ON BH-RELATIONS IN BH-ALGEBRAS

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Abstract. As a generalization of a BH-homomorphism, the notion of a relation on BH-algebras, called a BH-relation, is introduced. Some fundamental properties related to BH-subalgebras are discussed.

1. Introduction

It is well-known that the class of BCH-algebras is a generalization of the class of BCK/BCI-algebras. It is important for us to generalize some algebraic structures. Jun, Roh and Kim [2] introduced a new notion, called a BH-algebra, which is a generalization of BCK/BCI/BCH-algebras. In this paper, we introduce the notion of a relation on BH-algebras, called a BH-relation, which is a generalization of a BH-homomorphism, and then we discuss the fundamental properties related to BH-subalgebras.

2. Preliminaries

A BH-algebra is a nonempty set $X$ with a constant 0 and a binary operation $*$ satisfying the following conditions:

(I) $x * x = 0$,

(II) $x * y = 0$ and $y * x = 0$ imply $x = y$

(III) $x * 0 = x$

for all $x,y \in X$. A nonempty subset $S$ of a BH-algebra $X$ is called a BH-subalgebra of $X$ if $x * y \in S$ for all $x,y \in S$. A nonempty subset $J$ of a BH-algebra $X$ is called a BH-ideal of $X$ if it satisfies

• $0 \in J$,

• $\forall x,y \in X$, $x * y \in J$, $y \in J \Rightarrow x \in J$.

A mapping $f : X \rightarrow Y$ of BH-algebras is called a BH-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x,y \in X$. Note that if $f : X \rightarrow Y$ is a BH-homomorphism, then $f(0_X) = 0_Y$, where $0_X$ and $0_Y$ are constants of $X$ and $Y$, respectively.

3. BH-relations

Definition 3.1. Let $X$ and $Y$ be BH-algebras. A nonempty relation $\mathcal{H} \subseteq X \times Y$ is called a BH-relation if

(R1) for every $x \in X$ there exists $y \in Y$ such that $x \mathcal{H} y$,

(R2) $x \mathcal{H} a$ and $y \mathcal{H} b$ imply $(x * y) \mathcal{H} (a * b)$.

We usually denote such relation by $\mathcal{H} : X \rightarrow Y$. It is clear from (R1) and (R2) that $0_{X} \mathcal{H} 0_{Y}$. 

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Example 3.2. Consider a proper BH-algebra $X = \{0, a, b\}$ having the following Cayley table (see [2]):

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a relation $\mathcal{H} : X \to X$ as follows: $0\mathcal{H} 0$, $a\mathcal{H} a$, $b\mathcal{H} b$. It is easy to verify that $\mathcal{H}$ is a BH-relation. A relation $\mathcal{D} : X \to X$ given by $0\mathcal{D} 0$, $a\mathcal{D} a$, $a\mathcal{D} b$, $b\mathcal{D} 0$, and $b\mathcal{D} a$ is a BH-relation.

**Theorem 3.3.** Every BH-homomorphism is a BH-relation.

**Proof.** Let $\mathcal{H} : X \to Y$ be a BH-homomorphism. Clearly, $\mathcal{H}$ satisfies conditions (R1) and (R2).

Note that every diagonal BH-relation on a BH-algebra $X$ (i.e., a BH-relation satisfying $x\mathcal{H} x$ for all $x \in X$ in which $x\mathcal{H} y$ is false whenever $x \neq y$) is clearly a BH-homomorphism. But, in general, the converse of Theorem 3.3 need not be true as seen in the following example.

Example 3.4. The BH-relation $\mathcal{D}$ in Example 3.2 is not a BH-homomorphism.

Let $\mathcal{H} : X \to Y$ be a BH-relation. For any $x \in X$ and $y \in Y$, let

$$\mathcal{H}[x] := \{y \in H \mid x\mathcal{H} y\} \quad \text{and} \quad \mathcal{H}^{-1}[y] := \{x \in X \mid x\mathcal{H} y\}.$$ 

Note that $\mathcal{H}[x]$ and $\mathcal{H}^{-1}[y]$ are not BH-subalgebras of $Y$ and $X$, respectively, as seen in the following example:

Example 3.5. Let $\mathcal{H}$ be a BH-relation in Example 3.2(1). Then $\mathcal{H}^{-1}[b] = \{b\}$ (resp. $\mathcal{H}[a] = \{a\}$) is not a BH-subalgebra of $X$ (resp. $Y$).

**Theorem 3.6.** For any BH-relation $\mathcal{H} : X \to Y$, we have

(i) $\mathcal{H}[0_X]$, called the zero image of $\mathcal{H}$, is a BH-subalgebra of $Y$.

(ii) $\mathcal{H}^{-1}[0_Y]$, called the kernel of $\mathcal{H}$ and denoted by $\text{Ker}\mathcal{H}$, is a BH-subalgebra of $X$.

**Proof.** (i) Let $y_1, y_2 \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H} y_1$ and $0_X\mathcal{H} y_2$. It follows from (R2) and (I) that $0_X\mathcal{H} (y_1 + y_2)$, that is, $y_1 + y_2 \in \mathcal{H}[0_X]$.

(ii) Let $x_1, x_2 \in \text{Ker}\mathcal{H}$. Then $x_1\mathcal{H} 0_Y$ and $x_2\mathcal{H} 0_Y$. By using (R2) and (I), we get $(x_1 \ast x_2)\mathcal{H} 0_Y$ and so $x_1 \ast x_2 \in \text{Ker}\mathcal{H}$. This completes the proof.

**Proposition 3.7.** Let $\mathcal{H} : X \to Y$ be a BH-relation.

(i) If $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$ where $a, b \in X$, then $a \ast b \in \text{Ker}\mathcal{H}$.

(ii) If $\mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v] \neq \emptyset$ where $u, v \in Y$, then $u \ast v \in \mathcal{H}[0_X]$.

**Proof.** (i) Let $a, b \in X$ be such that $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$. Taking $y \in \mathcal{H}[a] \cap \mathcal{H}[b]$, we have $a\mathcal{H} y$ and $b\mathcal{H} y$. It follows from (R2) and (I) that $(a \ast b)\mathcal{H} (y \ast y) = (a \ast b)\mathcal{H} 0_Y$, so that $a \ast b \in \text{Ker}\mathcal{H}$.

(ii) Let $x \in \mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v]$. Then $x\mathcal{H} u$ and $x\mathcal{H} v$. Using (R2) and (I), we obtain $(x \ast x)\mathcal{H} (u \ast v) = 0_X\mathcal{H} (u \ast v)$, i.e., $u \ast v \in \mathcal{H}[0_X]$. This completes the proof.

**Theorem 3.8.** Let $\mathcal{H} : X \to Y$ be a BH-relation and let $S$ be a BH-subalgebra of $X$. Then

$$\mathcal{H}[S] := \{y \in H \mid x\mathcal{H} y \text{ for some } x \in S\}$$

is a BH-subalgebra of $Y$. 

Proof. Clearly, $\mathcal{H}[S] \neq \emptyset$ since $0_X \mathcal{H} 0_Y$. Let $y_1, y_2 \in \mathcal{H}[S]$. Then $x_1 \mathcal{H} y_1$ and $x_2 \mathcal{H} y_2$ for some $x_1, x_2 \in S$. Using (R2), we obtain $(x_1 \ast x_2) \mathcal{H} (y_1 \ast y_2)$ which implies that $y_1 \ast y_2 \in \mathcal{H}[S]$ since $x_1 \ast x_2 \in S$. Therefore $\mathcal{H}[S]$ is a BH-subalgebra of $Y$. \qed

Corollary 3.9. Let $\mathcal{H} : X \to Y$ be a BH-relation. Then

(i) $\mathcal{H}[X]$ is a BH-subalgebra of $Y$.
(ii) $\mathcal{H}[X] = \bigcup_{x \in X} \mathcal{H}[x]$.
(iii) The zero image of $\mathcal{H}$ is a BH-subalgebra of $\mathcal{H}[X]$.

Proof. (i) and (ii) are straightforward.
(iii) Let $a, b \in \mathcal{H}[0_X]$. Then $0_X \mathcal{H} a$ and $0_X \mathcal{H} b$, and hence $0_X \mathcal{H} (a \ast b)$, i.e., $a \ast b \in \mathcal{H}[0_X]$. Therefore $\mathcal{H}[0_X]$ is a BH-subalgebra of $\mathcal{H}[X]$. \qed

For any BH-relation $\mathcal{H} : X \to Y$, we know that there is a BH-ideal $J$ of $X$ in which $\mathcal{H}[J]$ is not a BH-ideal of $Y$. Indeed, consider the BH-relation $\mathcal{D}$ in Example 3.2. Note that $J := \{0, 2\}$ is a BH-ideal of $X$, but $\mathcal{H}[J] = \{0, 1\}$ is not a BH-ideal of $X$.

Theorem 3.10. Let $\mathcal{H} : X \to Y$ be a BH-relation and let $T$ be a BH-subalgebra of $Y$. Then

$$\mathcal{H}^{-1}[T] := \{x \in X \mid x \mathcal{H} y \text{ for some } y \in T\}$$

is a BH-subalgebra of $X$.

Proof. Obviously, $\mathcal{H}^{-1}[T] \neq \emptyset$ since $0_X \mathcal{H} 0_Y$. Let $x_1, x_2 \in \mathcal{H}^{-1}[T]$. Then there exist $y_1, y_2 \in T$ such that $x_1 \mathcal{H} y_1$ and $x_2 \mathcal{H} y_2$. Note that $y_1 \ast y_2 \in T$ since $T$ is a subalgebra of $Y$. It follows from (R2) that $(x_1 \ast x_2) \mathcal{H} (y_1 \ast y_2)$ so that $x_1 \ast x_2 \in \mathcal{H}^{-1}[T]$. Hence $\mathcal{H}^{-1}[T]$ is a BH-subalgebra of $X$. \qed

Corollary 3.11. Let $\mathcal{H} : X \to Y$ be a BH-relation. Then

(i) $\mathcal{H}^{-1}[Y]$ is a BH-subalgebra of $X$.
(ii) $\mathcal{H}^{-1}[Y] = \bigcup_{y \in Y} \mathcal{H}^{-1}[y]$.
(iii) The kernel of $\mathcal{H}$ is a BH-subalgebra of $\mathcal{H}^{-1}[Y]$.

Proof. (i) and (ii) are straightforward.
(iii) Let $x, y \in \text{Ker}\mathcal{H}$. Then $x \mathcal{H} 0_Y$ and $y \mathcal{H} 0_Y$. It follows from (R2) and (I) that

$$(x \ast y) \mathcal{H} (0_Y \ast 0_Y) = (x \ast y) \mathcal{H} 0_Y$$

so that $x \ast y \in \text{Ker}\mathcal{H}$. Hence Ker$\mathcal{H}$ is a BH-subalgebra of $\mathcal{H}^{-1}[Y]$. This completes the proof. \qed

Open Problem 3.12. In Theorem 3.10, if $T$ is a BH-ideal of $Y$, then is $\mathcal{H}^{-1}[T]$ a BH-ideal of $X$?

REFERENCES
