ALMOST POINTWISE ESTIMATE AND EXTRAPOLATION THEOREM

TAKUYA SOBUKAWA

Received August 7, 2003; revised July 9, 2004

Dedicated to Professor Kôzô YABUTA on his Sixtieth birthday

Abstract. We shall give a useful decomposition of function related to its non-increasing rearrangement in order to get some extrapolation estimates “nearer” to $L^1$ which contain Yano’s classical work.

1. Introduction and Result

Let $(\Omega, \mu)$ be a $\sigma$-finite measure space. In extrapolation theory on $L^p$-spaces, we treat the operator which satisfies the following assumptions, so called “Yano’s condition”:

Condition 1. Let $1 < p < \infty$ and fix it.

1. $T$ is a sub-additive operator on $L^p(\Omega, \mu)$ for any $p$, $1 < p < p_1$, i.e. $|T(f + g)| \leq |Tf| + |Tg|$ a.e. for any $f, g \in L^p(\Omega, \mu)$.

2. For any $f \in L^p(\Omega, \mu)$, $1 < p \leq p_1$,

$$
\|Tf\|_{L^p(\Omega)} \leq \frac{A}{(p-1)^{\alpha}} \|f\|_{L^p(\Omega)}
$$

Here, positive constants $A$ and $\alpha$ are independent of $p$ and $f$.

We can find many operators satisfying such conditions: Hilbert transform, Riesz transform, Calderon-Zygmund operators, multiple Wiener integral operators, Hardy-Littlewood many maximal operator, etc. For such operators, we cannot get $L^1$ boundedness but, instead of it, S.Yano proved that such $T$ is bounded from $L^1 \log^\alpha L$ to $L^1$ in the case $\mu(\Omega) < \infty$ (Yano’s extrapolation theorem [8]).

In general case, $\mu(\Omega) \leq \infty$, the author proved the following extrapolation estimates between some Orlicz spaces which include Yano’s result([6],[7]): For $1 < q \leq p_1$,

$$
\|Tf\|_{L^1 + L^q(\Omega)} \leq \frac{C}{(q-1)^{\alpha}} \|f\|_{L^1 \log^\alpha L + L^q(\Omega)}
$$

and, as its consequence,

$$
\|Tf\|_{L^{1} + L^{1} \log^{-\alpha-\varepsilon} L(\Omega)} \leq C_\varepsilon \|f\|_{L^{1} \log^\alpha L + L^{1} \log^{-\varepsilon} L(\Omega)},
$$

for $\varepsilon > 0$. Here we denote $L^{\Phi_0} + L^{\Phi_1} (\Omega) = L^{\Phi}(\Omega)$, $\Phi = \min\{\Phi_0, \Phi_1\}$ for any two Orlicz classes $L^{\Phi_0}(\Omega)$ and $L^{\Phi_1}(\Omega)$.

Our aim is to get some boundedness between function spaces “near to $L^1$”, however, counter examples are known to (1.3) for $\varepsilon = 0$. In this paper, instead of it, we shall show the following estimate:

2000 Mathematics Subject Classification. Primary 46B99, 46E30; Secondary 46B70.

Key words and phrases. Extrapolation theorem.

This work was partially supported by Bilateral Exchange Program of Japan Society of Promotion of Science and Academy of Science of the Czech Republic, 2000.
Theorem 1. Fix $\alpha > 0$ and $1 < p_1 < \infty$. Let $T$ be a sub-additive operator on $L^p(\Omega, \mu)$ for any $p, 1 < p \leq p_1$, i.e.

$$\|T(f + g)\| \leq \|Tf\| + \|Tg\| \quad a.e. \mu$$

for any simple function $f$ and $g$ and satisfy weak $L^p$ boundedness

(1.4) $$\|Tf\|(p, \infty) \leq \frac{A}{(p - 1)^\alpha} \|f\|_p,$$

for any $p, 1 < p < p_1$. Then, we have

(1.5) $$\|Tf\|(1, \infty, 0, -\alpha) \leq C\|f\|_{1, 1, 1, 0, 0}$$

for any $f \in L^1 + L^1 \log^\alpha L(\Omega)$. Here,

(1.6) $$\|g\|_{p,1,0,\infty} = \int_0^\infty t^\gamma (1 + \log^+ \frac{1}{t})^\alpha (1 + \log^+ t)^\alpha g^*(t) \frac{dt}{t},$$

for $\alpha_0, \alpha_\infty \in \mathbb{R}$, with

$$g^*(t) = \inf\{\lambda > 0; \mu(\{x \in \Omega : |g(x)| > \lambda\}) \leq t\} \text{ and } g^{**}(t) = \frac{1}{t} \int_0^t g^*(s)ds$$

for $t \in (0, \infty)$ and we write $\|g\|_{p,1} = \|g\|_{p,1,0,0}$ and $\|g\|_{(p,\infty)} = \|g\|_{(p,\infty,0,0)}$, simply.

Moreover, we can also show the following Koizumi type estimate after (1.2) (also see [4]), similarly and independently to Theorem 1.

Theorem 2. Under the same assumption of Theorem 1, we have

(1.7) $$\|Tf\|_{m(1, q), 1, 0, 0} \leq C\|f\|_{m(1, q), 1, 0, 0}$$

for any $f \in L^{m(1, q), 1, 0, 0}, 1 < q < p_1$ and

(1.8) $$\|Tf\|_{m(1, p_1), 1, 0, 0} \leq C\|f\|_{m(1, p_1), 1, 0, 1}$$

Here,

$$\|g\|_{m(p, q), 1, 0, \infty} = \int_0^\infty \min(t^\gamma, t^\gamma) (1 + \log^+ \frac{1}{t})^\alpha (1 + \log^+ t)^\alpha g^*(t) \frac{dt}{t},$$

for any $\alpha \geq 0$, we may prove

$$\|f\|_{1, 1, 0, 0} = \int_0^\infty (1 + \log^+ \frac{1}{t})^\alpha f^*(t) dt \approx \int_{\Omega} |f(x)|(1 + \log^+ |f(x)|)^\alpha d\mu(x)$$

and $L^1 + L^1 \log^\alpha L(\Omega) = \{f : \|f\|_{1, 1, 0, 0} < \infty\}$ (see [1]).

Remark 1. As is known, the condition (1.4) is weaker than (1.1). On the left hand side of (1.5),

$$\|Tf\|(1, \infty, 0, -\alpha) = \sup_{t > 0} \int_0^t (Tf)^*(s) ds \cdot (1 + \log^+ t)^\alpha = \int_{\Omega} (Tf)^*(s) ds + \sup_{t > 1} \int_{t}^1 (Tf)^*(s) ds \cdot \frac{1}{(1 + \log^+ t)^\alpha}$$

and

$$\int_0^1 (Tf)^*(s) ds \geq \int_{Tf \geq M} (Tf)(x) d\mu(x)$$

for some $M > 0$. Therefore, in the case $\mu(\Omega) < \infty$, our result implies Yano’s theorem. After Remark 1 above, it is easy to show that (1.7) or (1.8) does so, too.
2. Proof of the Theorems

We note that $f^*(t) \to 0 \ (t \to \infty)$ and $|f| < \infty$, $\mu$-a.e. for any $f \in L^1 + L^1 \log L(\Omega)$. Now, we shall decompose every function $f \in L^1 + L^1 \log L$ as follows.

First, we consider a family of pairwise disjoint measurable sets

$$E_n = \{x \in \Omega : f^*(2^{n+1}) < |f(x)| \leq f^*(2^n)\}, \quad n \in \mathbb{Z}.$$  

(2.1)

Here, if $f^*(2^n) = f^*(2^{n+1})$, we define $E_n = \emptyset$. Now, we put

$$f_n(x) = \begin{cases} f(x) & x \in E_n, \\ 0 & \text{otherwise}. \end{cases} \quad (n \in \mathbb{Z}).$$

(2.2)

It is easy to show

1. $f(x) = \sum_{n=-\infty}^{\infty} f_n(x)$ for any $x \in \Omega$,
2. $\mu(E_n) \leq 2^{n+1}$,
3. $|f_n(x)|, (f_n)^*(t) \leq f^*(2^n)$

for any $n$.

For simplicity, we shall write $\rho(p) = A(p-1)^{-\alpha}$. From the assumption (1.4), we may have

$$s^\sharp(Tf_n)^*(s) \leq \rho(p) \int_0^\infty t^\sharp f_n^*(t) \frac{dt}{t}$$

$$\approx \rho(p) \sum_{i=-\infty}^{\infty} (2^i)^\sharp f_n^*(2^i) = \rho(p) \sum_{i=-\infty}^{n+1} (2^i)^\sharp f_n^*(2^i)$$

$$\leq 2\rho(p)(2^{n+1})^\sharp f^*(2^n) \leq 4\rho(p)(2^n)^\sharp f^*(2^n)$$

for any $n \in \mathbb{Z}$. Put $s = 2^k, k \in \mathbb{Z}$, we have

$$(Tf_n)^*(2^k) \leq 4\rho(p)(2^{n-k})^\sharp f^*(2^n)$$

$(-\infty < k < \infty, -\infty < n < \infty)$. Taking infimum with respect to $p$, we may get

$$(2.3) \quad (Tf_n)^*(2^k) \leq 4 \inf_p \left(\rho(p)(2^{n-k})^\sharp\right) f^*(2^n)$$

Summing up with respect to $n$,

$$(Tf)^*(2^k) \leq \sum_{n=\infty}^{\infty} (Tf_n)^*(2^k)$$

(2.4)

$$\leq 4 \sum_{n=\infty}^{\infty} \inf_p \left(\rho(p)(2^{n-k})^\sharp\right) f^*(2^n)$$

and we call it “almost pointwise estimate”. Note,

$$(2.5) \quad \inf_{1 < p < p_1} \left(\frac{p}{p-1}\right)^\alpha (2^{n-k})^\sharp \approx \begin{cases} (k-n)^\alpha 2^{n-k} & (n < k) \\ (2^{n-k})^\sharp & (n \geq k), \end{cases}$$

we conclude

$$(Tf)^*(2^k) \leq \sum_{n=-\infty}^{n=\infty} (k-n)^\alpha 2^{n-k} f^*(2^n) + \sum_{n=\infty}^{\infty} (2^{n-k})^\sharp f^*(2^n).$$

(2.6)
Multiplying $2^k$, we have

\[ \sup_{0 < t < 1} t(Tf)^{(2^k)} \approx \sup_{k \leq 0} 2^k (Tf)^{(2^k)} \]

\[ \leq \sup_{k \leq 0} \left[ \sum_{n=-\infty}^{k-1} (k - n)^\alpha 2^n f^*(2^n) + \sum_{n=k}^{\infty} (2^k)^{1 - \frac{1}{p_1}} (2^n)^{\frac{1}{p_1}} f^*(2^n) \right] \]

\[ \leq \sup_{k \leq 0} \left[ \sum_{n=-\infty}^{k-1} (0 - n)^\alpha 2^n f^*(2^n) + \sum_{n=k}^{\infty} 2^n f^*(2^n) \right] \]

\[ \leq \sum_{n=-\infty}^{0} (1 - n)^\alpha 2^n f^*(2^n) + \sum_{n=1}^{\infty} 2^n f^*(2^n) \]

\[ \approx \int_{0}^{1} (1 - \log t)^\alpha f^*(t) dt + \int_{1}^{\infty} f^*(t) dt. \] (2.7)

On the other hand, multiplying $\frac{2^k}{k^\alpha}$, we get

\[ \sup_{k \geq 1} \frac{2^k}{k^\alpha} (Tf)^{(2^k)} \approx \sup_{1 \leq t < \infty} \frac{t(Tf)^{(2^k)}}{(1 + \log t)^\alpha} \]

\[ \leq \sup_{k \geq 1} \left[ \sum_{n=-\infty}^{k-1} \left(1 - \frac{n}{k}\right)^\alpha 2^n f^*(2^n) + \sum_{n=k}^{\infty} \frac{2^k}{k^\alpha} (2^{n-k})^{\frac{1}{p_1}} f^*(2^n) \right] \]

\[ \leq \sum_{n=-\infty}^{k-1} \left(1 - \frac{n}{k}\right)^\alpha 2^n f^*(2^n) + \sum_{n=0}^{\infty} \left(1 - \frac{n}{k}\right)^\alpha 2^n f^*(2^n) + \sum_{n=k}^{\infty} 2^n f^*(2^n) \]

\[ \approx \int_{0}^{1} (1 - \log t)^\alpha f^*(t) dt + \int_{1}^{\infty} f^*(t) dt. \] (2.8)

and Theorem 1 is proved.

Next, we assume $f \in L_{r(1,p_1),1;\alpha,1}(\Omega)$ and use the decomposition (2.6). Putting $k = 0$ in (2.6),

\[ (Tf)^{(1)} = \int_{0}^{1} s(Tf)^*(s) \frac{ds}{s} \]

\[ \leq \sum_{n=-\infty}^{-1} (1 - n)^\alpha 2^n f^*(2^n) + \sum_{n=0}^{\infty} (2^n)^{\frac{1}{p_1}} f^*(2^n) \]

\[ \approx \int_{0}^{1} (1 - \log t)^\alpha f^*(t) dt + \int_{1}^{\infty} t^{\frac{1}{p_1}} f^*(t) \frac{dt}{t}. \] (2.9)
On the other hand, multiplying \((2^k)^{1/k}\) to the left hand side of (2.6) and summing up with respect to \(k\), we have

\[
\sum_{k=0}^{\infty} (2^k)^{1/k} (T f)^{(k)}
\]

\[
\approx \sum_{k=0}^{\infty} (2^k)^{1/k} \frac{1}{2k} \sum_{n=-\infty}^{\infty} 2^n (T f)^{(k)} (2^n)
\]

(2.10)

\[
\geq \sum_{k=0}^{\infty} (2^k)^{1/k} 2^{-k} 2^n (T f)^{(k)} (2^n) = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} (2^k)^{1/k} 2^{-k} 2^n (T f)^{(k)} (2^n)
\]

\[
\approx \sum_{n=0}^{\infty} (2^n)^{1/n} (T f)^{(k)} (2^n) \approx \int_{1}^{\infty} t^{1/n} (T f)^{(k)} (t) \frac{dt}{t}
\]

and from the right handside,

\[
\sum_{k=0}^{\infty} \left[ \sum_{n=-\infty}^{k-1} (k-n)^{\alpha} 2^n (2^k)^{1-k} f^{(k)} (2^n) + \sum_{n=k}^{\infty} (2^n)^{1/k} f^{(k)} (2^n) \right]
\]

\[
= \sum_{k=0}^{\infty} \left[ \left( \sum_{n=-\infty}^{k-1} + \sum_{n=k}^{\infty} \right) (k-n)^{\alpha} 2^n (2^k)^{1-k} f^{(k)} (2^n) + \sum_{n=k}^{\infty} (2^n)^{1/k} f^{(k)} (2^n) \right]
\]

\[
\leq \sum_{n=-\infty}^{\infty} 2^n (1-n)^{\alpha} \sum_{k=1}^{\infty} (1+k)^{\alpha} (2^k)^{1-k} f^{(k)} (2^n)
\]

(2.11)

\[
+ \sum_{n=0}^{\infty} (2^n)^{1/n} \sum_{k=0}^{\infty} k^{\alpha} (2^k)^{1-k} f^{(k)} (2^n) + \sum_{n=0}^{\infty} n (2^n)^{1/n} f^{(k)} (2^n)
\]

\[
\approx \int_{0}^{1} (1 + \log \frac{1}{t})^{\alpha} f^{*}(t) dt + \int_{1}^{\infty} t^{1/n} (1 + \log t) f^{*}(t) dt.
\]

Therefore, we conclude

(2.12) \[
\int_{0}^{1} (T f)^{(k)} (t) dt + \int_{1}^{\infty} t^{1/n} (T f)^{(k)} (t) \frac{dt}{t}
\]

\[
\leq C \left[ \int_{0}^{1} (1 + \log \frac{1}{t})^{\alpha} f^{*}(t) dt + \int_{1}^{\infty} t^{1/n} (1 + \log t) f^{*}(t) \frac{dt}{t} \right]
\]

and (1.8) is proved.

For \(f \in L_{m(1,q),1,\alpha,0}(\Omega), 1 < q < p_1\), we may get

(2.13) \[
\int_{0}^{1} s(T f)^{(k)} (s) \frac{ds}{s} \leq C \left[ \int_{0}^{1} (1 - \log t)^{\alpha} f^{*}(t) dt + \int_{1}^{\infty} t^{1/n} f^{*}(t) \frac{dt}{t} \right]
\]

similarly to (2.9). Moreover, multiplying \((2^k)^{\frac{1}{\alpha}}\) to (2.6) and summing up with respect to \(k\), we get

(2.14) \[
\sum_{k=0}^{\infty} (2^k)^{\frac{1}{\alpha}} (T f)^{(k)} \leq C \left[ \int_{0}^{1} (1 + \log \frac{1}{t})^{\alpha} f^{*}(t) dt + \int_{1}^{\infty} t^{\frac{1}{\alpha}} f^{*}(t) \frac{dt}{t} \right]
\]

similarly to (2.10) and (2.11). Therefore (1.7), then, Theorem 2 is proved.
Acknowledgement: The author express his sincerely thanks to A. Gogatishvili for his encouragent of this work. He also thanks to the referee who read the manuscript carefully and pointed out several mistypings.

REFERENCES


Department of mathematics education, Okayama University, 3-1-1 Tsushima-naka, Okayama, 700-8530, JAPAN
E-mail address: sobu@cc.okayama-u.ac.jp