedean \( z \in A \) such that \( \overline{a} = \overline{b} \wedge z \wedge b_i \) is non-archimedean. If \( \overline{b} \wedge x \wedge b_i \) is quasi-archimedean for each \( x \), then \( \overline{b} \in H_i \) which gives \( 1 \in \overline{J} \), that is \( \overline{J} = A \). Conversely, if there is \( y \in A \) with \( \overline{b} \wedge y \wedge b_i = \overline{y} \) non-archimedean, then \( \overline{b} \wedge b_i > \overline{y}, \overline{b} \wedge b_i > \overline{y} \) and \( (\overline{b} \wedge b_i) \wedge (\overline{b} \wedge b_i) = 0 \). Since \( A \) is \( n \)-local, we obtain a contradiction.

Now we take a maximal ideal \( J \) of \( A \) different from \( J_1, J_2, ..., J_{n-1} \) and show that \( I_A \subseteq J \).
If, for each \( i = 1, 2, \ldots, n - 1 \), \( b_i \in J \), then \( (\bigvee b_i)^' \notin J \) which implies \( I_A \not\subseteq J \).

Hence we can suppose that there exists \( 1 \leq k \leq n - 1 \) such that \( b_k \notin J \). Since \( H_k \not\subseteq J \), we distinguish two cases:

1) There is \( b \in B_A \) such that \( b \land (x \land b_k) \) is quasi-archimedean, for each \( x \in A \), and \( b \notin J \);

2) There is \( x \land b_k \) non-archimedean such that \( x \land b_k \notin J \).

In the first case, being \( J \) a prime ideal of \( A \), the element \( b \land b_k \) lies in \( I_A \) but not in \( J \).

In the second case, by Theorem 4.7 in [2], there is a positive integer \( n \) such that \( n(x \land b_k)^' \in J \).

Thus, for some integer \( m \), \( b = m(x \land b_k)^' \in J \cap B_A \) that is \( b' \notin J \). We remark that \( b + b_k = m(x \land b_k)^' + b_k = mx' \lor b_k' + b_k = 1 \) that is \( b' \leq b_k \). Thus we have \( b_k \land x = [(b \land b_k) \lor b'] \land x = [(b \land b_k) \land x] \lor (b' \land x) \). Since \( b' \land x \) is quasi-archimedean, the above relation gives \( (b \land b_k) \land x \) non-archimedean. It follows that, for each \( y \in A \), the element \( b' \land y \) is quasi-archimedean, hence \( b' \in I_A \) and \( b' \notin J \).

We can conclude that \( A \) has \( n \) maximal ideals that is, by [4], it is semilocale.

\[ \square \]

**Corollary 8** A \( L \)-algebra \( A \) is quasi-local if and only if \( \frac{A}{I_A} \) is a local \( L \)-algebra.

**Theorem 9** A \( L \)-algebra \( A \) is finitely local if and only if it is isomorphic to a direct product of finitely many quasi local \( L \)-algebras.

**Proof.** We remark that if \( b \in B_A \), then \( I_{A_{b,b}} = \{ x \land b : x \in I_A \} \). Hence the claim follows by the propositions 1 and 6 and the corollary 1. \( \square \)

**References**


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