SOME REMARKS ON SEVEN CONDITIONS APPROXIMATING TO MEASURABLE NORMS

MICHE MAEDA, KEIKO HARAI AND MAYUMI SHIBUYA

Received August 6, 2003

ABSTRACT. In this paper, we treat seven conditions approximating to measurable norms. First we have the equivalency of these conditions except two for rotationally quasi-invariant cylindrical measures. Next it is shown that Yan-Gong’s condition is strictly weaker than Gross’ condition.

1 Introduction

In 1962 Gross introduced the concept of measurable norms ([5]), and in 1971 Dudley et al. introduced another measurability of norms ([3]). Badrikian and Chevet offered the following question ([1]). "Do these concepts of measurable norms coincide with each other for every cylindrical measure?" The first author solved this problem negatively ([8]). Also these two concepts were investigated from different directions and many results were gotten ([9, 10, 11]).

On the other hand, several condition, that are similar to these conditions, are given and studied by many mathematicians ([2, 4, 14]).

In this paper, we take out seven conditions approximating to measurable norms and research them with respect to some examples. First we have the equivalency of these conditions except (iv) and (vii) in Theorem 1 for rotationally quasi-invariant cylindrical measures. For four conditions induced in them, it was proved by the first author ([10]). Moreover it is shown that Yan-Gong’s condition is strictly weaker than Gross’ condition, where Yan-Gong’s and Gross’ conditions are included in the above seven conditions.

2 Preliminaries

Let $X$ be a locally convex Hausdorff space over the real field $\mathbb{R}$, $X'$ its topological dual, $(\cdot, \cdot)$ the natural pairing between $X$ and $X'$ and $\mathcal{B}(X)$ the Borel $\sigma-$algebra of $X$. Let $\{\xi_1, \ldots, \xi_n\}$ be a finite system of elements of $X'$. Then by $\Xi$ we denote the operator from $X$ into $\mathbb{R}^n$ mapping $x$ onto the vector $((x, \xi_1), \ldots, (x, \xi_n))$. A set $Z \subset X$ is said to be a cylindrical set if there are $\xi_1, \ldots, \xi_n \in X'$ and $B \in \mathcal{B}(\mathbb{R}^n)$ such that $Z = \Xi^{-1}(B)$. Let $\mathcal{C}_X$ denote the collection of all cylindrical sets of $X$.

A map $\mu$ from $\mathcal{C}_X$ into $[0, 1]$ is called a cylindrical measure if it satisfies the following conditions:

$$\mu(X) = 1;$$

2000 Mathematics Subject Classification. 28C20, 46G12, 60B11.

Key words and phrases. cylindrical measure, measurable norm.

*Corresponding author.
Let \( \mu \) into \( E \).

**Definition 5** Let rotationally quasi-invariant cylindrical measures. Cylindrical measures and rotationally quasi-invariant cylindrical measures are generalized cylindrical measure, then the converse is true ([11]). Of course, \( \mu \) generally ([8], this is the 1984-Example). If \( \mu \) is a generalized rotationally quasi-invariant cylindrical measure, then the converse is true ([11]). Of course, \( \gamma \) and rotationally invariant cylindrical measures and rotationally quasi-invariant cylindrical measures are generalized rotationally quasi-invariant cylindrical measures.

**Definition 5** Let \( \mu \) be a cylindrical measure on \( H \). If \( \mu(C) = \mu(u(C)) \) whenever \( C \) is a cylindrical set of \( H \) and \( u \) is a unitary operator of \( H \), \( \mu \) is said to be rotationally invariant.
Definition 6 If \( \mu \sim_c u(\mu) \) whenever \( u \) is a unitary operator of \( H \), \( \mu \) is called rotationally quasi-invariant.

Remark 1 \( \sim_c \) means to be cylindrically equivarent, and \( u(\mu) \) is the image of \( \mu \) under \( u \).

3 Seven conditions

In this section we treat with seven conditions that are similar to measurable norms. The following theorem is all.

Theorem 1 Let \( H \) be a real separable Hilbert space with norm \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \), \( \mu \) be a cylindrical measure on \( H \), \( \| \cdot \| \) be a continuous norm defined on \( H \), \( B \) be the completion of \( H \) with respect to \( \| \cdot \| \) and \( i \) be the inclusion map from \( H \) into \( B \). Moreover, let \( Y \) be the bidual \( B'' \) of \( B \) with weak*-topology \( \sigma(B'',B') \) and \( j \) be the inclusion map from \( H \) into \( Y \).

Then the following seven conditions satisfy the relations:

\[ (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv), \]
\[ (i) \Rightarrow (ii) \Rightarrow (v) \Leftrightarrow (vi) \Rightarrow (vii) \]

If \( \mu \) is continuous (this means that the characteristic function of \( \mu \) is continuous on \( H \)), then the following conditions satisfy the relations:

\[ (iii) \Rightarrow (vi) \quad \text{and} \quad (iv) \Rightarrow (vii) \]

(i) For any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \), where \( \mathbb{N} \) is the set of all natural numbers, such that \( n > m \geq N \) implies

\[
\mu(\{ x \in H ; \| P_n x - P_m x \| > \varepsilon \}) < \varepsilon
\]

for every sequence \( \{ P_n \} \subset \mathcal{F} \) such that \( P_n \) converges strongly to the identity map \( I \), we write it \( P_n \nearrow I \).

(ii) \( \| \cdot \| \) is a \( \mu - (G) \) measurable norm.

(iii) There exists a sequence \( \{ P_n \} \subset \mathcal{F} \) such that \( P_n \nearrow I \), which has the property that for any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( n > m \geq N \) implies

\[
\mu(\{ x \in H ; \| P_n x - P_m x \| > \varepsilon \}) < \varepsilon.
\]

(iv) There exists a sequence \( \{ P_n \} \subset \mathcal{F} \) such that \( P_n \nearrow I \), which has the property that for any \( \varepsilon > 0 \) there exist \( N \in \mathbb{N} \) and \( n \in \mathbb{N} \) such that \( N \geq N \) and \( n \geq n \) implies

\[
\mu(\{ x \in H ; \sup_{1 \leq k \leq n} \| P_k x \| > N \}) < \varepsilon.
\]

(v) \( \| \cdot \| \) is a \( \mu - (D) \) measurable norm.

(vi) \( i(\mu) \) (i.e. \( \mu \circ i^{-1} \)) is extensible to a measure.

(vii) \( j(\mu) \) is extensible to a measure.

Proof

(i) \( \Rightarrow \) (ii) (see Baxendale [2])

(ii) \( \Rightarrow \) (iii) (see Baxendale [2])

(iii) \( \Rightarrow \) (iv) (see Harai [6])

(v) \( \Leftrightarrow \) (vi) (see Dudley-Feldman-LeCam [3])

(iii) \( \Rightarrow \) (vi) (see Yan [14])

(iv) \( \Rightarrow \) (vii) (see Yan [14] and Gong [4])

\[ \square \]
Remark 2 (1) If $B$ is reflexive, then (vi) and (vii) are equivalent.

(2) The above (i), (ii), (iii), (v) and (vi) conditions are equivalent for $\gamma$ ([2]).

We have the following theorem.

**Theorem 2** Let $H$ be a real separable Hilbert space and $\mu$ be a rotationally quasi-invariant cylindrical measure. Then conditions (i), (ii), (iii), (v) and (vi) in Theorem 1 are equivalent.

**Proof** The first author showed that the conditions (ii), (iii), (v) and (vi) in Theorem 1 are equivalent for rotationally quasi-invariant cylindrical measures ([10]).

Here we have to show that (ii) $\Rightarrow$ (i). Since $\mu$ is a rotationally quasi-invariant cylindrical measure, there exists a rotationally invariant cylindrical measure $\lambda$ such that $\mu \sim_{c} \lambda([10])$.

If the statement "(ii) $\Rightarrow$ (i)" is satisfied for $\lambda$, then also for $\mu$. Therefore we have to show that "(ii) $\Rightarrow$ (i)" is satisfied for rotationally invariant cylindrical measures.

Here we need one lemma.

Let $\gamma^{t}$ be the Gauss cylindrical measure with parameter $t > 0$, i.e. $\gamma^{t}(C) = \gamma(C_{t})$ for every $C \in \mathcal{C}_{H}$ and $\gamma^{0} = \delta_{0}$. Let $\lambda$ be a rotationally invariant cylindrical measure. Then we have

$$
\lambda(\cdot) = \int_{[0, \infty)} \gamma^{t}(\cdot)d\sigma_{\lambda}(t)
$$

by Yamasaki’s theorem ([13]), where $\sigma_{\lambda}$ is a Borel probability measure on $[0, \infty)$ depending to $\lambda$. Let $\| \cdot \|$ be a norm defined on $H$. Obviously we can see that if $\| \cdot \|$ is $\gamma^{t_{0}} - (G)$ measurable for some $t_{0} > 0$, then $\| \cdot \|$ is $\gamma^{t} - (G)$ measurable for all $t > 0$.

**Lemma 1** Let $\lambda$ be a rotationally invariant cylindrical measure on $H$ and not $\delta_{0}$ and $\| \cdot \|$ be a norm defined on $H$. If $\| \cdot \|$ is $\lambda - (G)$ measurable, then $\| \cdot \|$ is $\gamma^{t} - (G)$ measurable for all $t > 0$.

**Proof** For any $\varepsilon > 0$, there exists $P_{0} \in \mathcal{F}$ such that $\lambda(\{x \in H; \|Px\| > \varepsilon\}) < \varepsilon$ for every $P \perp P_{0}$, $P \in \mathcal{F}$.

Recall that

$$
\lambda(\{x \in H; \|Px\| > \varepsilon\}) = \int_{[0, +\infty)} \gamma^{t}(\{x \in H; \|Px\| > \varepsilon\})d\sigma_{\lambda}(t)
$$

and

$$
= \int_{[0, u_{0})} \gamma^{t}(\{x \in H; \|Px\| > \varepsilon\})d\sigma_{\lambda}(t) + \int_{[u_{0}, +\infty)} \gamma^{t}(\{x \in H; \|Px\| > \varepsilon\})d\sigma_{\lambda}(t).
$$

It is clear that there exists $u_{0} > 0$ such that $\sigma_{\lambda}([u_{0}, +\infty)) > 0$, and denote it by $\alpha$ (i.e. $\alpha = \sigma_{\lambda}([u_{0}, +\infty))$).

Since $\int_{[u_{0}, +\infty)} \gamma^{t}(\{x \in H; \|Px\| > \varepsilon\})d\sigma_{\lambda}(t) < \varepsilon$ and $\alpha = \sigma_{\lambda}([u_{0}, +\infty))$, there exists $u'$ such that $u' \geq u_{0}$ and $\gamma^{u'}(\{x \in H; \|Px\| > \varepsilon\}) < \frac{\varepsilon}{\alpha}$.

$u'$ depends on $P$, however its existence in $[u_{0}, +\infty)$ is sure. And so we have

$$
\gamma^{u_{0}}(\{x \in H; \|Px\| > \varepsilon\}) = \gamma^{u'}(\{x \in H; \|Px\| > \frac{u'}{u_{0}}\varepsilon\}) \leq \gamma^{u'}(\{x \in H; \|Px\| > \varepsilon\}) < \frac{\varepsilon}{\alpha}.
$$
Therefore it follows that for any \( \varepsilon > 0 \) there exists \( P_0 \in \mathcal{F} \) such that
\[
\gamma_{u_0}^t(\{x \in H; \|Px\| > \varepsilon\}) < \frac{\varepsilon}{\alpha}
\]
for every \( P \perp P_0, P \in \mathcal{F} \).
This means \( \| \cdot \| \) is \( \gamma_{u_0}^t - (G) \) measurable. Then \( \| \cdot \| \) is \( \gamma_t - (G) \) measurable for all \( t > 0 \).

Here we return the proof of Theorem 2. Suppose that \( \| \cdot \| \) is \( \lambda - (G) \) measurable, where \( \lambda \) is a rotationally invariant cylindrical measure and not \( \delta_0 \).

It follows from Lemma 1 that \( \| \cdot \| \) is \( \gamma_t - (G) \) measurable for all \( t > 0 \).
Given arbitrary \( \varepsilon > 0 \). Let \( \{P_n\} \) be a sequence included in \( \mathcal{F} \) which strongly converges to \( I \), and \( u \) be a positive real number such that \( \sigma_\lambda([u, +\infty)) < \frac{\varepsilon}{2} \), where \( \sigma_\lambda \) is the induced Borel probability measure with \( \lambda \) on \([0, +\infty)\) (see Yamasaki [13]). Also we check that Remark 2 is satisfied for every \( \gamma_t \) (\( t > 0 \)).

Since \( \| \cdot \| \) is \( \gamma_u^t - (G) \) measurable, \( \| \cdot \| \) satisfies condition (i). This induces that there exists \( n_0 \in \mathbb{N} \) such that \( n > m \geq n_0 \) implies
\[
\gamma^n(\{x \in H; \|P_nx - P_mx\| > \varepsilon\}) < \frac{1}{2}\varepsilon.
\]

Therefore for every \( t \) such that \( 0 \leq t \leq u \), we have
\[
\gamma^t(\{x \in H; \|P_nx - P_mx\| > \varepsilon\}) = \gamma^n(\{x \in H; \|P_nx - P_mx\| > \frac{\varepsilon}{t}\})
\]
\[
\leq \gamma^n(\{x \in H; \|P_nx - P_mx\| > \varepsilon\})
\]
\[
< \frac{1}{2}\varepsilon.
\]

Then we have
\[
\lambda(\{x \in H; \|P_nx - P_mx\| > \varepsilon\}) = \int_{[0, +\infty)} \gamma^t(\{x \in H; \|P_nx - P_mx\| > \varepsilon\})d\sigma_\lambda(t)
\]
\[
= \int_{[0, u]} \gamma^t(\{x \in H; \|P_nx - P_mx\| > \varepsilon\})d\sigma_\lambda(t)
\]
\[
+ \int_{[u, +\infty)} \gamma^t(\{x \in H; \|P_nx - P_mx\| > \varepsilon\})d\sigma_\lambda(t)
\]
\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This completes the proof of Theorem 2. □

4 Some Examples and seven conditions

The 1984-Example([8]) showed that Gross’ condition(ii) in Theorem 1) is strictly stronger than D.F.L.’s condition((vi) in Theorem 1).

In this section we treat some examples and clarify the relation between (ii) and (iii) (or (iv)).

First we introduce several continuous norms and several cylindrical measures on \( \ell^2 \).
Let \( e_n = (0, \ldots, 0, 1, 0, \ldots) \), where 1 appears in the \( n \)–th place. It is clear that \( \{e_n\}_{n=1,2,\ldots} \)
is a complete orthonormal system (CONS) on $\ell^2$.

(i) Construction of $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_3$ and $\|\cdot\|_4$

Let $\{\alpha_n\}_{n=1,2,\ldots}$ be the sequence of non-negative real numbers such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots$ and $\alpha_n \to \infty$ as $n \to \infty$. Denote by $\Gamma_1$ the convex hull of the set 
$\{\pm \alpha_n(e_1 + e_2 + \ldots + e_n); n = 1, 2, \ldots\}$, $B_1$ the open unit ball of $\ell^2$ and $U_1 = \Gamma_1 + B_1$. It is obvious that $U_1$ is open, convex, absorbing and circled. We denote by $\|x\|_1$ gauge of $U_1$ at $x \in \ell^2$. $\|\cdot\|_1$ is the continuous norm defined on $\ell^2$.

Let $\{\beta_n\}_{n=1,2,\ldots}$ be the sequence of non-negative real numbers such that $\beta_{2m} = 0$ for $m = 1, 2, \ldots$, $\beta_{2m-1} > 0$ for $m = 1, 2, \ldots$, and the sequence $\{\beta_{2m-1}\}_{m=1,2,\ldots}$ is increasing and tends to $\infty$ as $m \to \infty$. Let $\Gamma_2$ be the convex hull of the set $\{\pm \beta_n(e_1 + e_2 + \ldots + e_n); n = 1, 2, \ldots\}$ and $U_2 = \Gamma_2 + B_1$. $U_2$ is also open, convex, absorbing and circled. We denote by $\|x\|_2$ the gauge of $U_2$ at $x \in \ell^2$. $\|\cdot\|_2$ is continuous, too.

Let $\{\eta_n\}_{n=1,2,\ldots}$ be the sequence satisfying the same condition as it of $\{\beta_n\}_{n=1,2,\ldots}$. Let $\Gamma_3$ be the convex hull of the set $\{\pm \eta_n(e_1 + e_2 + \ldots + ne_n); n = 1, 2, \ldots\}$, $B_2$ the open set $x = (x_n) \in \ell^2; \sqrt{\sum_{n=1}^{\infty} (\frac{x_n}{n})^2} < 1$ and $U_3 = \Gamma_3 + B_2$. The gauge of $U_3$ at $x \in \ell^2$ defines the continuous norm $\|x\|_3$.

For $x = (x_n) \in \ell^2$, we denote $\|x\|_4 = \sqrt{\sum_{n=1}^{\infty} (\frac{x_n}{n})^2}$.

(ii) Construction of $\mu_a$ and $\mu_b$

Let $(\ell^2)^*$ be the algebraic dual of $\ell^2$, equipped with its weak topology $\sigma((\ell^2)^*, \ell^2)$, and $(\cdot, \cdot)$ be the natural pairing $(\ell^2)^* \times \ell^2 \to \mathbb{R}$. Then a cylindrical set in $(\ell^2)^*$ and in $\ell^2$ can be described as

$$Z = \{x \in (\ell^2)^*; (x, \xi_1), \ldots, (x, \xi_n) \in D\}$$

and

$$\tilde{Z} = \{x \in \ell^2; <x, \xi_1>, \ldots, <x, \xi_n> \in D\}$$

where $\xi_1, \ldots, \xi_n \in \ell^2$ and $D \in \mathcal{B}(\mathbb{R}^n)$, respectively.

We choose an algebraic basis $\mathcal{J}$ of $\ell^2$ containing $\{e_n\}_{n=1,2,\ldots}$.

Define $a$ and $b \in (\ell^2)^*$ as follows:

$$\langle a, e_n \rangle = 1 \text{ for } n = 1, 2, \ldots,$$

$$\langle a, e_n \rangle = 0 \text{ for } e_n \in \mathcal{J} \setminus \{e_n\}_{n=1,2,\ldots},$$

$$\langle b, e_n \rangle = n \text{ for } n = 1, 2, \ldots,$$

$$\langle b, e_n \rangle = 0 \text{ for } e_n \in \mathcal{J} \setminus \{e_n\}_{n=1,2,\ldots}.$$

Let $\delta_a$ and $\delta_b$ denote the Dirac measures at the fixed point $a$ and $b$ in $(\ell^2)^*$ respectively.

Then the induced measures $\mu_a$ and $\mu_b$ on $\ell^2$ are defined by

$$\mu_a(Z) = \delta_a(Z),$$

$$\mu_b(Z) = \delta_b(Z),$$

respectively.
It is known that \( \| \cdot \|_2 \) is \( \mu_a -(D) \) measurable but not \( \mu_a -(G) \) measurable ([8], this is the 1984 Example).

Later, we have that \( \| \cdot \|_2 \) is not \( \gamma - \) measurable([12]). Then we construct \( \| \cdot \|_3 \) and \( \mu_b \).

\[ \| \cdot \|_3 \] is \( \mu_b -(D) \) measurable but not \( \mu_b -(G) \) measurable ([12]), however it is \( \gamma - \) measurable.

\[ \| \cdot \|_4 \] is the well known example that is \( \gamma - \) measurable.

The following theorem is the main result in this section.

**Theorem 3**

(i) \( \| \cdot \|_2 \) satisfies the condition (iii) in Theorem 1 with respect to \( \mu_a \).

(ii) \( \| \cdot \|_3 \) satisfies (iii) in Theorem 1 with respect to \( \mu_b \).

**Remark 3**

This theorem implies that Gross’ condition ((ii) in Theorem 1) is strictly stronger than Yan-Gong’s condition (iii) and (iv) in Theorem 1.

**Proof**

Let \( P_n \) be the orthogonal projection from \( \ell^2 \) onto the linear span of \( \{ e_1, e_2, \ldots, e_{2n+1} \} \). Clearly \( P_n \) strongly converges to \( I \).

For \( n > m \), we have

\[
\mu_a \circ (P_n - P_m)^{-1} = (P_n - P_m)(\mu_a) = \delta_{e_{2m+2} + \ldots + e_{2n+1}}
\]

Now we show that for any \( \varepsilon > 0 \)

\[
\lim_{n,m \to \infty} \mu_a(\{ x \in \ell^2; \| P_n x - P_m x \|_2 \leq \varepsilon \}) = 1.
\]

It is enough to prove that for any \( \varepsilon > 0 \) exists \( n_0 \in \mathbb{N} \) such that \( \| e_{2m+2} + \ldots + e_{2n+1} \|_2 \leq \varepsilon \) for every \( n > m \geq n_0 \).

Put \( k = \frac{1}{\beta_{2m+1} + \beta_{2n+1}} \), we have

\[
k(e_{2m+2} + \ldots + e_{2n+1}) = k(e_1 + \ldots + e_{2n+1}) - k(e_1 + \ldots + e_{2m+1}) = \frac{k}{\beta_{2n+1}}\{\beta_{2n+1}(e_1 + \ldots + e_{2n+1})\} + \frac{k}{\beta_{2m+1}}\{-\beta_{2m+1}(e_1 + \ldots + e_{2m+1})\}.
\]

It is sure that \( \frac{k}{\beta_{2m+1}} + \frac{k}{\beta_{2n+1}} = 1 \).

Resure that \( \Gamma_2 \) is the convex hull of \( \{ \pm \beta_{2j+1}(e_1 + \ldots + e_{2j+1}); j = 1, 2, \ldots \} \) and so \( k(e_{2m+2} + \ldots + e_{2n+1}) \in \Gamma_2 \subset U_2 \).

This means

\[
\| e_{2m+2} + \ldots + e_{2n+1} \|_2 \leq \frac{1}{k} = \frac{1}{\beta_{2m+1}} + \frac{1}{\beta_{2n+1}}.
\]

For arbitrary \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \frac{2}{\beta_{2n_0+1}} \leq \varepsilon \), so we have

\[
\| e_{2m+2} + \ldots + e_{2n+1} \|_2 \leq \frac{2}{\beta_{2n_0+1}} \leq \varepsilon
\]

This completes the proof.

The case (ii) is shown by similar methods of the above proof (see [6]). \( \square \)
The following theorem shows the connection between the cylindrical measure \( \mu_a \), and the norms \( \| \cdot \|_1, \| \cdot \|_3 \) and \( \| \cdot \|_4 \).

**Theorem 4** The norms \( \| \cdot \|_1, \| \cdot \|_3 \) and \( \| \cdot \|_4 \) satisfies the condition (iii) in Theorem 1 for the cylindrical measure \( \mu_a \).

**Proof** First we show the case of \( \| \cdot \|_1 \). Let \( P_n \) be the orthogonal projection from \( \ell^2 \) onto the linear span of \( \{e_1, e_2, \ldots, e_n\} \). \( P_n \) converges strongly to \( I \) as \( n \to \infty \).

For \( n > m \), \( (P_n - P_m) \) is the orthogonal projection onto the linear span of \( \{e_{m+1}, \ldots, e_n\} \).

Put \( k = \frac{1}{\alpha_m + \alpha_n} \).

After is similar to the proof of Theorem 3.

Second we consider the case of \( \| \cdot \|_4 \).

Let \( P_n \) be the same as above.

For any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that

\[
\sum_{j=m+1}^{n} (\frac{1}{j})^2 \leq \varepsilon^2 \text{ for every } n > m \geq N.
\]

Since \( (P_n - P_m) \mu_a = \delta_{e_{m+1} + \cdots + e_n} \) and \( \|e_{m+1} + \cdots + e_n\|_4 = \sqrt{\sum_{j=m+1}^{n} (\frac{1}{j})^2} \leq \varepsilon \), we have

\[
\mu_a(\{x \in \ell^2 : \|P_n x - P_m x\|_4 \leq \varepsilon \}) = \delta_{e_{m+1} + \cdots + e_n}(\{y \in (P_n - P_m)\ell^2 : \|y\|_4 \leq \varepsilon \}) = 1
\]

This means that \( \| \cdot \|_4 \) satisfies the condition (iii) for \( \mu_a \).

Now check that \( \|x\|_3 \leq \|x\|_4 \) for all \( x \in \ell^2 \). Then \( \| \cdot \|_3 \) satisfies (iii) for \( \mu_a \), too. \( \square \)

We close this section by the next Theorem.

**Theorem 5** \( \| \cdot \|_4 \) is not \( \mu_b - (D) \) measurable.

**Proof** It is sufficient to show that there exists a positive number \( \varepsilon_0 \) satisfying that any \( G \in FD(\ell^2) \) there exists \( F \in FD(\ell^2) \) such that \( F \perp G \) and \( \mu_b(F \cap N_{\varepsilon_0}(P_F(G))) < 1 - \varepsilon_0 \), where \( N_{\varepsilon_0} = \{x \in \ell^2 : \|x\|_4 \leq \varepsilon_0 \} \).

Let \( \varepsilon_0 = \frac{1}{2} \) be given.

Let \( G \) be an arbitrary finite dimensional subspace of \( \ell^2 \), and \( \{\xi^i\}_{i=1,2,\ldots,n} \) be a CONS of \( G \). Then each \( \{\xi^i\} \) is of the form \( \xi^i = \sum_{j=1}^{\infty} \alpha^j_i e_i \) where \( \alpha^j_i \in \mathbb{R} \) for \( j = 1,2,\ldots,n \) and \( i = 1,2,\ldots,n \).

Then we have the following matrix \( A \):

\[
A = \begin{pmatrix}
\alpha^1_1 & \ldots & \alpha^1_n & \ldots & \alpha^1_{n+m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha^n_1 & \ldots & \alpha^n_n & \ldots & \alpha^n_{n+m}
\end{pmatrix}
\]

where \( m \) is chosen such that \( \text{rank } A = n \). Suppose \( N > n + m \). Then the next equation has its solution in \( \mathbb{R}^{n+m} \).

\[
A \begin{pmatrix}
x_1 \\
\vdots \\
x_n \\
x_{n+m}
\end{pmatrix} = \begin{pmatrix}
-\alpha^1_N \\
\vdots \\
-\alpha^n_N
\end{pmatrix} \quad (\ast)
\]
By construction we know that $\alpha_j^i \to 0$ as $i \to \infty$ for $j = 1, 2, \ldots, n$. Therefore, for every $\delta > 0$, we may choose a positive integer $N(> n + m)$, $N$ sufficiently large, such that the equation (*) has the solution $x_1 = \eta_1, \ldots, x_{n+m} = \eta_{n+m}$ satisfying

$$\max_{1 \leq l \leq n+m} |\eta_l| < \delta \ldots (**).$$

Let $\tau = \eta_1 e_1 + \ldots + \eta_{n+m} e_{n+m} + e_N$ and $F$ be the one dimensional subspace of $\ell^2$ generated by $\tau$. Then $F \perp G$.

Upon putting $\phi = \frac{\tau}{|\tau|}$, where $|\cdot|$ is the original norm of $\ell^2$, we obtain

$$(b, \phi) = \frac{(b, \tau)}{|\tau|} = (\eta_1 + 2\eta_2 + \ldots + (n + m)\eta_{n+m} + N)/|\tau|$$

We have to show that $(b, \phi) \phi \not\in P_F(N_{\varepsilon_0})$. Suppose that $(b, \phi) \phi \in P_F(N_{\varepsilon_0})$.

Let $\delta < \frac{1}{n+m}$. If $(b, \phi) \phi \in P_F(N_{\varepsilon_0})$, then there exists $x_0 \in N_{\varepsilon_0}$ such that $P_F x_0 = (b, \phi) \phi$.

Now for any $x \in N_{\varepsilon_0}$, we have

$$|\eta_1| < x, e_1 > < \delta \varepsilon_0,$$
$$|\eta_2| < x, e_2 > < 2\delta \varepsilon_0,$$
$$\vdots$$
$$|\eta_{n+m}| < x, e_{n+m} > < (n + m)\delta \varepsilon_0$$

and

$$|< x, e_N > | < N \varepsilon_0.$$

Then

$$|< \tau, x >| = |\eta_1 < x, e_1 > + \ldots + \eta_{n+m} < x, e_{n+m} > + < x, e_N > |$$
$$\leq |\eta_1| |< x, e_1 >| + \ldots + |\eta_{n+m}| |< x, e_{n+m} >| + |< x, e_N > |$$
$$< \varepsilon_0 + \varepsilon_0 N$$
$$= (1 + N)\varepsilon_0$$
$$= \frac{1}{2}(1 + N)$$

On the other hand, we have

$$|< \tau, x >| = |\eta_1 + 2\eta_2 + \ldots + (n + m)\eta_{n+m} + N|$$
$$\geq N - |\eta_1| - 2|\eta_2| - \ldots - (n + m)|\eta_{n+m}|$$
$$> N - \{1 + 2 + \ldots + (n + m)\} \delta$$
$$> N - \frac{1}{2} \frac{n + m + 1}{n + m}$$
$$> N - 1$$

This means contradiction. \hfill \Box

**Appendix**

By the above results and [8] and [12], we have the following table.

The sign ○ means the norm satisfies the condition with respect to the cylindrical measure, the sign × means it does not satisfy and the blank space means indefiniteness.
<table>
<thead>
<tr>
<th>norm $| \cdot |_1$</th>
<th>cylindrical measure</th>
<th>(i)</th>
<th>(ii)</th>
<th>(iii)</th>
<th>(iv)</th>
<th>(v)</th>
<th>(vi)</th>
<th>(vii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$\mu_a$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$| \cdot |_2$</td>
<td>$\gamma$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$\mu_a$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$| \cdot |_3$</td>
<td>$\mu_a$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$| \cdot |_4$</td>
<td>$\mu_b$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$\mu_a$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$\mu_b$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
</tbody>
</table>

Most blanks except a few ones will be filled by $\bigcirc$ or $\times$ in the near future. It will appear elsewhere.

**References**


**Department of Mathematics, Faculty of Science, OCHANOMIZU University**

2-1-1, Otsuka, Bunkyo-ku, Tokyo112-8610, Japan

E-mail address : maedamic@cc.ocha.ac.jp