ON SCATTERED COUNTABLE METRIC SPACES

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Abstract. The compact countable metric spaces are topologically classified simply by the classical Mazurkiewicz-Sierpiński theorem. Our concern is non-compact case. After viewing the scattered countable metric spaces of length 2 and the locally compact countable metric spaces, we shall prove Theorem 2, the main theorem of the present paper. Theorem 2 presents a topological classification of a class of scattered countable metric spaces which is far from the class of locally compact countable metric spaces.

1. Preliminaries. Let $X$ be a topological space. The following is Cantor's well-known process of deriving which is done by transfinite induction. (cf. Kuratowski [1])

Let $X^{(0)} = X$ and $X^{(1)}$ the set of the isolated points of $X^{(0)}$. If $\beta$ is a non-limit ordinal, let $X^{(\beta)} = X^{(\beta-1)} - X^{(\beta-1)}$ and $X^{(\beta)}$ the set of the isolated points of $X^{(\beta)}$, where $\beta - 1$ means the ordinal preceding $\beta$. If $\beta$ is a limit ordinal, let $X^{(\beta)} = \cap_{\gamma<\beta} X^{(\gamma)}$ and $X^{(\beta)}$ the set of the isolated points of $X^{(\beta)}$.

Each $X^{(\beta)}$ is a closed subset of $X$, and each $X^{(\beta)}$ is a discrete open subset of $X^{(\beta)}$.

A space $X$ is called scattered if $X^{(\alpha)} = \emptyset$ for some $\alpha$. A scattered space is also characterized as a space in which every non-empty (closed) subspace has an isolated point. The first ordinal $\alpha$ for which $X^{(\alpha)}$ vanishes is called the length of the scattered space $X$ and is denoted by $\text{length}(X)$. For a point $x$ of $X$, we write rank $x = \beta$ if $x \in X^{(\beta)}$. A scattered space $X$ has the following properties which will be used in this paper implicitly and frequently.

Let $\beta$ be an ordinal and $U$ an open set of $X$.

(a) $X^{(\beta)} \cap U = U^{(\beta)}$ and $X^{(\beta)} \cap U = U^{(\beta)}$ (and hence we have the following two).

(b) $\text{length}(U) = \beta$ if and only if $U \cap X^{(\beta)} = \emptyset$ and $U \cap X^{(\gamma)} \neq \emptyset$ for every $\gamma < \beta$.

(c) $X^{(\beta)}$ is dense in $X^{(\beta)}$.

A scattered countable metric space $X$ of length $\alpha$ has in addition the following properties.

(d) The length $\alpha$ is a countable or finite ordinal. (For compact case, $\alpha$ is in addition a non-limit ordinal)

(e) If $\beta + 1 < \alpha$ then $|X^{(\beta)}| = \omega$ with $\omega$ the first countable ordinal. If $\beta + 1 = \alpha$ then $|X^{(\beta)}| = |X^{(\beta)}| \leq \omega$. (For compact case, $|X^{(\beta)}| = |X^{(\beta)}| < \omega$.)

If the length $\alpha > 0$ is a non-limit ordinal and $|X^{(\alpha-1)}| = m$, $1 \leq m \leq \omega$, we write type $X = (<\alpha, m)$.

The following is the well-known Mazurkiewicz-Sierpiński theorem which caused and led us to write the present paper.

Theorem 1. (Mazurkiewicz-Sierpiński [2]) A compact countable metric space $X$ of type $(\alpha, n)$ is homeomorphic to the ordinals $(0, \omega^{\alpha-1}n]$ with the order topology. Hence the

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topological type of a compact countable metric space is uniquely determined by its type \((\alpha, n)\), and the number of topological types of compact countable metric spaces is \(\aleph_1\).

The compact countable metric space of type \((\alpha, n)\) is denoted by \(MS(\alpha, n)\).

2. Scattered countable metric spaces of length 2. We start with type \((2, 1)\).

**Proposition 1.** Let \(X\) be a scattered countable metric space of type \((2, 1)\) with \(X^{(1)} = X_{(1)} = \{p\}\). Then \(X\) admits precisely three topological types. Each type is characterized by the existence of a clopen neighborhood base \(X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots\) of \(p\) satisfying

\[
\begin{align*}
(r) & \quad |U_m - U_{m+1}| = 1 \text{ for every } m, \text{ or} \\
(r') & \quad |U_1 - U_2| = \omega \text{ and } |U_m - U_{m+1}| = 1 \text{ for every } m \geq 2, \text{ or} \\
(s) & \quad |U_m - U_{m+1}| = \omega \text{ for every } m.
\end{align*}
\]

The \(X\)’s which admit clopen neighborhood bases satisfying \((r), (r'), (s)\) are respectively denoted by \(r, r', s\). Typical spaces are as follows:

<table>
<thead>
<tr>
<th>(r)</th>
<th>(r')</th>
<th>(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(MS(2, 1), [0, \omega])</td>
<td>(MS(2, 1) \oplus \mathbb{N}, [0, \omega]^2 - {\omega})</td>
<td>(MS(3, 1) - MS(3, 1)_{(1)}, [0, \omega^2] - {\omega_n</td>
</tr>
</tbody>
</table>

, where \(\oplus\) means the topological sum and \(\mathbb{N}\) denotes the countable discrete space.

If \(1 \leq n < \omega\), the space of the form

\[
\underbrace{\underbrace{\cdots}_{r} \oplus \cdots}_{r} \cdots \oplus \underbrace{\cdots}_{s}
\]

is denoted by \(nr\) (resp. \(ns\)) and the space of the form

\[
\underbrace{\underbrace{\cdots}_{r} \oplus \cdots}_{r} \cdots \oplus \underbrace{\cdots}_{s}
\]

by \(\omega r\) (resp. \(\omega s\)).

**Definition 1.** A space \(X\) is said to absorb a space \(Y\) if \(X \cong X \oplus Y\), where \(\cong\) means the left side is homeomorphic to the right side.

A finite points space is absorbed by \(nr, \omega r, ns, \omega s\). The countable discrete space \(\mathbb{N}\) is absorbed by \(\omega r, ns, \omega s\) but not by \(nr\).

**Proposition 2.** Let \(X\) be a scattered countable metric space of type \((2, n)\), \(1 \leq n < \omega\). Then \(X\) admits precisely \(n + 2\) topological types as follows:

\[
nr, ns, kr \oplus (n - k)s, 1 \leq k \leq n - 1, nr \oplus \mathbb{N}.
\]

**Proposition 3.** Let \(X\) be a scattered countable metric space of type \((2, \omega)\). Then \(X\) is homeomorphic to one and only one of the following spaces:

\[
\omega r, \omega s, kr \oplus \omega s, 1 \leq k < \omega, ks \oplus \omega r, 1 \leq k < \omega, \omega r \oplus \omega s.
\]
ON SCATTERED COUNTABLE METRIC SPACES

Proof. Using 0-dimensionality we can take a discrete family \( \{ U_x : x \in X(1) \} \) of clopen sets of \( X \) so that \( U_x \cap X(1) = \{ x \} \). We may assume \( U_x \approx r \) or \( s \). Put \( U = \bigcup_{x \in X(1)} U_x \) and \( R = X \setminus U \). Then \( R \) is homeomorphic to \( \emptyset \), a finite points space or \( \mathbb{N} \). A finite points space is absorbed by \( U \) and can be vanished. The residue \( \mathbb{N} \) is not absorbed by \( U \) only when \( U \approx n \). This completes the proof.

3. Non-compact locally compact countable metric spaces. The non-compact locally compact countable metric spaces can be topologically classified easily by using Alexandrov’s one-point compactification \( X^* = X \cup \{ p \} \).

Proposition 4. If \( \alpha \) is a limit ordinal with \( \alpha < \omega_1 \), then a locally compact countable metric space \( X \) of length \( \alpha \) has the unique topological type.

Proof. In this case, type \( X^* = (\alpha + 1, 1) \) and the point \( p \) has the highest rank \( \alpha \) in \( X^* \). Theorem 1 says that \( X^* \) has the unique topological type so that \( X \) does because the rank of a point is preserved under homeomorphisms. This completes the proof.

Proposition 5. If \( \alpha \) is a non-limit ordinal with \( 0 < \alpha < \omega_1 \), then a locally compact countable metric space \( X \) of type \( (\alpha, \omega) \) has the unique topological type.

Proof. In this case, it is also true that type \( X^* = (\alpha + 1, 1) \) and the point \( p \) has the highest rank \( \alpha \) in \( X^* \) because if not, \( X^{(\alpha-1)} \) would not have an accumulation point in the compact space \( X^* \). Theorem 1 says again that \( X^* \) has the unique topological type so that \( X \) does. This completes the proof.

If \( \alpha \) is a limit ordinal (resp a non-limit ordinal), \( LC(\alpha) \) denotes the unique locally compact countable metric space of length \( \alpha \) (resp of type \( (\alpha, \omega) \)) assured by the propositions above. \( LC(0) \) denotes the empty set for convenience.

Proposition 6. Let \( \alpha \) be a non-limit ordinal with \( 0 < \alpha < \omega_1 \) and let \( X \) be a non-compact locally compact countable metric space \( X \) of type \( (\alpha, n), 1 \leq n < \omega \). Then the topological type of \( X \) is uniquely determined by the rank of \( p \) in \( X^* \) and is homeomorphic to

\[
MS(\alpha, n) \oplus LC(\beta), \beta < \alpha,
\]

with \( \beta \) the rank of \( p \).

Proof. Taking a clopen set \( U \) of \( X^* \) so that \( U \cap (X^*)^{(\beta)} = \{ p \} \) we have

\[
X^* = (X^* - U) \cup U \approx MS(\alpha, n) \oplus MS(\beta + 1, 1)
\]

which implies \( X \approx X^* - \{ p \} \approx MS(\alpha, n) \oplus LC(\beta) \). This completes the proof.

Remark. Though a detailed description as above was not given in [2], it was proved there that the number of topological types of locally compact countable metric spaces is \( \aleph_1 \) because every locally compact countable metric space is obtained from \( MS(\alpha, n), \alpha < \omega_1, 1 \leq n < \omega \), by removing a point.

4. Main theorem.

Definition 2. Let \( X \) be a scattered countable metric space. A non-isolated point \( x \) of \( X \) whose rank \( \beta \) is a non-limit ordinal is called a regular point if \( x \) has a clopen neighborhood
base \( U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \) satisfying
\[
\|U_m - U_{m+1}\| \cap X_{(\beta-1)} < \omega \quad \text{for every } m.
\]

If not, \( x \) is called a singular point.

A point \( x \) of \( X \) whose rank \( \beta \) is a limit ordinal is called a regular point if \( x \) has a clopen neighborhood base \( U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \) in \( X \) satisfying
\[
\text{len}(U_m - U_{m+1}) < \beta \quad \text{for every } m.
\]

If not, \( x \) is called a singular point.

**Remark.** If a point \( x \), whose rank \( \beta \) is a non-limit ordinal, is a regular point, then using 0-dimensionality we can choose \( U_m \) so that
\[
\|U_m - U_{m+1}\| \cap X_{(\beta-1)} = 1 \quad \text{for every } m.
\]

If a point \( x \), whose rank \( \beta \) is a non-limit ordinal, is a singular point, then we can also choose \( U_m \) so that
\[
\|U_m - U_{m+1}\| \cap X_{(\beta-1)} = \omega \quad \text{for every } m.
\]

If a point \( x \), whose rank \( \beta \) is a limit ordinal, is a singular point, then we can choose \( U_m \) so that
\[
\text{len}(U_m - U_{m+1}) = \beta \quad \text{for every } m.
\]

The term ‘regular’ comes from the following fact.

**Proposition 7.** Every non-isolated point of a locally compact countable metric space \( X \) is a regular point.

**Proof.** Let \( x \) be a non-isolated point of \( X \). If rank \( x = \beta \) is a non-limit ordinal, take a compact clopen set \( U \) so that \( U \cap X^{(\beta)} = \{x\} \) and take a clopen neighborhood base \( U = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \) of \( x \). Then \( \|U_m - U_{m+1}\| \cap X_{(\beta-1)} < \omega \) for every \( m \), because if \( \|U_m - U_{m+1}\| \cap X_{(\beta-1)} = \omega \) for some \( m \) then \( (U_m - U_{m+1}) \cap X_{(\beta-1)} \) would not have an accumulation point. If \( \beta \) is a limit ordinal, take \( U \) and \( U_m, m = 1, 2, 3, \ldots \), as above. Then each \( U_m - U_{m+1} \) is compact so that \( \text{len}(U_m - U_{m+1}) < \beta \) for every \( m \). This completes the proof.

**Definition 3.** Let \( X \) be a scattered countable metric space of length \( \alpha \geq 2 \). Let \( \Phi \) be a function, with no continuity assumed, of the interval \( (0, \alpha) \) to the two points set \( \{r, s\} \). We define \( X \) to have rankwise uniform type \( \Phi \) if
\[
\begin{cases}
\text{every point of } X_{(\beta)} \text{ is a regular point} & \text{if } \Phi(\beta) = r, \\
\text{every point of } X_{(\beta)} \text{ is a singular point} & \text{if } \Phi(\beta) = s.
\end{cases}
\]

Let \( X \) be a scattered countable metric space of length \( \alpha \) having rankwise uniform type \( \Phi \). Propositions 1, 2 and 3 tell us the following: For each non-limit ordinal \( 0 < \beta < \alpha \),
\[
\Phi(\beta) = r \quad \text{is equivalent to}
\begin{cases}
X_{(\beta)} \cup X_{(\beta-1)} \approx n \cdot r \quad \text{or} \quad n \cdot r \oplus \mathbb{N} & \text{if } \beta + 1 = \alpha \text{ and } |X^{(\beta)}| = n, 1 \leq n < \omega, \\
X_{(\beta)} \cup X_{(\beta-1)} \approx \omega \cdot r & \text{if otherwise},
\end{cases}
\]
and \( \Phi(\beta) = s \) is equivalent to
\[
\begin{cases}
X_{(\beta)} \cup X_{(\beta-1)} \approx n \cdot s & \text{if } \beta + 1 = \alpha \text{ and } |X^{(\beta)}| = n, 1 \leq n < \omega, \\
X_{(\beta)} \cup X_{(\beta-1)} \approx \omega \cdot s & \text{if otherwise}.
\end{cases}
\]
Remark. By Proposition 7 every locally compact countable metric space has the rankwise uniform type $\Phi$ taking the value $r$ constantly. To go beyond the length 2, one might expect that this kind of rankwise uniform type is easy to deal with. Unfortunately this is not so. Indeed, a scattered countable metric space of type $(3, 1)$ with the rankwise uniform type $\Phi(2) = \Phi(1) = r$ admits just five topological types (see [3, Table 1]), three of which are locally compact and the other non-locally compact two are described as below:

$$T = [0, \omega^2] - \{\omega(2n - 1) | 1 \leq n < \omega\},$$

$$T' = [0, \omega^2 2] - \{\omega(2n - 1) | 1 \leq n < \omega\},$$

with the topologies induced from the order topology of $[0, \omega_1)$. Furthermore a scattered countable metric space of type $(4, 1)$ with the rankwise uniform type $\Phi(3) = \Phi(2) = \Phi(1) = r$ admits infinitely many topological types. In fact, $MS(4, 1) \approx nT$, $1 \leq n < \omega$, give, with $n$ varying, countably many topological types.

Being far away from locally compact spaces, we have the following results.

**Theorem 2.** Let $X$ be a scattered countable metric space of type $(\alpha, m)$, $2 \leq \alpha < \omega_1$, $1 \leq m \leq \omega$, having a rankwise uniform type $\Phi$. Assume

$$(*) \quad \Phi(\beta) = \Phi(\beta + 1) = r \quad \text{does not occur.}$$

(1) In the exceptional case where $\alpha$ is a non-limit ordinal, $m < \omega$ and $\Phi(\alpha - 1) = r$, then $X$ has precisely two topological types.

(2) If otherwise, the topological type of $X$ is uniquely determined.

**Corollary 1.** Let $X$ be a scattered countable metric space of type $(\alpha, m)$, $2 \leq \alpha, 1 \leq m \leq \omega$, with the rankwise uniform type $\Phi$ taking the value $s$ constantly. Then the topological type of $X$ is uniquely determined.

Remark. Let $X$ be a scattered countable metric space with a rankwise uniform type $\Phi$ satisfying $(*)$ and $U$ a clopen set of $X$ of length $\beta \geq 2$. It follows from (a) in Preliminaries that $U$ has the rankwise uniform type $\Phi(0, \beta)$, the restriction to $(0, \beta)$ of $\Phi$, and $\Phi(0, \beta)$ satisfies $(*)$ as well. This fact will be used here and there in the proof without explicit mention.

Proof of Theorem 2. We shall prove the theorem by the transfinite induction on length $\alpha$. If $\alpha = 2$ the theorem is assured by Propositions 1, 2 and 3. Let $\gamma$ be an ordinal. Assume that the theorem has been proved for every $\alpha < \gamma$.

In case $\gamma$ and $\gamma - 1$ are both non-limit ordinals and $\Phi(\gamma - 1) = s$ : Let $X, Y$ be scattered countable metric spaces of type $(\gamma, m)$ with a common rankwise uniform type $\Phi$ satisfying $\Phi(\gamma - 1) = s$ and $(*)$. To show that $X \cong Y$, first consider the case $m = 1$. Let $a, b$ be the points of $X, Y$ respectively having the highest rank $\gamma - 1$ and let

$$X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \quad \text{and} \quad Y = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

be clopen neighborhood bases of $a, b$ respectively satisfying

$$|(U_m - U_{m+1}) \cap X_{(\gamma - 1)}| = \omega \quad \text{and} \quad |(V_m - V_{m+1}) \cap Y_{(\gamma - 1)}| = \omega$$

for every $m$. Note that

$$\text{type}(U_m - U_{m+1}) = \text{type}(V_m - V_{m+1}) = (\gamma - 1, \omega),$$
which implies that these sets are outside the exceptional case (1) even if \( \Phi(\gamma - 2) = r \).

The induction hypothesis is now applied to give \( U_m - U_{m+1} \approx V_m - V_{m+1} \) for every \( m \).

Taking a homeomorphism \( h_m: U_m - U_{m+1} \rightarrow V_m - V_{m+1} \) we can define a homeomorphism \( h: X \rightarrow Y \) by

\[
h(x) = \begin{cases} h_m(x) & \text{if } x \in U_m - U_{m+1} \\
b & \text{if } x = a.
\end{cases}
\]

The continuity of \( h \) at \( a \) (or of \( h^{-1} \) at \( b \)) is because \( \{ U_1, U_2, U_3, \ldots \} \) and \( \{ V_1, V_2, V_3, \ldots \} \) are open neighborhood bases of \( a, b \) respectively.

If \( 2 \leq m \leq \omega \), we can decompose \( X \), using 0-dimensionality, as \( X \approx \bigoplus_{\lambda \in \Lambda} X_\lambda \), where \( |\Lambda| = m \) and each \( X_\lambda \) is of type \((\gamma, 1)\). Since, as proved above, each \( X_\lambda \) admits the unique topology, so does \( X \). This completes the proof.

**In case \( \gamma \) and \( \gamma - 1 \) are both non-limit ordinals and \( \Phi(\gamma - 1) = r \):** Let \( X \) be a scattered countable metric space of type \((\gamma, 1)\) with a rankwise uniform type \( \Phi \) satisfying \( \Phi(\gamma - 1) = r \) and (8). Since \( \Phi(\gamma - 1) = r \), one and only one of the following two cases occurs:

\( X_{(\gamma-1)} \cup X_{(\gamma-2)} \approx r \) or \( r' \)

Let us denote, for the time being, by \( R \) the \( X \) in which the former happens and by \( R' \) the \( X \) in which the latter happens. To show the uniqueness of \( R \) let \( X, Y \) be scattered countable metric spaces of type \((\gamma, 1)\) with the rankwise uniform type \( \Phi \) satisfying

\( X_{(\gamma-1)} \cup X_{(\gamma-2)} \approx r \approx Y_{(\gamma-1)} \cup Y_{(\gamma-2)} \).

Let \( a, b \) be the points of \( X, Y \) respectively having the highest rank \( \gamma - 1 \) and let

\[ X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \quad \text{and} \quad Y = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \]

be open neighborhood bases of \( a, b \) respectively satisfying

\[ |(U_m - U_{m+1}) \cap X_{(\gamma-2)}| = |(V_m - V_{m+1}) \cap Y_{(\gamma-2)}| = 1 \]

for every \( m \). Note that

\[ \text{type } (U_m - U_{m+1}) = \text{type } (V_m - V_{m+1}) = (\gamma - 1, 1) \]

and \( \Phi(\gamma - 2) = s \) by condition (8). This implies that these sets are outside the exceptional case (1) so that \( U_m - U_{m+1} \approx V_m - V_{m+1} \) by the induction hypothesis. Now in a similar way to that in the preceding case, we can define a homeomorphism \( h: X \rightarrow Y \). The uniqueness of \( R \) has been thus assured.

To show the uniqueness of \( R' \), decompose \( R' \) as \( R' \approx R \oplus J \) so that

\[ \text{type } R = (\gamma, 1), \quad R_{(\gamma-1)} \cup R_{(\gamma-2)} \approx r \quad \text{and type } J = (\gamma - 1, \omega). \]

Then \( R \) is unique as proved above and \( J \) is unique by the induction hypothesis. Thus \( R' \) is unique as desired.

As for \( X \) of type \((\gamma, n)\), \( 1 \leq n < \omega \), write \( X = \bigoplus_{i=1}^{n} X_i \) with \( X_i \) of type \((\gamma, 1)\). Then, as proved above, each \( X_i \) is homeomorphic to \( R \) or \( R \oplus J \). If \( X_i \approx R \) for every \( i \) then \( X \approx nR \).

If \( X_i \approx R \oplus J \) for some \( i \), note that the topological sum of finitely (or countably) many \( J \)'s is homeomorphic to \( J \) because of its uniqueness. Thus \( X \approx nR \oplus J \). Consequently \( X \) has precisely two topological types because \( nR \) can not absorb \( J \).

To show the uniqueness of \( X \) of type \((\gamma, \omega)\), write \( X = \bigoplus_{i=1}^{\omega} X_i \) with \( X_i \approx R \) or \( R \oplus J \) so that \( X \approx \omega R \) or \( \omega R \oplus J \). To hide \( J \) into \( \omega R \), write \( J = \bigoplus_{i=1}^{\omega} J_i \) with \( J_i \) of type \((\gamma - 1, 1)\),
Since \( \text{type}(R \oplus J_i) = (\gamma, 1) \) and \( (R \oplus J_i)_{\gamma-1} \cup (R \oplus J_i)_{\gamma-2} \approx r \) it follows from the uniqueness of \( R \) that \( R \oplus J_i \approx R \) which yields

\[
\omega R \oplus J \approx (R \oplus J_1) \oplus (R \oplus J_2) \oplus (R \oplus J_3) \oplus \cdots \approx \omega R
\]
as desired. This completes the proof.

**In case \( \gamma \) is a limit ordinal:** Let \( X, Y \) be scattered countable metric spaces of length \( \gamma \) with a common rankwise uniform type \( \Phi \) satisfying (8). Write for each \( \beta < \gamma \)

\[
X - X(\beta) = \bigcup_{m=1}^{\infty} U_m^\beta, \quad Y - Y(\beta) = \bigcup_{m=1}^{\infty} V_m^\beta
\]

by clopen sets \( U_m^\beta, m = 1, 2, 3, \ldots \), of \( X \) and \( V_m^\beta, m = 1, 2, 3, \ldots \), of \( Y \). Rewrite

\[
\{ U_m^\beta \mid \beta < \gamma, \ m = 1, 2, 3, \ldots \} = \{ U_1, U_2, U_3, \ldots \},
\]

\[
\{ V_m^\beta \mid \beta < \gamma, \ m = 1, 2, 3, \ldots \} = \{ V_1, V_2, V_3, \ldots \}.
\]

Then \( \bigcup_{m=1}^{\infty} U_m = X, \bigcup_{m=1}^{\infty} V_m = Y \) and \( \text{leng}(U_m) < \gamma \) and \( \text{leng}(V_m) < \gamma \) for every \( m \).

Put \( \Sigma = \{ \sigma \in (0, \gamma) \mid |\Phi(\sigma) = s\} \). Note that \( \Sigma \) is cofinal in the interval \( (0, \gamma) \) by the condition (8). We can thus take a function \( \rho : \{ 1, 2, 3, \ldots \} \to \Sigma \) satisfying

\[
\rho(m) > \max\{ \text{leng}(U_m), \text{leng}(V_m) \} \quad \text{and} \quad \rho(m + 1) > \rho(m) \quad \text{for every} \ m.
\]

For each \( \sigma \in \Sigma \), fix \( x_\sigma \in X(\sigma), y_\sigma \in Y(\sigma) \), a clopen set \( A_\sigma \) of \( X \) and a clopen set \( B_\sigma \) of \( Y \) so that

\[
A_\sigma \cap X(\sigma) = \{ x_\sigma \} \quad \text{and} \quad B_\sigma \cap Y(\sigma) = \{ y_\sigma \}.
\]

Now define

\[
E_1 = U_1 \cup A_{\rho(1)}, \quad E_m = (U_m \cup A_{\rho(m)}) - \bigcup_{i=1}^{m-1} (U_i \cup A_{\rho(i)}),
\]

\[
F_1 = V_1 \cup B_{\rho(1)}, \quad F_m = (V_m \cup B_{\rho(m)}) - \bigcup_{i=1}^{m-1} (V_i \cup B_{\rho(i)}),
\]

for each \( m = 2, 3, 4, \ldots \). Then \( \{ E_1, E_2, E_3, \ldots \} \) and \( \{ F_1, F_2, F_3, \ldots \} \) are disjoint clopen covers of \( X \) and \( Y \) respectively. Note that \( \text{leng}(E_m) = \text{leng}(F_m) = \rho(m) + 1 < \gamma \) and that \( x_{\rho(m)} \) is the only point having the highest rank \( \rho(m) \) in the space \( E_m \) and so is \( y_{\rho(m)} \) in \( F_m \).

Thus we have type \( E_m = \text{type } F_m = (\rho(m) + 1, 1) \). Since \( \Phi(\rho(m)) = s \), \( E_m \) and \( F_m \) are outside the exceptional case. The induction hypothesis is now applied to obtain \( E_m \approx F_m \) for every \( m \), which yields \( X \approx Y \). This completes the proof.

**In case \( \gamma \) is a non-limit ordinal, \( \gamma - 1 \) is a limit ordinal and \( \Phi(\gamma - 1) = r \) Let \( X \) be a scattered countable metric space of type \( (\gamma, 1) \) with a rankwise uniform type \( \Phi \) satisfying \( \Phi(\gamma - 1) = r \) and (8). Let \( a \) be the point of \( X \) having the highest rank \( \gamma - 1 \). Since \( \gamma - 1 \) is a limit ordinal and \( \Phi(\gamma - 1) = r \), one and only one of the following two cases occurs:

\[
(r) \quad a \text{ has a clopen neighborhood base } X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \text{ satisfying } \text{leng}(U_m - U_{m+1}) < \gamma - 1 \text{ for every } m \text{ or}
\]

\[
(r') \quad a \text{ has a clopen neighborhood base } X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \text{ satisfying } \text{leng}(U_1 - U_2) = \gamma - 1 \text{ and } \text{leng}(U_m - U_{m+1}) < \gamma - 1 \text{ for every } m \geq 2.
\]

Let us denote, for the time being, by \( R \) the \( X \) in which the former happens and by \( R' \) the \( X \) in which the latter happens. To show the uniqueness of \( R \) let \( X, Y \) be scattered countable
metric spaces of type \( (\gamma, 1) \) with the rankwise uniform type \( \Phi \) and with \( a, b \) the points of \( X, Y \) respectively having the highest rank \( \gamma - 1 \), and let

\[
X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \quad \text{and} \quad Y = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots
\]

be open neighborhood bases of \( a, b \) respectively satisfying

\[
leng(U_m - U_{m+1}) < \gamma - 1 \quad \text{and} \quad leng(V_m - V_{m+1}) < \gamma - 1
\]

for every \( m \).

Hereafter the proof goes a way somewhat similar to that in the preceding case. Put

\[
\Sigma = \{ \sigma \in (0, \gamma - 1) \mid \Phi(\sigma) = s \}
\]

and take a function \( \rho : \{1, 2, 3, \ldots\} \rightarrow \Sigma \) satisfying

\[
\rho(m) > \max\{\leng(U_m - U_{m+1}), \leng(V_m - V_{m+1})\} \quad \text{and} \quad \rho(m + 1) > \rho(m)
\]

for every \( m \).

For each \( \sigma \in \Sigma \), fix \( x_\sigma \in X_\sigma \), \( y_\sigma \in Y_\sigma \), a clopen set \( A_\sigma \) of \( X \) and a clopen set \( B_\sigma \) of \( Y \) so that

\[
A_\sigma \cap X(\sigma) = \{ x_\sigma \} \quad \text{and} \quad B_\sigma \cap Y(\sigma) = \{ y_\sigma \}.
\]

Putting \( U'_m = U_m - U_{m+1} \) and \( V'_m = V_m - V_{m+1} \), define

\[
E_1 = U'_1 \cup A_{\rho(1)} \quad \text{and} \quad E_m = (U'_m \cup A_{\rho(m)}) - \bigcup_{i=1}^{m-1} (U'_i \cup A_{\rho(i)}),
\]

\[
F_1 = V'_1 \cup B_{\rho(1)} \quad \text{and} \quad F_m = (V'_m \cup B_{\rho(m)}) - \bigcup_{i=1}^{m-1} (V'_i \cup B_{\rho(i)}),
\]

for each \( m = 2, 3, 4, \ldots \). Then \( \{E_1, E_2, E_3, \ldots\} \) and \( \{F_1, F_2, F_3, \ldots\} \) are disjoint families of clopen sets of \( X, Y \) respectively, and

\[
\{X - \bigcup_{i=1}^{m} E_i \mid m = 1, 2, 3, \ldots\} \quad \text{and} \quad \{Y - \bigcup_{i=1}^{m} F_i \mid m = 1, 2, 3, \ldots\}
\]

are clopen neighborhood bases of \( a, b \) respectively. Since type \( E_m = \text{type} \quad F_m = (\rho(m)+1, 1) \) and \( \Phi(\rho(m)) = s \) it follows from the induction hypothesis that \( E_m \approx F_m \) for every \( m \). Taking a homeomorphism \( h_m : E_m \rightarrow F_m \) we can define a homeomorphism \( h : X \rightarrow Y \) by

\[
h(x) = \begin{cases} 
  h_m(x) & \text{if} \quad x \in E_m \\
  b & \text{if} \quad x = a.
\end{cases}
\]

Thus \( X \approx Y \). The uniqueness of \( R \) has been verified.

To show the uniqueness of \( R' \) recall that, in \( (\text{r}') \) above, \( \leng(U_1 - U_2) = \gamma - 1 \) and \( \gamma - 1 \) is a limit ordinal. Thus \( U_1 - U_2 \) has the unique topological type by the induction hypothesis. Denoting this type of space by \( J \), we have \( R' \approx R \oplus J \) which assurs the uniqueness of \( R' \).

As for \( X \) of type \( (\gamma, n) \), \( 1 \leq n < \omega \), write \( X = \bigoplus_{i=1}^{n} X_i \) with \( X_i \) of type \( (\gamma, 1) \). Then, as proved above, each \( X_i \) is homeomorphic to \( R \) or \( R \oplus J \). If \( X_i \approx R \) for every \( i \) then \( X \approx nR \).

If \( X_i \approx R \oplus J \) for some \( i \), note that the topological sum of finitely (or countably) many \( J \)'s is homeomorphic to \( J \) because of its uniqueness. We thus have \( X \approx nR \oplus J \). Consequently \( X \) has precisely two topological types because \( nR \) can not absorb \( J \).

To show the uniqueness of \( X \) of type \( (\gamma, \omega) \), write \( X = \bigoplus_{i=1}^{\omega} X_i \) with \( X_i \approx R \) or \( R \oplus J \) so that \( X \approx \omega R \) or \( \omega R \oplus J \). To vanish \( J \), write \( J = \bigoplus_{i=1}^{\omega} J_i \) with \( \leng(J_i) < \gamma - 1 \), and put \( J_1 = U_1 \) and \( J_i = U_i - \bigcup_{i=1}^{m-1} U_m \) for \( i \geq 2 \). Since type \((R \oplus J_1) = (\gamma, 1) \) and the case \( (\text{r}) \) happens in \( R \oplus J \), it follows from the uniqueness of \( R \) that \( R \oplus J \approx R \) so that

\[
\omega R \oplus J \approx (R \oplus J_1) \oplus (R \oplus J_2) \oplus (R \oplus J_3) \oplus \cdots \approx \omega R
\]
as desired.

**In case \( \gamma \) is a non-limit ordinal, \( \gamma - 1 \) is a limit ordinal and \( \Phi(\gamma - 1) = s \):**

This is the easiest case. Let \( X \) be a scattered countable metric space of type \((\gamma, 1)\) with a rankwise uniform type \( \Phi \) satisfying \( \Phi(\gamma - 1) = s \) and (\( \ast \)). Let \( \{a\} = X^{(\gamma - 1)} \) and take a clopen neighborhood base \( X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \) of \( a \) satisfying \( \text{length}(U_n - U_{n+1}) = \gamma - 1 \) for every \( m \). Then by the induction hypothesis, each \( U_m - U_{m+1} \) has the unique topological type so that \( X \) does.

As for the type \((\gamma, m)\), \( 2 \leq m \leq \omega \), we have only to decompose \( X \) as \( X \simeq \bigoplus_{\lambda \in \Lambda} X_{\lambda} \), where \( |\Lambda| = m \) and each \( X_{\lambda} \) is of type \((\gamma, 1)\). The uniqueness of \( X_{\lambda} \) implies the uniqueness of \( X \), which completes the proof.

**Existence.** Let \( \alpha \) be an ordinal with \( 2 \leq \alpha < \omega \) and let \( \Phi : (0, \alpha) \to \{r, s\} \) be a function (which does not necessarily satisfy the condition \( \ast \)). A scattered countable metric space of length \( \alpha \) with the rankwise uniform type \( \Phi \) is given in the following way.

Let \( \mathfrak{p} \) denote the first limit ordinal not smaller than \( \alpha \) and let \( \varphi : [0, \alpha) \to [0, \mathfrak{p}) \) be a function satisfying

1. \( \varphi(0) = 0 \),
2. if \( \beta \) is a non-limit ordinal, then
   \[
   \varphi(\beta) = \begin{cases} 
   \varphi(\beta - 1) + 1 & \text{if } \Phi(\beta) = r \\
   \varphi(\beta - 1) + 2 & \text{if } \Phi(\beta) = s,
   \end{cases}
   \]
3. if \( \beta \) is a limit ordinal, then
   \[
   \varphi(\beta) = \begin{cases} 
   \beta & \text{if } \Phi(\beta) = r \\
   \beta + 1 & \text{if } \Phi(\beta) = s.
   \end{cases}
   \]

It is easily proved that such a function \( \varphi \) exists and is unique. Putting \( K = MS(\mathfrak{p} + 1, 1) \),

\[
X = \bigcup_{\beta \in [0, \alpha)} K_{\varphi(\beta)}
\]

with the topology induced from \( K \). Then \( X \) has the rankwise uniform type \( \Phi \), \( \text{length} X = \alpha \), and type \( X = (\alpha, \omega) \) if \( \alpha \) is a non-limit ordinal.

To obtain \( X' \) of type \((\alpha, m)\) with \( \alpha \) a non-limit ordinal and \( 1 \leq m < \omega \), take \( m \) many points \( x_1, x_2, \ldots, x_m \) in \( K_{\varphi(\alpha - 1)} \) and take a clopen set \( U \) of \( K \) so that

\[
U \cap K_{\varphi(\alpha - 1)} = \{x_1, x_2, \ldots, x_m\}.
\]

Put \( X' = U \cap X \).

As for our exceptional case, the spaces \( R \) in the proof of Theorem 2 can be obtained in this way. If one requires the spaces \( R \oplus J \), define

\[
X'' = X' \oplus (\bigcup_{\beta \in [0, \alpha - 1)} K_{\varphi(\beta)}).
\]

We have thus finished the proof of Theorem 2.

**Remark.** Mazurkiewicz and Sierpiński constructed in \( [2] \) \( 2^{\omega} \) many distinct scattered countable metric spaces of length \( \omega \) by using the notion of ‘lacunae’. However the spaces constructed there do not have rankwise uniform types in our terminology.
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