INFINITE PRODUCT PROBLEMS ON $\delta\theta$-REFINABLE SPACES

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Abstract. Suppose that $X=\prod_{\alpha<\omega}X_{\alpha}$, if each space $\prod_{\alpha<\omega}X_{\alpha}$ is $\delta\theta$-refinable (i.e., submetaindelsity), is $X$ also $\delta\theta$-refinable? K.Chiba asked in [1]. This paper first show that an inverse limit theorem for $\delta\theta$-refinable spaces. Using this, we obtain the result: Let $X=\prod_{\alpha \in \Lambda}X_{\alpha}$ be $|\Lambda|$-paracompact, $X$ is $\delta\theta$-refinable if $\prod_{\alpha \in F}X_{\alpha}$ is $\delta\theta$-refinable for each $F \in [\Lambda]^{<\omega}$. Then, the above problem is answered positively. Next, we show that there are similar results on hereditarily $\delta\theta$-refinable spaces.

In the paper [1], K.Chiba asked: Suppose that $X=\prod_{\alpha<\omega}X_{\alpha}$, if each space $\prod_{\alpha<\omega}X_{\alpha}$ is $\delta\theta$-refinable (i.e., submetaindelsity), is $X$ also $\delta\theta$-refinable? This paper first prove respectively the following:

**Theorem 1.** Let $X$ be the inverse limit of an inverse system $\{X_{\alpha}, \pi_{\alpha}\}$ and let the projection $\pi_{\alpha}$ be an open and onto map for each $\alpha \in \Lambda$. If $X$ is $|\Lambda|$-paracompact and each $X_{\alpha}$ is $\delta\theta$-refinable, then $X$ is $\delta\theta$-refinable.

**Theorem 2.** Let $X$ be the inverse limit of an inverse system $\{X_{\alpha}, \pi_{\alpha}\}$ and let the projection $\pi_{\alpha}$ be an open and onto map for each $\alpha \in \Lambda$. If $X$ is hereditarily $|\Lambda|$-paracompact and each $X_{\alpha}$ is hereditarily $\delta\theta$-refinable, then $X$ is also hereditarily $\delta\theta$-refinable.

Using the above, we obtain the results:

**Theorem 3.** Let $X=\prod_{\alpha \in \Lambda}X_{\alpha}$ be $|\Lambda|$-paracompact (resp. hereditarily $|\Lambda|$-paracompact), $X$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) if $\prod_{\alpha \in F}X_{\alpha}$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) for each $F \in [\Lambda]^{<\omega}$.

Therefore, the following holds trivially:

**Theorem 4.** Let $X=\prod_{i \in \Gamma}X_{i}$ be countable paracompact (resp. hereditarily countable paracompact), then the following are equivalent:

1. $X$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable).
2. $\prod_{i \in F}X_{i}$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) for each $F \in [\Gamma]^{<\omega}$.
3. $\prod_{i \in \omega}X_{i}$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) for each $n \in \omega$.

We use that $N_{Y}(x)$ denotes the neighbourhood system of a point $x$ of a subspace $Y$ of a space $X$. Equally, $N(x)$ denotes $N_{Y}(x)$ when $Y=X$. $|[\Lambda]|$, $d_{\Lambda}$, Int$A$ and $A'$ denote respectively the cardinality, the closure, the interior and the complementary set of a set $A$; $(U)x$, $(U)|_{A}$ and $\bigwedge_{n \in F}H_{x}$ denote respectively $\{U \in \mathcal{U}: x \in U\}$, $\{U \cap A: U \in \mathcal{U}\}$ and $\{\bigwedge_{n \in F}H_{x}: H_{x} \in H_{x}\}$; $\omega$ and $[\omega]^{<\omega}$ denote respectively, the first infinite ordinal number and the collection of all non-empty finite subsets of a non-empty set $\omega$. And assume that all spaces are Hausdorff spaces throughout this paper.

**Definition 1.** Let $\kappa$ be a cardinal number, $A$ is $\kappa$-paracompact if its every open cover $\mathcal{U}$ of cardinal $|\mathcal{U}| \leq \kappa$ has a locally finite open refinement; $A$ is $[\omega]^{\omega}$-paracompact if it is $\kappa$-paracompact, where $\kappa = |\omega|$.

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Definition 2. A space $X$ is said to be $\delta\theta$-refinable (submetalindefel) if its every open cover $\mathcal{U}$ has a sequence $(G_n)_{n \in \omega}$ of open refinements such that for every $x \in X$ there is an $n \in \omega$ with $\text{ord}(x,G_n) \leq \omega$; a space $X$ is said to be weakly $\delta\theta$-refinable if its every open cover $\mathcal{U}$ has an open refinement $G = \bigcup_{n \in \omega} G_n$ such that for every $x \in X$ there is an $n \in \omega$ such that $1 \leq \text{ord}(x,G_n) \leq \omega$.

Lemma 1. Let $\lambda$ be a cardinal number. Suppose $X$ is $\lambda$-paracompact, $\lambda$ is a directed set with $|\lambda| = \lambda$, and $\mathcal{H} = \{H_\alpha : \alpha \in \lambda\}$ is an open cover of $X$ such that $H_\alpha \subseteq H_\beta$ for each $\alpha, \beta \in \lambda$ satisfying $\alpha \leq \beta$. Then there is an open cover $K = \{K_\alpha : \alpha \in \lambda\}$ of $X$ such that $\text{cl}K_\alpha \subseteq H_\alpha$ for each $\alpha \in \lambda$ and $K_\alpha \cap K_\beta$ for each $\alpha, \beta \in \lambda$ satisfying $\alpha \leq \beta$.

Lemma 2. A space $X$ is hereditarily $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable) iff each open subspace of $X$ is $\delta\theta$-refinable (resp. weakly $\delta\theta$-refinable).

This lemma is a direct result of Definition 2. Now we prove main theorems of this paper.

Proof of Theorem 1. Let $\mathcal{U} = \{U_\xi : \xi \in \Xi\}$ be an arbitrary open cover of $X$. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let us put

$$V_{\alpha,\xi} = \{V : V \subseteq X_\alpha \text{ and } \pi^{-1}_\alpha(V) \subseteq U_\xi\},$$

and put $V_\alpha = \bigcup\{V_{\alpha,\xi} : \xi \in \Xi\}$, then

1. $\bigcup\{\pi^{-1}_\alpha(V_{\alpha,\xi}) : \xi \in \Xi\} = V_\alpha$, and $\pi^{-1}_\beta(V_{\alpha,\xi}) \subseteq \pi^{-1}_\beta(V_{\alpha,\beta})$ if $\alpha \leq \beta$.

Since $X$ is $|\Lambda|$-paracompact, there is an open cover $\{W_\alpha : \alpha \in \Lambda\}$ of $X$ such that

2. $\text{cl}W_\alpha \subseteq \pi^{-1}_\beta(V_{\alpha,\xi})$ for each $\alpha \in \Lambda$, and $W_\alpha \subseteq W_\beta$ if $\alpha \leq \beta$.

For each $\alpha \in \Lambda$, let us put $T_\alpha = X_\alpha - \pi^{-1}_\alpha(X - \text{cl}W_\alpha)$, then $T_\alpha$ is closed in $X_\alpha$ because $\pi^{-1}_\alpha$ is an open map. Again let $C_\alpha = \text{Int}T_\alpha$ for each $\alpha \in \Lambda$.

3. $\{C_\alpha : \alpha \in \Lambda\}$ is an open cover of $X$.

In fact, for each $x \in X$ there is $\alpha \in \Lambda$ such that $x \in W_\alpha$. There are some $\beta \in \Lambda$ and some open set $V$ in $X_\beta$ such that $x \in \pi^{-1}_\beta(V) \subseteq W_\alpha$ since $W_\alpha$ is open in $X$. We choose a $\gamma \in \Lambda$ satisfying $\gamma \geq \alpha$ and $\gamma \geq \beta$, then $x \in C_\gamma$ because $\pi^{-1}_\beta(V) \subseteq \pi^{-1}_\gamma(T_\gamma)$. To show this, let $y = (x_\delta)_{\delta \in \lambda} \in \pi^{-1}_\gamma(V) - \pi^{-1}_\gamma(T_\gamma)$, then $\gamma \in Y \subseteq V$ and $y_\gamma \in \pi^{-1}_\gamma(X - \text{cl}W_\gamma)$, i.e., there is an element $z = (z_\delta)_{\delta \in \lambda} \in X - \text{cl}W_\gamma$ such that $y_\gamma = \pi_\gamma(z) = z_\gamma$, $y_\beta = \pi_\beta(z_\beta) \subseteq V$, $x \in \pi^{-1}_\beta(V) = \pi^{-1}_\gamma(V) \subseteq W_\alpha$, then $x \in W_\alpha$. This is a contradiction.

By $|\Lambda|$-paracompactness of $X$, there is a locally finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of $X$ such that $O_\alpha \subseteq C_\alpha$ for each $\alpha \in \Lambda$. Since $T_\alpha \subseteq V_\alpha = \bigcup\{V_{\alpha,\xi} : \xi \in \Xi\}$ and $T_\alpha$ is closed in $X_\alpha$, then there is a sequence $(G_n(\alpha))_{n \in \omega}$ of open sets of $X_\alpha$, satisfying

4. Each $G_n(\alpha)$ is a part refinement of $\{V_{\alpha,\xi} : \xi \in \Xi\}$ and $T_\alpha \subseteq \bigcup G_n(\alpha)$ for each $n \in \omega$.

5. For each $x \in T_\alpha$ there is an $n \in \omega$ such that $\text{ord}(x,G_n(\alpha)) \leq \omega$ and $G_n \cap G_m \subseteq G_n(\alpha)$ if $G_1 \cap G_2 \subseteq G_n(\alpha)$ if $G_1, G_2 \subseteq G_n(\alpha)$.

For each $n \in \omega$, let $H_n = \{\pi^{-1}_\alpha(G) \cap \text{cl}(\bigcup_{\alpha \in \Lambda} G_n(\alpha)) : \alpha \in \Lambda\}$, then

6. $H_n$ is an open refinement of $\mathcal{U}$ for each $n \in \omega$.

In fact, for each $x \in X$, there is $\alpha \in \Lambda$ such that $x \in \text{cl}(\bigcup_{\alpha \in \Lambda} G_n(\alpha))$ and there is $G \subseteq G_n(\alpha)$ such that $x \in \pi^{-1}_\alpha(G) \cap \bigcup_{\alpha \in \Lambda} G_n(\alpha)$, i.e., $H_n$ is a cover of $X$. Again since for each $\alpha \in \Lambda$ and each $G \subseteq G_n(\alpha)$ there is some $\xi(G) \in \Xi$ such that $G \subseteq V_{\alpha,\xi(G)}$, then $\pi^{-1}_\alpha(G) \cap \bigcup_{\alpha \in \Lambda} G_n(\alpha)$.

Finally, we prove:

8. For each $x \in X$, there is a $F \subseteq [\omega]^{<\omega}$ such that $\text{ord}(x,H_F) \leq \omega$.

Let $x \in X$, since $\{O_\alpha : \alpha \in \Lambda\}$ is a locally open cover of $X$, a nonempty finite set. And for each $\alpha \in \Delta$, since $x \in \text{cl}(\bigcup_{\alpha \in \Lambda} G_n(\alpha))$, there is some $n_\alpha \in \omega$ such that $\text{ord}(x,G_n(\alpha)) \leq \omega$. Put $F = \{n_\alpha : \alpha \in \Delta\}$ and let $G_n^{-1}(\alpha) = \{\pi^{-1}_\alpha(G) \cap \bigcup_{\alpha \in \Lambda} G_n(\alpha) : \alpha \in \Delta\}$, then

$$(H_F)_x \subseteq \bigcap_{\alpha \in \Delta}(\bigcup_{n_\alpha \in \omega} G_n^{-1}(\alpha))_x \text{ and } \Delta' \in [\omega]^{<\omega}$$
Therefore, \(\text{ord}(x, \mathcal{H}_F) \leq \omega\). □

**Proof of Theorem 2.** Let \(\mathcal{U} = \{U_\xi : \xi \in \Xi\}\) be an open cover of open subspace \(Y\) of \(X\). For each \(\alpha \in \Lambda\) and each \(\xi \in \Xi\), we put \(V_{\alpha \xi} = \bigcup \{V : V \text{ is in } X_{\alpha} \text{ and } \pi_{\alpha}^{-1}(V) \subseteq U_\xi\}\) and \(V_{\alpha} = \bigcup \{V_{\alpha \xi} : \xi \in \Xi\}\), then

1. \(\{\pi_{\alpha}^{-1}(V_{\alpha}) : \alpha \in \Lambda\}\) is an open cover of \(Y\) and \(\pi_{\alpha}^{-1}(V_{\alpha}) \subseteq \pi_{\beta}^{-1}(V_{\beta})\) if \(\alpha \leq \beta\).

Since \(X\) is hereditarily \(|\cdot|\)-paracompact, the open cover \(\{\pi_{\alpha}^{-1}(V_{\alpha}) : \alpha \in \Lambda\}\) of \(Y\) has an open refinement \(\{W_\alpha : \alpha \in \Lambda\}\) such that

2. \(dW_\alpha \subseteq \pi_{\alpha}^{-1}(V_{\alpha})\) for each \(\alpha \in \Lambda\), and \(W_\alpha \subseteq W_\beta\) if \(\alpha \leq \beta\).

For each \(\alpha \in \Lambda\), put \(E_\alpha = \bigcup \{E : E\text{ is open in } X_{\alpha} \text{ and } \pi_{\alpha}^{-1}(E) \subseteq W_\alpha\}\), then

3. \(\pi_{\alpha}^{-1}(E_\alpha) \subseteq W_\alpha\) for each \(\alpha \in \Lambda\) and \(\pi_{\alpha}^{-1}(E_\alpha) \subseteq \pi_{\beta}^{-1}(E_\beta)\) if \(\alpha \leq \beta\).

Now, we assert that

4. \(\{\pi_{\alpha}^{-1}(E_\alpha) : \alpha \in \Lambda\}\) is an open cover of \(Y\).

In fact, for each \(x \in Y\) there is a \(\alpha \in \Lambda\) such that \(x \in W_\alpha\). There are some \(\beta \in \Lambda\) and some open set \(V\) in \(X_{\beta}\) such that \(x \in \pi_{\beta}^{-1}(V) \subseteq W_\alpha\) by [4, Theorem 2.5.5]. Let us put \(\gamma \in \Lambda\) such that both \(\gamma \geq \alpha\) and \(\alpha \geq \beta\), then \(x \in \pi_{\beta}^{-1}(V) = (\pi_{\beta}^{-1}(\pi_{\gamma})^{-1}(V)) = (\pi_{\gamma})^{-1}(V) \subseteq W_\alpha \subseteq W_\gamma\), from which \(x \in \pi_{\gamma}^{-1}(E_\gamma)\).

Put \(F_\alpha = \text{cl}(E_\alpha) \cap \bigcap \{\text{cl}(V_{\alpha \xi}) : \xi \in \Xi\}\) for each \(\alpha \in \Lambda\), we assert that

5. \(\pi_{\alpha}^{-1}(F_\alpha) \cap Y = \phi\).

Indeed, if there is some \(x = (x_\alpha)_{\alpha \in \Lambda} \in \pi_{\alpha}^{-1}(F_\alpha) \cap Y\), then \(x_\alpha \in F_\alpha \subseteq \text{cl}(V_{\alpha \xi}) \subseteq V_{\alpha \xi}\). Since \(x_\alpha \not\in \pi_{\alpha}^{-1}(E_\alpha)\) we have \(x \in \pi_{\alpha}^{-1}(E_\alpha) \subseteq F_\alpha\). To prove this, let us put \(H = \pi_{\alpha}^{-1}(E_\alpha) \cap \mathcal{H}_F\), then there are some \(\beta \in \Lambda\) and some open set \(V\) in \(X\) such that \(x \in \pi_{\beta}^{-1}(V) \subseteq H\). Pick \(\gamma \geq \alpha\), \(\gamma \geq \beta\) and let \(V' = (\pi_{\beta}^{-1}(V))\), then

\[
x \in \pi_{\gamma}^{-1}(V') = \pi_{\gamma}^{-1}(\pi_{\beta}^{-1}(V)) = (\pi_{\gamma}^{-1}(\pi_{\beta}^{-1}(V))) = \pi_{\gamma}^{-1}(V) \subseteq H.
\]

Since \(x_\alpha \in F_\alpha \subseteq \text{cl}(E_\alpha)\) and \(x_\beta \in V'\), then \(\pi_{\gamma}^{-1}(V') \cap E_\alpha \neq \phi\). Let us put \(b \in \pi_{\gamma}^{-1}(V') \cap E_\alpha\), then there is \(c \in V'\) such that \(\pi_{\gamma}^{-1}(c) = b\). There is \(y_\gamma = (y_\alpha)_{\alpha \in \Lambda} \in X\) such that \(y_\gamma = \pi_{\gamma}(y) = c\).

6. \((X_{\alpha}, F_\alpha) \cap \text{cl}(E_\alpha) \subseteq \bigcup \{V_{\alpha \xi} : \xi \in \Xi\}\) for each \(\alpha \in \Lambda\).

In fact, for each \(x \in (X_{\alpha}, F_\alpha) \cap \text{cl}(E_\alpha)\), we have \(\pi_{\gamma}^{-1}(c) = b \in \pi_{\gamma}^{-1}(V') \cap E_\alpha\). Since \(t \in \text{cl}(V_{\alpha \xi}) \subseteq V_{\alpha \xi}\) and \(E_\alpha \subseteq V_{\alpha \xi}\), then \(t \in V_{\alpha \xi}\).

By \(\delta\)-refinability of \(X_{\alpha}, F_\alpha\), there is a sequence \(\langle G_\alpha(x) \rangle_{n \in \omega}\) of open covers of \((X_{\alpha}, F_\alpha) \cap \text{cl}(E_\alpha)\) such that

7. Each \(G_\alpha(x)\) is a fine cover of \(\{V_{\alpha \xi} : \xi \in \Xi\}\) and \(G_1 \cap G_2 \subseteq G_\alpha(x)\) if \(G_1, G_2 \subseteq G_\alpha(x)\)

8. For each \(x \in (X_{\alpha}, F_\alpha) \cap \text{cl}(E_\alpha)\) there is a \(n \in \omega\) such that \(\text{ord}(x, G_\alpha(x)) \leq \omega\).

Next, since \(X\) is hereditarily \(|\cdot|\)-paracompact, the open cover \(\{\pi_{\alpha}^{-1}(E_\alpha) : \alpha \in \Lambda\}\) of the subspace \(Y\) has a locally fine dense refinement \(\{O_\alpha : \alpha \in \Lambda\}\) such that \(O_\alpha \subseteq \pi_{\alpha}^{-1}(V_{\alpha \xi})\) for each \(\alpha \in \Lambda\)

Define \(\mathcal{H}_F = \{O_\alpha \cap \pi_{\alpha}^{-1}(G) : G \in G_\alpha(x), \alpha \in \Lambda\}\) and let \(\mathcal{H}_F = \bigwedge_{n \in \omega} \mathcal{H}_n\) for each \(F \in [\omega]^\omega\), then

9. Each \(\mathcal{H}_F\) is an open refinement of \(\mathcal{U}\).

In fact, for each \(x \in Y\) and each \(n \in \omega\), there is some \(\alpha \in \Lambda\) such that \(x \in O_\alpha \subseteq \pi_{\alpha}^{-1}(E_\alpha)\), then \(x_\alpha \in (X_{\alpha}, F_\alpha) \cap \text{cl}(E_\alpha)\). There is \(G_\alpha(x)\) such that \(x_\alpha \in G_\alpha(x), x_\alpha \in O_\alpha \subseteq \pi_{\alpha}^{-1}(G)\), i.e., \(\mathcal{H}_n\) is an open cover of \(Y\). Since for each \(G \in G_\alpha(x)\) there is \(\xi \in \Xi\) such that \(G \subseteq V_{\alpha \xi}\), then \(O_\alpha \subseteq \pi_{\alpha}^{-1}(G) \subseteq \pi_{\alpha}^{-1}(V_{\alpha \xi}) \subseteq V_{\alpha \xi}\), hence \(\mathcal{H}_F\) is an open refinement of \(\mathcal{U}\) for each \(F \in [\omega]^\omega\).

Finally, we assert that

10. For each \(x \in Y\), there is some \(F \in [\omega]^\omega\) such that \(\text{ord}(x, \mathcal{H}_F) \leq \omega\).
Let $x \in Y$, $\Delta = \{ \alpha \in \Lambda : x \in O_\alpha \}$ is an nonempty finite set. For each $\alpha \in \Delta$, $x \in O_\alpha \subseteq \pi^{-1}_\alpha(E_\alpha)$, we have $x_\alpha \in (X_\alpha \cap F_\alpha) \setminus E_\alpha$ by (5), there is some $n_\alpha \in \omega$ such that $\text{ord}(x, G_{n_\alpha}(\alpha)) \leq \omega$. Put $F = \{ n_\alpha : \alpha \in \Delta \}$, then $(\mathcal{H}_F x) \subset (\bigcap_{\alpha \in \Delta} O_\alpha \setminus G^{-1}_{n_\alpha}(\alpha)) x$ and $\Delta' \in [\Delta]^{<\omega}$, i.e., $\text{ord}(x, \mathcal{H}_F) \leq \omega$.

So, $X$ is a hereditarily $\delta\theta$-refine space. □

Now, we discuss Tychonoff products of infinite factors about both $\delta\theta$-refinable spaces and hereditarily $\delta\theta$-refinable spaces.

**Proof of Theorem 3.** ($\Leftarrow$) When $|\Lambda| < \omega$, it is obvious that $X = \prod_{\alpha \in \Lambda} X_\alpha$ is $\delta\theta$-refinable since $F = \Lambda \in [\Lambda]^{<\omega}$. Without the loss of generality, we suppose $|\Lambda| \geq \omega$. Define the relation $\leq$: $F \leq E$ if and only if $F \subseteq E$ for each $F, E \in [\Lambda]^{<\omega}$. Then $[\Lambda]^{<\omega}$ is a directed set on the relation $\leq$. Put $X_F = \prod_{\alpha \in F} X_\alpha$ for each $F \in [\Lambda]^{<\omega}$ and define the projection: $$\pi^F_F : X_F \to X_E$$ when $F \leq E$, where $\pi^F_F(x) = (x_\alpha)_{\alpha \in F}$ for each $x = (x_\alpha)_{\alpha \in E} \in X_E$.

It is easy to prove that $\pi^F_F$ is an open and onto map. $\{X_F, \pi^F_F, [\Lambda]^{<\omega} \}$ is an inverse system of spaces $X_F$ with bounding maps $\pi^F_F : X_E \to X_F$ when $E \subseteq F$.

Let $X'$ be the inverse limit of the inverse system $\{X_F, \pi^F_F, [\Lambda]^{<\omega} \}$, by [4, 2.5.3 Example], $X'$ is homeomorphic to $X = \prod_{\alpha \in \Lambda} X_\alpha$.

In other respects, since each $X_F = \prod_{\alpha \in F} X_\alpha$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable), the inverse system $\{X_F, \pi^F_F, [\Lambda]^{<\omega} \}$ satisfies the condition of Theorem 1. $X'$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). Therefore, so is $X = \prod_{\alpha \in \Lambda} X_\alpha$ also.

($\Rightarrow$) Assume that the product $X = \prod_{\alpha \in \Lambda} X_\alpha$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). For every $F \in [\Lambda]^{<\omega}$, let us put a point $x_\alpha \in X_\alpha$ when $\alpha \in \Lambda \setminus F$, then the closed subspace $X_F = \prod_{\alpha \in F} X_\alpha \times \prod_{\alpha \in \Lambda \setminus F} \{ x_\alpha \}$ of $X$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). Thus, $X_F$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). □

**Proof of Theorem 4.** The equivalence of (1) and (2) is direct by Theorem 3. (2)$\Rightarrow$(3) hold obviously. Now, we prove (3)$\Rightarrow$(2). In fact, for each $F \in [\Lambda]^{<\omega}$, let $m = \max F$ since $F \neq \emptyset$. We pick a fixed $x_\alpha \in X_\alpha$ when $\alpha \in \{0, 1, \ldots, m\} \setminus F$, then $\prod_{\alpha \in F} X_\alpha \times \prod_{\alpha \in \{0, 1, \ldots, m\} \setminus F} \{ x_\alpha \}$ is a closed set of $\prod_{\alpha \in \Lambda} X_\alpha$. So, $\prod_{\alpha \in F} X_\alpha$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). □

Finally, we point out that there are similar results about both weakly $\delta\theta$-refinable spaces and hereditarily weakly $\delta\theta$-refinable spaces.

**Corollary 1.** Let $X$ be the inverse limit of an inverse system $\{X_\alpha, \pi^*_\alpha, \Lambda\}$ and let the projection $\pi_\alpha$ be an open and onto map for each $\alpha \in \Lambda$. If $X$ is $[\Lambda]$-paracompact (resp. hereditarily $[\Lambda]$-paracompact) and each $X_\alpha$ is weakly $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable), then $X$ is weakly $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable).

**Proof.** We only prove the situation of weakly $\delta\theta$-refinable spaces, the Proof of hereditarily weakly $\delta\theta$-refinable spaces is similar to Theorem 2.

Let $\mathcal{U} = \{ U_\xi : \xi \in \Xi \}$ be an arbitrary open cover of $X$. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, the following are the same as the symbols in the proof of the above theorem: $V_{\alpha \xi}$, $V_\alpha$, $W_\alpha$, $T_\alpha$, $C_\alpha$ and $O_\alpha$. And there are the results which are same as (1)-(3) in Theorem 1.

Since $T_\alpha \subseteq V_\alpha = \bigcup \{ V_{\alpha \xi} : \xi \in \Xi \}$, there is an open cover $\bigcup_{n \in \omega} G_{n}(\alpha)$ of $T_\alpha$ such that

(4') For each $G \in \bigcup_{n \in \omega} G_{n}(\alpha)$, there is some $\xi \in \Xi$ such that $G \subseteq V_{\alpha \xi}$, and $G_1 \cap G_2 \subseteq G_{n}(\alpha)$ for each $G_1, G_2 \in G_{n}(\alpha)$

(5') For each $x \in T_\alpha$ there is some $n_\alpha \in \omega$ such that $1 \leq \text{ord}(x, G_{n_\alpha}(\alpha)) \leq \omega$.

For each $n \in \omega$ and each $F \in [\omega]^{<\omega}$, let us put $\mathcal{H}_n = \{ \pi^{-1}_n(G) \setminus O_\alpha : G \in G_{n}(\alpha) \}$ and $\alpha \in \Lambda$ and $\mathcal{H}_F = \bigwedge_{n \in F} \mathcal{H}_n$, then

(6') Each $\mathcal{H}_F$ is an open part refinement of $\mathcal{U}$.

Finally, we prove:

(7') For each $x \in X$ there is some $F \in [\omega]^{<\omega}$ such that $\text{ord}(x, \mathcal{H}_F) \leq \omega$. 

Let \( x \in X \), since \( \{ O_\alpha : \alpha \in \Lambda \} \) is a locally open cover of \( X \), \( \Delta = \{ \alpha \in \Lambda : x \in O_\alpha \} \) is an nonempty finite set. And for each \( \alpha \in \Delta \), since \( x \in \bigcap_{n=1}^{\omega} (T \alpha) \), then \( s_\alpha \in T \alpha \). There is some \( n_\alpha \in \omega \) such that \( 1 \leq \text{ord}(x, G_{n_\alpha}(\alpha)) \leq \omega \). Put \( F = \{ n_\alpha : \alpha \in \Delta \} \), then
\[
\phi \notin \mathcal{H}_F x \subset \bigcap_{n \in \Delta} \left( G_{n_\alpha}(\alpha) \right) x \text{ and } \Delta' \in [\Delta]^{<\omega}.
\]
So, \( 1 \leq \text{ord}(x, \mathcal{H}_F) \leq \omega \). □

**Corollary 2.** Let \( X = \prod_{\alpha \in \Lambda} X_\alpha \) be \( \Lambda \)-paracompact, \( X \) is \( \delta \theta \)-refinable (resp. \( \delta \theta \)-refinable) iff \( \prod_{\alpha \in F} X_\alpha \) is \( \delta \theta \)-refinable (resp. \( \delta \theta \)-refinable) for each \( F \in [\Sigma]^{<\omega} \).

**Corollary 3.** Let \( X = \prod_{i \in \omega} X_i \) is countable paracompact, then the following are equivalent:

1. \( X \) is weakly \( \delta \theta \)-refinable (resp. hereditarily weakly \( \delta \theta \)-refinable).
2. \( \prod_{i \in F} X_i \) is weakly \( \delta \theta \)-refinable (resp. hereditarily weakly \( \delta \theta \)-refinable) for each \( F \in [\Sigma]^{<\omega} \).
3. \( \prod_{i \in n} X_i \) is weakly \( \delta \theta \)-refinable (resp. hereditarily weakly \( \delta \theta \)-refinable) for each \( n \in \omega \).

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