Abstract. We consider a two-person zero-sum game where the players alternate their moves until each of them has made a total of $n$ moves. A move of either player consists of instructing a referee to move a chip either clockwise or counterclockwise to the next node around a three-node board. These three nodes are arranged in a circle and are labeled $+1$, $+2$ and $-3$. The main feature here is neither player is informed of any of his opponent’s past or current moves. Whenever the chip visits a node there is an intermediate payoff equal to the label on that node. The payoff is taken to be the sum of these intermediate payoffs at termination. Each player can remember all his own past moves and therefore may use a history of such moves to decide his next move. This game is solved for all positive integral values of $n$.

1 Introduction

In the early 1950’s Andrew Gleason of Harvard proposed an interesting two-person zero-sum game. This stochastic game with an information lag for both players has a very simple description but turns out to be quite difficult to solve. Ferguson and Shapley[2] described Gleason’s Game as follows. The two players move a chip around a three-node board (see Figure 1). The nodes are arranged in a circle, and are labeled $+1$, $+2$ and $-3$. Initially the chip rests on node $+1$ and player 1 starts. Thereafter, the players move alternately. There is a one move delay in informing the players of the position of the chip, so that, except for the first move, the players make their move only knowing the node from which the opponent has just moved. A move consists of instructing a referee to move the chip either clockwise or counterclockwise to the next node; the players are not allowed to leave the chip where it is. After each move is given to the referee, the referee announces the node that the chip has just left, and requires player 2 to pay player 1 an amount equal to the label of that node. The problem is for player 1 to maximize and for player 2 to minimize the limiting average payoff.

Ferguson and Shapley explained why Gleason’s Game seems easy but is actually hard: “When the referee announces the state just vacated, both players know the history of the game up to that point. Indeed, this information is common knowledge (both players know the other knows, both know the other knows he knows, etc.). At first sight, it might be thought that one needs not remember back past that point in choosing a strategy. This is
not so because when the opponent made his last move, he had to choose it not knowing the actual state, and you should be able to take advantage of that. And the opponent chose his strategy trying to take advantage of your lack of knowledge of the previous state, so that should be taken into account, and so on." They solved Gleason’s Game by first converting it to a stochastic game using the notion of generalized subgames. The functional equations associated with these generalized subgames are then solved with an iterative method that involves alternately solving a finite game with perfect information and a Markov decision problem with limiting average payoff. This leads to $-0.09336 < v < -0.09323$, where $v$ is the value of Gleason’s Game. It was also shown by them that no strategy that remembers only a bounded number of past moves can be optimal. Games in which a player needs to remember a history of all its past moves in order to play optimally are difficult to analyze, and only a very small number of them had ever been solved. The best-known example is undoubtedly the Big Match due to Blackwell and Ferguson[1].

In this paper we consider a game that differs from Gleason’s Game in terms of the information made available to the players. More specifically, we assume neither player is informed of any moves, past or current, made by the other player. It is surprising there exist optimal strategies that use a mixture of at most four pure strategies irrespective of the duration of the game. Games in which each player is not informed of any of the opponent’s moves are sometimes called games with no information. The published literature on games with no information is limited. Because of their nature, these games are almost exclusively certain search games where a mobile seeker searches a mobile hider in darkness over some specified region. A good example is the Princess and Monster Game; see, for example, Foreman[3], Gal[4], Garnaev[5], Wilson[8], and Worsham[9]. It is therefore of interest to find a multi-stage game with no information that is not a search game and that can be solved exactly.

2 The Game $\Gamma_n$ We consider the two-person zero-sum game $\Gamma_n$ where $n$ is any positive integer. A referee with the board shown in Figure 1 is stationed in one room. The players, called player 1 and player 2, are isolated from one another and also from the referee in another two rooms. Using an intercom, the referee can talk to either player. The players know $n$, how the board looks like, and that the chip is initially at node +1. Here is how the game proceeds. The referee calls player 1 and asks him to make his first move. Player 1 is only allowed to decide between moving clockwise or counterclockwise. If player 1 chooses clockwise (counterclockwise), the referee moves the chip clockwise (counterclockwise) from its initial position at node +1 to the next node on the board and records its label as the intermediate payoff. That is, the intermediate payoff is taken to be the label of the node that the chip visits as the result of a player’s move. For example, if the first move of player 1 is to move clockwise, the referee will move the chip to node +2, and he will record +2 as the intermediate payoff. The referee next calls player 2 and asks him to make his first move. Player 2 also has to decide between moving clockwise or counterclockwise. If player 2 chooses clockwise (counterclockwise), the referee moves the chip clockwise (counterclockwise) from its current position to the next node on the board. He then records the label of that node as another intermediate payoff. The referee then calls player 1 and asks him to make his second move. This process continues with the players moving alternately. While the game is still in progress the referee reveals nothing else to the players. The game terminates after each player has completed $n$ moves, making a total of $2n$ moves between the two players. After the game terminates, the payoff is taken to be the sum of the $2n$ intermediate payoffs recorded by the referee. The objective in $\Gamma_n$ is for player 1 to maximize and for player 2 to minimize this payoff.

There are certain differences between $\Gamma_n$ and Gleason’s Game. The former is a finite
game for each $n$ while the latter is an infinite game. We avoid matrix games with countably many pure strategies since such games exhibit several undesirable properties not found among finite games. Furthermore, since $n$ can be made arbitrarily large, nothing of significance is lost by considering finite games in our case. In Gleason’s Game, the payoff is the average of the intermediate payoffs, chosen out of necessity to avoid unbounded payoffs. Such a problem does not arise in $\Gamma_n$, so the payoff is chosen to be the sum of the intermediate payoffs for ease of exposition. The main difference between the two games occurs in how the information of the opponent’s moves is revealed. In Gleason’s Game, this information is given one move late while in $\Gamma_n$ it is not given at all.

Although a player has no access to his opponent’s moves we assume he can remember all his own past moves. Hence he may use a history of such moves to decide his next move.

3 A Solution of $\Gamma_n$ To obtain a solution, we first represent $\Gamma_n$ in extensive form as a tree with its information sets. In standard terminology, this representation is a game of perfect recall. We next examine the set of pure strategies, and reduce their number by using the reduced normal form of an extensive game. This method of reducing the number of pure strategies is due to Kuhn[6, 7]. After performing such reduction to our case, we are still left with $2^n$ pure strategies for each player. This is the major obstacle to solve a game in extensive form since the size of a mixed strategy generally grows exponentially in the size of the game tree. Such a huge increase in the size often renders a problem computationally intractable.

Each pure strategy may be identified by a path of length $n$. Let $m$ be a move of a player. We take it that a move has the value 1 or -1: $m = 1$ ($m = -1$) stands for a clockwise (counterclockwise) move. An ordered tuple $(m_1, m_2, \ldots , m_n)$ of $n$ moves is called a path of length $n$. All the moves in a path belong to the same player. In the above path, $m_i$ denotes his $i^{th}$ move. A path for a player is a description how he moves. For example, the path $(−1, 1, 1)$ for player 2 means his first move is counterclockwise, his second and third moves are clockwise. For brevity, we hereafter refer a pure strategy as a path.

We next construct the game matrix of size $2^n \times 2^n$. We adopt the standard convention that player 1 chooses a row and player 2 chooses a column. Let $\text{node} = +1,+2,$ or $-3,$ $s_n = (m_1, m_2, \ldots , m_n)$ be a path of player 1, $t_n = (m'_1, m'_2, \ldots , m'_n)$ be a path of player 2, and $m$ and $m'$ be moves. The length of a given path is indicated by its subscript. Let

\[
\langle s_n, t_n \rangle \quad \text{Payoff to player 1 in } \Gamma_n \text{ if player 1 uses } s_n \text{ and player 2 uses } t_n
\]

\[
\text{leaf}(s_n, t_n) \quad \text{Node where the chip is found at termination in } \Gamma_n
\]

\[
\langle \text{node} : m, m' \rangle \quad \text{Sum of the two intermediate payoffs if player 1 chooses } m \text{ and then player 2 chooses } m', \text{ assuming that the chip is at } \text{node} \text{ just before player 1 chooses } m
\]

\[
s_n \circ m \quad \sum_{i=1}^{n} (m_i + m'_i)
\]

\[
\text{sum} (s_n, t_n) \quad \text{Payoff to player 1 in } \Gamma_n \text{ if player 1 uses } s_n \text{ and player 2 uses } t_n
\]

Suppose $s_3 = (1, 1, -1)$ and $t_3 = (-1, 1, 1).$ Then $\langle s_3, t_3 \rangle = 2 + 1 + 2 + (-3) + 2 + (-3) = 1; \leaf (s_3, t_3) = -3; \langle +2 : -1, -1 \rangle = 1 + (-3) = -2; \langle -3 : -1, 1 \rangle = 2 + (-3) = -1; s_3 \circ 1 = (1, 1, -1, 1); \text{sum} (s_3, t_3) = 2.$ The following relations are easy to establish.

\[
\langle s_n \circ m, t_n \circ m' \rangle = \langle s_n, t_n \rangle + \langle \text{leaf} (s_n, t_n) : m, m' \rangle,
\]

(1) $\text{sum} (s_n, t_n) \equiv 0 \pmod{3}$ if and only if $\text{leaf} (s_n, t_n) = +1,$

(2) $\text{sum} (s_n, t_n) \equiv 1 \pmod{3}$ if and only if $\text{leaf} (s_n, t_n) = -2,$

(3) $\text{sum} (s_n, t_n) \equiv 2 \pmod{3}$ if and only if $\text{leaf} (s_n, t_n) = -3.$
Proposition 1  The value of $\Gamma_1$ is $-1$, and the value of $\Gamma_n$ is $-\frac{2}{9}$ for $n \geq 2$.
Define the paths of length $n \geq 1$ by

\[
\begin{align*}
    s_1^0 &= (1, 1, 1, -1, 1, \ldots, (-1)^0), & s_1^2 &= (1, 1, -1, 1, -1, 1, \ldots, (-1)^0), \\
    s_2^3 &= (1, -1, 1, 1, -1, 1, \ldots, (-1)^0), & s_2^4 &= (1, -1, -1, 1, -1, 1, \ldots, (-1)^0), \\
    t_1^3 &= (1, 1, 1, -1, 1, -1, 1, \ldots, (-1)^0+1), & t_1^2 &= (1, 1, 1, -1, 1, -1, 1, \ldots, (-1)^0), \\
    t_2^3 &= (1, -1, 1, -1, 1, -1, 1, \ldots, (-1)^0+1), & t_2^4 &= (1, -1, 1, -1, 1, -1, 1, \ldots, (-1)^0).
\end{align*}
\]

Let $x_n^* = \frac{1}{3} s_1^n + \frac{1}{3} s_2^n + \frac{1}{3} s_3^n$ and $y_n^* = \frac{1}{3} t_1^n + \frac{1}{3} t_2^n + \frac{1}{3} t_3^n$.
Then $x_n^*$ is an optimal strategy of player 1 and $y_n^*$ is an optimal strategy of player 2 in $\Gamma_n$ for $n \geq 1$.

We clarify certain points in Proposition 1. We only discuss the case for player 1; similar discussion applies to player 2. (1) The four paths $s_1^n$ to $s_2^n$ of player 1 are defined with respect to $n$ starting from the left end of the tuples. For example, $s_2^2 = (1, -1)$. (2) Beginning with the fourth move, the moves in each path always alternate between 1 and -1. (3) When $n = 1$ or $n = 2$, $s_1^n$ to $s_2^n$ are not distinct. In these cases, $x_n^*$ are interpreted as follows. For $n = 2$,

\[
x_2^* = \frac{1}{3} s_1^2 + \frac{1}{3} s_2^2 + \frac{1}{3} s_3^2 = \frac{1}{3} (1, 1) + \frac{1}{3} (1, 1) + \frac{1}{3} (1, -1) + \frac{1}{3} (1, -1) = \frac{4}{9} (1, 1) + \frac{5}{9} (1, -1).
\]

For $n = 1$, it is easy to see $x_1^* = 1(1)$, that is, player 1 always moves clockwise.

![Figure 2: Optimal behavior strategies for player 1(left) and player 2(right) in $\Gamma_n$.](image)

To better visualize the optimal strategies in Proposition 1, we represent them as behavior strategies in Figure 2. In this figure, an outgoing edge towards the lower right (left) is a clockwise (counterclockwise) move. If a node has only one outgoing edge, that edge is clearly chosen with certainty. If there are two outgoing edges, the probabilities of choosing them sum to one; we only show the probability choosing the clockwise move. One interesting observation is that the optimal strategies embed among themselves. By this is meant that if a player removes the last move from his optimal strategy in $\Gamma_n$, he will obtain an optimal strategy in $\Gamma_{n-1}$. This implies the players do not even need to know the value of $n$ to play optimally. We now derive some results required to prove Proposition 1.

Lemma 1  Let $n \geq 3$ and let $s_1^n$ to $s_2^n$ be defined as in Proposition 1. For any path $t_n$ of
player 2, one of the following three cases holds.

\[
\text{leaf}(s_n^1, t_n) = +1, \text{leaf}(s_n^2, t_n) = +2, \text{leaf}(s_n^3, t_n) = +2, \text{leaf}(s_n^4, t_n) = -3. \\
\text{(4)} \quad \text{leaf}(s_n^1, t_n) = +2, \text{leaf}(s_n^2, t_n) = -3, \text{leaf}(s_n^3, t_n) = -3, \text{leaf}(s_n^4, t_n) = +1. \\
\text{leaf}(s_n^1, t_n) = -3, \text{leaf}(s_n^2, t_n) = +1, \text{leaf}(s_n^3, t_n) = +1, \text{leaf}(s_n^4, t_n) = +2.
\]

**Proof** Since the chip has to terminate at a node, \(\text{leaf}(s_n^1, t_n) = +1, +2\) or \(-3\).

Suppose \(\text{leaf}(s_n^1, t_n) = +1\). We need to show \(\text{leaf}(s_n^2, t_n) = +2, \text{leaf}(s_n^3, t_n) = +2,\) and \(\text{leaf}(s_n^4, t_n) = -3\). This is the first case in (4). Let \(\text{sum}(s_n^1, t_n) = r\). Then \(\text{sum}(s_n^1, t_n) = r - 2, \text{sum}(s_n^3, t_n) = r - 2,\) and \(\text{sum}(s_n^4, t_n) = r - 4\). To see that \(\text{sum}(s_n^2, t_n) = r - 2\), we just need to examine those moves that differ in \(s_n^1\) and \(s_n^2\). Only their third moves are different, so the result follows.

\(\text{From (1)}, \text{leaf}(s_n^1, t_n) = +1 \Rightarrow \text{sum}(s_n^1, t_n) \equiv 0 \pmod{3}\), that is, \(r \equiv 0 \pmod{3}\).

\(\text{sum}(s_n^2, t_n) = r - 2 \equiv r + 1 \equiv 1 \pmod{3}\) implies from \(\text{(2)}\) that \(\text{leaf}(s_n^2, t_n) = +2\).
\(\text{sum}(s_n^3, t_n) = r - 2 \equiv r + 1 \equiv 1 \pmod{3}\) implies from \(\text{(2)}\) that \(\text{leaf}(s_n^3, t_n) = +2\).
\(\text{sum}(s_n^4, t_n) = r - 4 \equiv r + 2 \equiv 2 \pmod{3}\) implies from \(\text{(3)}\) that \(\text{leaf}(s_n^4, t_n) = -3\).

The second and third cases of (4) can be treated similarly.\(\diamondsuit\)

**Lemma 2** Let \(n \geq 3\) and \(t_n^1, t_n^2, t_n^3, t_n^4\) be defined as in Proposition 1. Let \(s_n\) be any path of player 1. If \(n\) is odd, one of the following three cases holds.

\[
\text{leaf}(s_n, t_n^1) = +1, \text{leaf}(s_n, t_n^2) = +2, \text{leaf}(s_n, t_n^3) = +2, \text{leaf}(s_n, t_n^4) = -3. \\
\text{(5)} \quad \text{leaf}(s_n, t_n^1) = +2, \text{leaf}(s_n, t_n^2) = -3, \text{leaf}(s_n, t_n^3) = -3, \text{leaf}(s_n, t_n^4) = +1. \\
\text{leaf}(s_n, t_n^1) = -3, \text{leaf}(s_n, t_n^2) = +1, \text{leaf}(s_n, t_n^3) = +1, \text{leaf}(s_n, t_n^4) = +2.
\]

If \(n\) is even, one of the following three cases holds.

\[
\text{leaf}(s_n, t_n^1) = +1, \text{leaf}(s_n, t_n^2) = +1, \text{leaf}(s_n, t_n^3) = +2, \text{leaf}(s_n, t_n^4) = +2. \\
\text{(6)} \quad \text{leaf}(s_n, t_n^1) = +2, \text{leaf}(s_n, t_n^2) = +2, \text{leaf}(s_n, t_n^3) = -3, \text{leaf}(s_n, t_n^4) = -3. \\
\text{leaf}(s_n, t_n^1) = -3, \text{leaf}(s_n, t_n^2) = -3, \text{leaf}(s_n, t_n^3) = +1, \text{leaf}(s_n, t_n^4) = +1.
\]

**Proof** Proceed as in the proof of Lemma 1. If \(\text{sum}(s_n, t_n^1) = r\), we need to verify that:

For \(n\) odd, \(\text{sum}(s_n, t_n^2) = r - 2, \text{sum}(s_n, t_n^3) = r - 2,\) and \(\text{sum}(s_n, t_n^4) = r - 4\).

For \(n\) even, \(\text{sum}(s_n, t_n^2) = r, \text{sum}(s_n, t_n^3) = r - 2,\) and \(\text{sum}(s_n, t_n^4) = r - 2\).\(\diamondsuit\)

Let

\[
U = \{(+1, +1, +2, +2, -3), (+2, -3, -3, +1), (-3, +1, +1, +2)\}, \\
V = \{(+1, +1, +1, +2, +2), (+2, +2, -3, +3), (-3, +1, +1, +1)\}.
\]

Each ordered tuple in \(U\) is a case in (4) or (5), and each tuple in \(V\) is a case in (6). The proof of Lemma 3 below is by straightforward exhaustive evaluation and is omitted.

**Lemma 3** Let \(m\) and \(m'\) be any moves.

\(\text{(i)}\) For each \((k_1, k_2, k_3, k_4)\) in \(U\),

\[
\frac{1}{3} \langle k_1 : m, m' \rangle + \frac{1}{9} \langle k_2 : m, m' \rangle + \frac{2}{9} \langle k_3 : m, m' \rangle + \frac{1}{3} \langle k_4 : m, m' \rangle = 0.
\]
(ii) For each \((k_1, k_2, k_3, k_4)\) in \(U\),

\[
\begin{align*}
2/9(k_1 : m, -1) + 1/3(k_2 : m, 1) + 1/3(k_3 : m, -1) + 1/9(k_4 : m, 1) &= 0.
\end{align*}
\]

(iii) For each \((k_1, k_2, k_3, k_4)\) in \(V\),

\[
\begin{align*}
2/9(k_1 : m, 1) + 1/3(k_2 : m, -1) + 1/3(k_3 : m, 1) + 1/9(k_4 : m, -1) &= 0.
\end{align*}
\]

Let \(E(x_n, t_n)\) denote the expected payoff (to player 1) if player 1 uses the mixed strategy \(x_n\) and player 2 uses the path \(t_n\). Let \(E(s_n, y_n)\) denote the expected payoff if player 1 uses the path \(s_n\) and player 2 uses the mixed strategy \(y_n\).

**Lemma 4** Let \(n \geq 3\). Let \(x^n_1\) and \(y^n_0\) be defined as in Proposition 1. Then

\[
\begin{align*}
E(x^n_{n+1}, t_n \circ m) &= E(x^n_n, t_n) \quad \text{for all paths } t_n \text{ and for all moves } m',
E(s_n \circ m, y^n_{n+1}) &= E(s_n, y^n_0) \quad \text{for all paths } s_n \text{ and for all moves } m.
\end{align*}
\]

**Proof** We only prove (11). Let \(n \geq 3\), \(s_n\) be any path, and \(m\) be any move. Let \(\tilde{m}\) denote the \((n+1)\)th move in the paths \(t^n_{i} (i = 1, 2, 3, 4)\) that are defined in Proposition 1. When \(i = 1, 3\), this \(\tilde{m}\) move is clockwise (counterclockwise) for \(n\) even (odd). When \(i = 2, 4\), this move is clockwise (counterclockwise) for \(n\) odd(even). That is, for \(i = 1, 3\), \(\tilde{m} = (-1)^n\), and for \(i = 2, 4\), \(\tilde{m} = (-1)^{n+1}\). The above observation may also be seen from the right diagram in Figure 2 where these paths are numbered from right to left. Hence for \(i = 1, 3\),

\[
t^n_{n+1} = t^n_i \circ (-1)^n, \quad \text{and for } i = 2, 4, \ t^n_{n+1} = t^n_i \circ (-1)^{n+1}.
\]

We have

\[
E(s_n \circ m, y^n_{n+1}) = E(s_n \circ m, 2/9 t^n_{1} + 1/3 t^n_{3} + 1/3 t^n_{3} + 1/9 t^n_{4})
\]

\[
= 2/9 \left( s_n \circ m, t^n_{1} \right) + 1/3 \left( s_n \circ m, t^n_{3} \right) + 1/3 \left( s_n \circ m, t^n_{3} \right) + 1/9 \left( s_n \circ m, t^n_{4} \right)
\]

\[
= 2/9 \left( s_n \circ m, t^n_{1} \right) + 1/3 \left( s_n \circ m, t^n_{1} \right) + 1/3 \left( s_n \circ m, t^n_{1} \right) + 1/9 \left( s_n \circ m, t^n_{1} \right)
\]

\[
+ 1/3 \left( s_n \circ m, t^n_{1} \right) + 1/9 \left( s_n \circ m, t^n_{1} \right)
\]

\[
+ 2/9 \left( s_n \circ m, t^n_{1} \right) + 1/3 \left( s_n \circ m, t^n_{1} \right) + 1/3 \left( s_n \circ m, t^n_{1} \right) + 1/9 \left( s_n \circ m, t^n_{1} \right)
\]

\[
= E(s_n, y^n_0) + \theta \quad \text{where}
\]

\[
\theta = 2/9 \left( \text{leaf}(s_n, t^n_{1}) : m, (-1)^n \right) + 1/3 \left( \text{leaf}(s_n, t^n_{2}) : m, (-1)^{n+1} \right)
\]

\[
+ 1/3 \left( \text{leaf}(s_n, t^n_{3}) : m, (-1)^n \right) + 1/9 \left( \text{leaf}(s_n, t^n_{4}) : m, (-1)^{n+1} \right).
\]

We are done if we can show \(\theta = 0\).

Let \(n\) be odd. From (5),

\[
\theta = 2/9 \left( k_1 : m, -1 \right) + 1/3 \left( k_2 : m, 1 \right) + 1/3 \left( k_3 : m, -1 \right) + 1/9 \left( k_4 : m, 1 \right)
\]
for some \((k_1, k_2, k_3, k_4)\) in \(U\). Using (8), \(\theta = 0\).

Let \(n\) be even. From (6),

\[
\theta = \frac{2}{9} (k_1 : m, 1) + \frac{1}{3} (k_2 : m, -1) + \frac{1}{3} (k_3 : m, 1) + \frac{1}{9} (k_4 : m, -1)
\]

for some \((k_1, k_2, k_3, k_4)\) in \(V\). Using (9), \(\theta = 0\).

When constructing the payoff matrix of \(\Gamma_n\), it is useful to enumerate the \(2^n\) paths of a player in some consistent way from path 1 to path \(2^n\). For \(1 \leq i \leq 2^n\), we define path \(i\) as follows. First convert \(i - 1\) to binary as a string of 0 and 1. If necessary, left pad by adding extra 0 in front until we have a string of length \(n\). Replace all 1 by -1, and then replace all 0 by 1. As an illustration suppose \(n = 5\) and \(i = 14\). In binary, 13 = 1101 and we need to add one extra 0 in front to get 01101. Replace all 1 by -1 to obtain 0, -1, -1, 0, -1. Replace all 0 by 1 to obtain 1, -1, -1, 1, -1. Hence path 14 is (1, -1, -1, 1, -1).

**Proof of Proposition 1** When \(n = 1\), the payoff matrix is

\[
\begin{pmatrix}
-1 & 3 \\
-2 & -1
\end{pmatrix}
\]

This matrix has a saddle-point in pure strategies; the -1 in bold is its saddle-point value.

Now let \(n \geq 2\). The conclusion of Proposition 1 follows if we can prove

\[
E(x^*_n, t_n) \geq -\frac{2}{9} \quad \text{for all paths } t_n,
\]

(12)

\[
E(s^*_n, y^*_n) \leq -\frac{2}{9} \quad \text{for all paths } s_n.
\]

(13)

We only prove (12) since the proof of (13) is similar. When \(n = 2, 3\), the payoff matrices are respectively

\[
\begin{pmatrix}
2 & -3 & 2 & 6 \\
-2 & 2 & 1 & 2 \\
-3 & 1 & -3 & -2 \\
-4 & -3 & 2 & -3
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & -5 & 5 & 0 & 5 & -9 \\
5 & 0 & -4 & 0 & 1 & 5 & 4 & 5 \\
1 & -4 & 1 & 5 & 0 & 4 & 0 & 1 \\
-3 & 1 & 0 & 1 & -1 & 0 & 5 & 0 \\
0 & -5 & 0 & 4 & -4 & 0 & -4 & -3 \\
-4 & 0 & -1 & 0 & -5 & -4 & 1 & -4 \\
-5 & -1 & -5 & -4 & 0 & 1 & 0 & -5 \\
-6 & -5 & 0 & -5 & 5 & 0 & -4 & 0
\end{pmatrix}
\]

We verify directly, using the above matrices, that (12) is true for \(n = 2, 3\).

Assume now (12) is true for \(n, k \geq 3\).

Using (10) with \(n\) replaced by \(k\), for all paths \(t_k\) and all moves \(m'\),

\[
E(x^*_k, t_k \circ m') = E(x^*_k, t_k).
\]

Hence for all moves \(m'\),

\[
E(x^*_{k+1}, t_k \circ m') = E(x^*_k, t_k) \geq -\frac{2}{9} \quad \text{for all paths } t_k.
\]
If \( t_k \) runs through all the \( 2^k \) paths of length \( k \), and \( m' \) runs through the two moves \( m' = 1 \) and \( m' = -1 \), then \( t_k \circ m' \) will run through all the \( 2^{k+1} \) paths of length \( k + 1 \). Writing \( t_{k+1} \) for \( t_k \circ m' \), we obtain

\[
E(x_k^{t_{k+1}}, t_{k+1}) \geq -\frac{2}{9} \text{ for all paths } t_{k+1}.
\]

By induction (12) is true for \( n \geq 3 \). Including the case for \( n = 2 \) that has been verified by direct calculation, (12) is true for \( n \geq 2 \). \( \diamond \)

There are many optimal strategies. We restrict the discussion to those optimal strategies that address the following problem: What are some alternate optimal strategies if all we can change are the fourth or later moves in the paths in Proposition 1? The weights in Proposition 1 cannot be changed.

Using (7) it can be shown the following sets of paths also work for player 1:

\[
\begin{align*}
\Gamma_n^1 &= (1, 1, 1, -1, 1, -1, \ldots, (-1)^{n+1}), \\
\Gamma_n^2 &= (1, 1, -1, 1, -1, 1, \ldots, (-1)^n), \\
\Gamma_n^3 &= (1, -1, 1, 1, -1, 1, \ldots, (-1)^n), \\
\Gamma_n^4 &= (1, -1, -1, 1, -1, 1, \ldots, (-1)^n).
\end{align*}
\]

Alternate optimal strategies with at most 4 paths seem to occur less frequent for player 2. Besides the one given in Proposition 1, the only other set of paths that we are able to find is

\[
\begin{align*}
\hat{\Gamma}_n^1 &= (1, 1, 1, -1, 1, -1, \ldots, (-1)^n), \\
\hat{\Gamma}_n^2 &= (1, 1, -1, 1, -1, -1, \ldots, (-1)^n), \\
\hat{\Gamma}_n^3 &= (1, -1, 1, 1, -1, -1, \ldots, (-1)^n), \\
\hat{\Gamma}_n^4 &= (1, -1, -1, 1, -1, -1, \ldots, (-1)^n).
\end{align*}
\]

For a (mixed) strategy \( x_n \) and a path \( s_n \), let \( x_n(s_n) \) denote the probability that \( x_n \) assigns to \( s_n \). The support of \( x_n \) is defined as the set of paths \( s_n \) with \( x_n(s_n) > 0 \). The size of \( x_n \) is then defined to be the cardinality of its support. Proposition 1 says each player in \( \Gamma_n \) has an optimal strategy of size 4 or less for all positive \( n \). We show this remains true even when we change the initial position of the chip at the start of the game.

### 4 Games with Other Initial Positions

In \( \Gamma_n \), the chip rests initially at node +1 when the game starts. Let \( \Gamma_n^2 \) and \( \Gamma_n^3 \) denote the game where the chip rests initially at node +2 and at node -3 respectively. All other aspects of \( \Gamma_n \) are assumed to remain unchanged, like player 1 still making the first move.

First consider \( \Gamma_n^2 \). We solve \( \Gamma_n^2 \) in the same way as we have solved \( \Gamma_n \). The basic method is to solve \( \Gamma_n^2 \) numerically for small values of \( n \), make a guess, and then verify the guess.

**Proposition 2** The value of \( \Gamma_n^2 \) is \( -\frac{7}{6} \), and the value of \( \Gamma_n^3 \) is \( -\frac{3}{5} \) for \( n \geq 2 \).

In \( \Gamma_n^2 \), an optimal strategy of player 1 is to choose the clockwise move with probability \( \frac{4}{5} \), and an optimal strategy of player 2 is to choose the clockwise move with probability \( \frac{3}{5} \).

Define the paths of length \( n \geq 2 \) by

\[
\begin{align*}
\hat{\Gamma}_n^1 &= (1, -1, 1, 1, -1, \ldots, (-1)^n), \\
\hat{\Gamma}_n^2 &= (1, -1, 1, 1, -1, \ldots, (-1)^n), \\
\hat{\Gamma}_n^3 &= (1, -1, 1, 1, -1, \ldots, (-1)^n), \\
\hat{\Gamma}_n^4 &= (1, -1, 1, 1, -1, \ldots, (-1)^n).
\end{align*}
\]

Let \( \hat{x}_n = \frac{1}{3} \hat{\Gamma}_n^1 + \frac{1}{9} \hat{\Gamma}_n^2 + \frac{2}{9} \hat{\Gamma}_n^3 + \frac{2}{9} \hat{\Gamma}_n^4 \) and \( \hat{y}_n = \frac{1}{9} x_n^1 + \frac{1}{9} x_n^2 + \frac{1}{9} x_n^3 + \frac{2}{9} x_n^4 \).

Then \( \hat{x}_n \) is an optimal strategy of player 1 and \( \hat{y}_n \) is an optimal strategy of player 2 in \( \Gamma_n^2 \) for \( n \geq 2 \).
We have given the optimal strategies separately for \( n = 1 \) since we cannot extend the definition of \( \tilde{x}_n \) and \( \tilde{y}_n \) above to include this case. Let us see what happen if we do otherwise. For example, \( \tilde{x}_1 = \frac{4}{9} (1) + \frac{5}{9} (-1) \), that is, to choose the clockwise move with probability \( \frac{4}{9} \). It is easily seen that this strategy is not optimal. We can prove Proposition 2 with the same method we have used to prove Proposition 1. Since the chip now starts initially from node +2, we have to make some minor adjustments to reflect this fact. For example, \( (1) \) has to be replaced by
\[
\sum (s_n, t_n) \equiv 0 \pmod{3} \; \text{if and only if leaf} (s_n, t_n) = +2.
\]
We omit the proof of Proposition 2.

Now consider \( \Gamma_n^{-3} \). Here we may guess a solution of \( \Gamma_n^{-3} \) based on the known solution of \( \Gamma_n \). Consider the game \( \Gamma_n+1 \). Using the optimal strategies in Proposition 1 both players choose clockwise for their first move. Thus the referee moves the chip to node -3 and he is about to call player 1 to choose his second move. From that moment the game to be played is precisely \( \Gamma_n^{-3} \). Hence the guess is that for \( n \geq 1 \), value \( (\Gamma_{n+1}) = 2 + (-3) + \text{value} (\Gamma_n^{-3}) \), or value \( (\Gamma_n^{-3}) = \frac{2}{9} \). Furthermore, if we delete the first move from each player's optimal strategy in \( \Gamma_n+1 \) we should obtain his optimal strategy in \( \Gamma_n^{-3} \). It turns out the above guess is correct. For completeness we summarize the results in Proposition 3.

**Proposition 3** The value of \( \Gamma_n^{-3} \) is \( \frac{2}{9} \) for \( n \geq 1 \). Define the paths of length \( n \geq 1 \) by
\[
\tilde{s}_n = (1, 1, 1, -1, 1, -1, \ldots, (-1)^{n+1}), \quad \tilde{s}_n^2 = (1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}),
\]
\[
\tilde{s}_n^3 = (-1, 1, 1, -1, 1, -1, \ldots, (-1)^{n+1}), \quad \tilde{s}_n^4 = (-1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}),
\]
\[
\tilde{t}_n = (1, 1, 1, -1, 1, -1, \ldots, (-1)^n), \quad \tilde{t}_n^2 = (1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}),
\]
\[
\tilde{t}_n^3 = (-1, 1, 1, -1, 1, -1, \ldots, (-1)^{n+1}), \quad \tilde{t}_n^4 = (-1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}).
\]

Let \( \tilde{x}_n = \frac{1}{3} \tilde{s}_n^1 + \frac{1}{3} \tilde{s}_n^2 + \frac{2}{3} \tilde{s}_n^3 + \frac{1}{3} \tilde{s}_n^4 \) and \( \tilde{y}_n = \frac{2}{9} \tilde{t}_n^1 + \frac{1}{3} \tilde{t}_n^2 + \frac{1}{3} \tilde{t}_n^3 + \frac{1}{3} \tilde{t}_n^4 \).

Then \( \tilde{x}_n \) is an optimal strategy of player 1 and \( \tilde{y}_n \) is an optimal strategy of player 2 in \( \Gamma_n^{-3} \) for \( n \geq 1 \).

5 The Game \( \Omega_n(a, b, c) \) We may generalize further by adding a chance move at the very beginning before player 1 chooses his first move. Here is how the chance move may be implemented. The referee uses a probability distribution to select the node where the chip will rest initially. Let the probabilities of selecting nodes 1, 2 and -3 be \( a, b \) and \( c \) respectively where \( a + b + c = 1 \). Both players are informed the values of \( a, b \) and \( c \) but are not informed the outcome of this chance move. All other aspects of the game remain unchanged. Let \( \Omega_n(a, b, c) \) denote this game. We have already solved three special cases of \( \Omega_n(a, b, c) \): \( \Omega_n(1, 0, 0) = \Gamma_n \), \( \Omega_n(0, 1, 0) = \Gamma_n^2 \), and \( \Omega_n(0, 0, 1) = \Gamma_n^{-3} \).

It is helpful to recall what we have done earlier. Associated with each of \( \Gamma_n \), \( \Gamma_n^2 \) or \( \Gamma_n^{-3} \) is an \( 2^0 \times 2^0 \) payoff matrix where each entry is obtained by summing 2n intermediate payoffs. Let these matrices be called \( A(\Gamma_n) \), \( A(\Gamma_n^2) \) and \( A(\Gamma_n^{-3}) \) respectively. \( A(\Omega_n(a, b, c)) \), the payoff matrix of \( \Omega_n(a, b, c) \), is clearly given by
\[
A(\Omega_n(a, b, c)) = aA(\Gamma_n) + bA(\Gamma_n^2) + cA(\Gamma_n^{-3}).
\]

For any specific values of \( a, b \) and \( c \), it is possible but laborious to use the above relation to solve \( \Omega_n(a, b, c) \) by applying the same method as we have done for \( \Gamma_n \). Instead of doing that, we restrict our attention now to explore when \( \Omega_n(a, b, c) \) may be solved easily. Since we already know the solutions of \( \Gamma_n \), \( \Gamma_n^2 \) and \( \Gamma_n^{-3} \), it is natural to investigate how to reduce \( \Omega_n(a, b, c) \) to a game that is similar in some sense to one of these three games. Towards this goal we first prove the following lemma.
Lemma 5 For $n \geq 1$, 

\begin{equation}
A(\Gamma_n) + A(\Gamma_n^2) + A(\Gamma_n^{-3}) = 0.
\end{equation}

Proof The 0 on the right side of (14) is an $2^n \times 2^n$ matrix whose entries are all zeroes. Let $s_n$ and $t_n$ be any paths of player 1 and player 2 respectively. We wish to show that, corresponding to these two paths, 

\begin{equation}
\text{payoff in } \Gamma_n + \text{payoff in } \Gamma_n^2 + \text{payoff in } \Gamma_n^{-3} = 0.
\end{equation}

The proof becomes obvious if we play the games when the chip initially rests at node -1, node -2 and node +3 respectively. For Lemma 5 implies $\Omega_n(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a trivial game since 

\begin{equation}
A(\Omega_n(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})) = \frac{1}{3} A(\Gamma_n) + \frac{1}{3} A(\Gamma_n^2) + \frac{1}{3} A(\Gamma_n^{-3}) = 0.
\end{equation}

Its value is zero, and any path of length $n$ is an optimal strategy for either player.

Let $T_n$ denote one of $\Gamma_n, \Gamma_n^2, \Gamma_n^{-3}$ or their variants. For any positive real number $\alpha$, we define $\Omega_n(a, b, c) \sim \alpha T_n$ if $A(\Omega_n(a, b, c)) = \alpha A(T_n)$. If $\Omega_n(a, b, c) \sim \alpha T_n$, a basic result in game theory states the games $\Omega_n(a, b, c)$ and $T_n$ have the same optimal strategies and the value of $\Omega_n(a, b, c)$ is $\alpha$ times the value of $T_n$.

Proposition 4 Let $\frac{1}{3} < a \leq 1$. Then 

\begin{equation}
\Omega_n(\frac{1-a}{2}, \frac{1-a}{2}) \sim \frac{3a-1}{2} \Gamma_n,
\end{equation}

\begin{equation}
\Omega_n(\frac{1-a}{2}, a, \frac{1-a}{2}) \sim \frac{3a-1}{2} \Gamma_n^2,
\end{equation}

\begin{equation}
\Omega_n(\frac{1-a}{2}, a, a) \sim \frac{3a-1}{2} \Gamma_n^{-3}.
\end{equation}

Proof To prove (16), 

\begin{align*}
A(\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2})) &= a A(\Gamma_n) + \frac{1-a}{2} A(\Gamma_n^2) + \frac{1-a}{2} A(\Gamma_n^{-3}) \\
&= \frac{3a-1}{2} A(\Gamma_n) + \frac{1-a}{2} A(\Gamma_n) + A(\Gamma_n^2) + A(\Gamma_n^{-3})
\end{align*}

so that 

\begin{equation}
\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) \sim \frac{3a-1}{2} \Gamma_n.
\end{equation}

Suppose we reverse the sign of all the node labels, that is, node +1 becomes node -1, node +2 becomes node -2, and node -3 becomes node +3. Let $\Gamma_n^{-1}, \Gamma_n^{-2}$ and $\Gamma_n^{-3}$ denote the games when the chip initially rests at node -1, node -2 and node +3 respectively.
**Proposition 5** Let $0 \leq a < \frac{1}{3}$. Then

\[
\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) \sim \frac{1-3a}{2} \hat{\Gamma}_n^{-1}, \\
\Omega_n(\frac{1-a}{2}, a, \frac{1-a}{2}) \sim \frac{1-3a}{2} \hat{\Gamma}_n^{-2}, \\
\Omega_n(\frac{1-a}{2}, 1-a, \frac{1-a}{2}) \sim \frac{1-3a}{2} \hat{\Gamma}_n^3.
\]

**Proof** To prove (17),

\[
A(\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2})) = a A(\Gamma_n) + \frac{1-a}{2} A(\Gamma_n^2) + \frac{1-a}{2} A(\Gamma_n^{-3})
\]

\[
= \frac{1-3a}{2} \{ -A(\Gamma_n) \} + \frac{1-a}{2} \left( A(\Gamma_n) + A(\Gamma_n^2) + A(\Gamma_n^{-3}) \right)
\]

\[
= \frac{1-3a}{2} A(\hat{\Gamma}_n^{-1})
\]

so that

\[
\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) \sim \frac{1-3a}{2} \hat{\Gamma}_n^{-1}.\n\]

A solution of $\hat{\Gamma}_n^{-1}$ cannot be deduced from a solution of $\Gamma_n$. We need to repeat the whole process in Section 3 to obtain a solution.

**6 Conclusion** Motivated by Gleason’s Game we formulate and solve a class of finite games. The surprising thing about these games is that they have optimal strategies with a small support. It is uncertain whether this is due to the fact that the labels on the nodes sum to zero. In the original formulation, Gleason probably chose the labels with a zero sum to make the game appeared fair to both the players. As pointed out by Ferguson and Shapley[2], Gleason’s Game is not a fair game since it favors player 2. Talking about fairness, $\Omega_n(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a fair game, both $\Gamma_n$ and $\Gamma_n^2$ favor player 2 while $\Gamma_n^{-3}$ favors player 1.

We make one final remark. The left diagram in Figure 2 shows an optimal strategy of player 1 in $\Gamma_n$. From the fourth move onwards, player 1 always moves clockwise for his even moves and counterclockwise otherwise. This implies player 2 knows exactly how player 1 is going to move. But player 2 cannot gain any advantage from this knowledge because he does not know the current position of the chip. Player 1, by randomizing appropriately when he chooses his second and third moves, ensures the chip would move in a pattern that prevents player 2 from exploiting his future deterministic moves.

**References**


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