CONGRUENCES ON BCC-ALGEBRAS

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ABSTRACT. Using fuzzy BCC-ideals, the quotient structure of BCC-algebras is discussed. We show that (1) If \( f : G \rightarrow H \) is an onto homomorphism of BCC-algebras, and if \( \bar{B} \) is a fuzzy BCC-ideal of \( H \), then \( G/f^{-1}(\bar{B}) \) is isomorphic to \( H/\bar{B} \); (2) If \( \bar{A} \) and \( \bar{B} \) are fuzzy BCC-ideals of BCC-algebras \( G \) and \( H \), respectively, then \( \frac{G/\bar{A}}{G/\bar{B}} \cong G/\bar{A} \times H/\bar{B} \); and (3) If \( \bar{A} \) is a fuzzy BCC-ideal of \( G \), and if \( J \) is a BCC-ideal of \( G \) such that \( J/\bar{A} \) is a BCC-ideal of \( G/\bar{A} \), then \( \frac{G/\bar{A}}{J/\bar{A}} \cong G/J \).

1. Introduction

In 1966, Y. Imai and K. Iséki ([8]) defined a class of algebras of type (2,0) called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra ([9]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [11]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori ([10]) introduced a notion of BCC-algebras, and W. A. Dudek ([1, 2]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [6], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun ([3]) considered the fuzzification of BCC-ideals in BCC-algebras. They showed that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing an example. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra, and in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. W. A. Dudek, Y. B. Jun and Z. Stojaković ([3]) described several properties of fuzzy BCC-ideals in BCC-algebras, and discussed an extension of fuzzy BCC-ideals. In this paper we consider the quotient structure of BCC-algebras using fuzzy BCC-ideals. We show that (1) If \( f : G \rightarrow H \) is an onto homomorphism of BCC-algebras, and if \( \bar{B} \) is a fuzzy BCC-ideal of \( H \), then \( G/f^{-1}(\bar{B}) \) is isomorphic to \( H/\bar{B} \); (2) If \( \bar{A} \) and \( \bar{B} \) are fuzzy BCC-ideals of BCC-algebras \( G \) and \( H \), respectively, then \( \frac{G/\bar{A}}{G/\bar{B}} \cong G/\bar{A} \times H/\bar{B} \); and (3) If \( \bar{A} \) is a fuzzy BCC-ideal of \( G \), and if \( J \) is a BCC-ideal of \( G \) such that \( J/\bar{A} \) is a BCC-ideal of \( G/\bar{A} \), then \( \frac{G/\bar{A}}{J/\bar{A}} \cong G/J \).

2. Preliminaries

Recall that a BCC-algebra is an algebra \((G, \ast, 0)\) of type \( (2,0) \) satisfying the following axioms:

(C1) \((x \ast_y) \ast (z \ast_y)) \ast (x \ast z) = 0,

(C2) \(0 \ast x = 0\).

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\[(C3)\; x \ast 0 = x, \]
\[(C4)\; x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y. \]

For every \(x, y, z \in G\). For any \(BCC\)-algebra \(G\), the relation \(\leq\) defined by \(x \leq y\) if and only if \(x \ast y = 0\) is a partial order on \(G\). In a \(BCC\)-algebra \(G\), the following holds (see [7]).

1. \(x \leq x\)
2. \(x \leq y \Rightarrow x \ast y \leq y \ast z \leq y \ast z \leq y \ast x\)

for all \(x, y, z \in G\). Any \(BCK\)-algebra is a \(BCC\)-algebra, but there are \(BCC\)-algebras which are not \(BCK\)-algebras (see [2]). Note that a \(BCC\)-algebra is a \(BCK\)-algebra if and only if it satisfies

- \((x \ast y) \ast z = (x \ast z) \ast y, \forall x, y, z \in G.\)

A non-empty subset \(A\) of a \(BCC\)-algebra \(G\) is called a \(BCC\)-\textit{ideal} of \(G\) if it satisfies

- \(0 \in A,\)
- \(\forall x, y, z \in G, y \in A, (x \ast y) \ast z \in A \Rightarrow x \ast z \in A.\)

Note that any \(BCC\)-ideal of a \(BCC\)-algebra is a \(BCC\)-subalgebra (see [6]).

**Definition 2.1.** [3] A fuzzy set \(\tilde{A}\) in a \(BCC\)-algebra \(G\) is called a \(fuzzy BCC\)-\textit{ideal} of \(G\) if it satisfies

1. \(\tilde{A}(0) \geq \tilde{A}(x), \forall x \in G,\)
2. \(\tilde{A}(x \ast y) \geq \min\{\tilde{A}(x \ast a) \ast y, \tilde{A}(a)\}, \forall a, x, y \in G.\)

**Definition 2.2.** [3] A fuzzy set \(\tilde{A}\) in a \(BCC\)-algebra \(G\) is called a \(fuzzy BCK\)-\textit{ideal} of \(G\) if it satisfies (F1) and

3. \(\tilde{A}(x) \geq \min\{\tilde{A}(x \ast y), \tilde{A}(y)\}, \forall x, y \in G.\)

**Lemma 2.3.** [3, Theorem 4.3] In a \(BCC\)-algebra, every fuzzy \(BCC\)-\textit{ideal} is a fuzzy \(BCK\)-\textit{ideal}.

3. **Congruence relations**

In what follows, let \(G\) denote a \(BCC\)-algebra unless otherwise specified. Let \(\tilde{A}\) be a fuzzy \(BCC\)-\textit{ideal} of \(G\) and \(\alpha \in [0, 1)\). We consider a relation on \(G\) as follows:

\(\mathcal{R}_{\tilde{A}, \alpha} := \{(x, y) \in G \times G \mid \tilde{A}(x \ast y) > \alpha, \tilde{A}(y \ast x) > \alpha\}.\)

**Lemma 3.1.** Let \(\tilde{A}\) be a fuzzy \(BCC\)-\textit{ideal} of \(G\) and \(\alpha \in [0, 1)\). If \(\mathcal{R}_{\tilde{A}, \alpha} \neq \emptyset\), then \(\tilde{A}(0) > \alpha.\)

**Proof.** If \(\mathcal{R}_{\tilde{A}, \alpha} \neq \emptyset\), then there exists \((x, y) \in G \times G\) such that \(\tilde{A}(x \ast y) > \alpha.\) It follows from (F1) that \(\tilde{A}(0) \geq \tilde{A}(x \ast y) > \alpha.\) This completes the proof. \(\square\)

**Proposition 3.2.** Let \(\tilde{A}\) be a fuzzy \(BCC\)-\textit{ideal} of \(G\) and \(\alpha \in [0, 1)\). If \(\mathcal{R}_{\tilde{A}, \alpha} \neq \emptyset\), then \(\mathcal{R}_{\tilde{A}, \alpha}\) is a congruence on \(G\)

**Proof.** Note from (p1) and Lemma 3.1 that \(\tilde{A}(x \ast x) = \tilde{A}(0) > \alpha\) for all \(x \in G\). Hence \((x, x) \in \mathcal{R}_{\tilde{A}, \alpha}\) for all \(x \in G\), and so \(\mathcal{R}_{\tilde{A}, \alpha}\) is reflexive. Obviously, \(\mathcal{R}_{\tilde{A}, \alpha}\) is symmetric. Let \(x, y, z \in G\) be such that \((x, y) \in \mathcal{R}_{\tilde{A}, \alpha}\) and \((y, z) \in \mathcal{R}_{\tilde{A}, \alpha}\). Then \(\tilde{A}(x \ast y) > \alpha, \tilde{A}(y \ast z) > \alpha,\) \(\tilde{A}(y \ast z) > \alpha,\) and \(\tilde{A}(z \ast y) > \alpha.\) Since \(((x \ast z) \ast (y \ast z)) \ast (x \ast y) = 0\) by (C1), we have

\[\tilde{A}(((x \ast z) \ast (y \ast z)) \ast (x \ast y)) = \tilde{A}(0) > \alpha.\]

Since \(\tilde{A}\) is a fuzzy \(BCK\)-\textit{ideal} by Lemma 2.3, it follows from (F3) that

\[\tilde{A}(((x \ast z) \ast (y \ast z)) \ast (x \ast y)) \geq \min\{\tilde{A}(((x \ast z) \ast (y \ast z)) \ast (x \ast y)), \tilde{A}(x \ast y)\} > \alpha\]

so that \(\tilde{A}(x \ast z) \geq \min\{\tilde{A}((x \ast z) \ast (y \ast z)), \tilde{A}(y \ast z)\} > \alpha.\) Similarly we have \(\tilde{A}(z \ast x) > \alpha,\) and thus \((x, z) \in \mathcal{R}_{\tilde{A}, \alpha}\). Therefore \(\mathcal{R}_{\tilde{A}, \alpha}\) is transitive, and hence \(\mathcal{R}_{\tilde{A}, \alpha}\) is an equivalence.
relation on $G$. Now, let $x, y, u, v \in G$ be such that $(x, u) \in \mathcal{R}_\alpha$ and $(y, v) \in \mathcal{R}_\alpha$. Then $\bar{A}(x \cdot u) > \alpha$, $\bar{A}(u \cdot x) > \alpha$, $\bar{A}(y \cdot v) > \alpha$, and $\bar{A}(v \cdot y) > \alpha$. Since $((x \cdot y) \cdot (u \cdot v)) \cdot (x \cdot u) = 0$, we have

$$\bar{A}((x \cdot y) \cdot (u \cdot v)) \geq \min\{\bar{A}(((x \cdot y) \cdot (u \cdot v)) \cdot (x \cdot u)), \bar{A}(x \cdot u)\}$$

$$= \min\{\bar{A}(0), \bar{A}(x \cdot u)\} > \alpha.$$

Similarly, $\bar{A}((u \cdot x) \cdot (v \cdot y)) > \alpha$. Hence $(x \cdot y, u \cdot v) \in \mathcal{R}_\alpha$. On the other hand, since $((u \cdot x) \cdot (v \cdot y)) \cdot (u \cdot v) = 0$, it follows from (F2) and Lemma 3.1 that

$$\bar{A}((u \cdot x) \cdot (v \cdot y)) \geq \min\{\bar{A}(((u \cdot x) \cdot (v \cdot y)) \cdot (u \cdot v)), \bar{A}(v \cdot y)\}$$

$$= \min\{\bar{A}(0), \bar{A}(v \cdot y)\} > \alpha.$$

Similarly, we get $\bar{A}((u \cdot x) \cdot (v \cdot y)) > \alpha$. Therefore $(u \cdot x, u \cdot v) \in \mathcal{R}_\alpha$. Using the transitivity of $\mathcal{R}_\alpha$, we conclude that $(x \cdot y, u \cdot v) \in \mathcal{R}_\alpha$. Consequently, $\mathcal{R}_\alpha$ is a congruence on $G$. □

**Corollary 3.3.** Let $\bar{A}$ be a fuzzy BCC-ideal of $G$ and $\alpha \in [0, 1)$. If $\bar{A}(0) > \alpha$, then $\mathcal{R}_\alpha$ is a congruence on $G$.

Let $\bar{A}$ be a fuzzy BCC-ideal of $G$ and let $\alpha \in [0, 1)$. Denote by $\{x\}^\alpha$ the set $\{y \in G \mid (x, y) \in \mathcal{R}_\alpha\}$ and by $G/\bar{A}$ the set $\{[x]^\alpha \mid x \in G\}$. Define a binary operation $\bar{\circ}$ on $G/\bar{A}$ by

$$[x]^\alpha \circ [y]^\alpha = [x \cdot y]^\alpha$$

for all $x, y \in G$. First we shall verify that the operation $\bar{\circ}$ is well-defined. Assume that $[x]^\alpha = [u]^\alpha$ and $[y]^\alpha = [v]^\alpha$, i.e., $(x, u) \in \mathcal{R}_\alpha$ and $(y, v) \in \mathcal{R}_\alpha$. Then $(x \cdot y, u \cdot v) \in \mathcal{R}_\alpha$, since $\mathcal{R}_\alpha$ is a congruence on $G$. Let $w \in [x]^\alpha \circ [y]^\alpha$. Then $(w, x \cdot y) \in \mathcal{R}_\alpha$, and so $(w, u \cdot v) \in \mathcal{R}_\alpha$. Hence $w \in [u]^\alpha \circ [v]^\alpha$, and therefore $[x]^\alpha \circ [y]^\alpha = [u]^\alpha \circ [v]^\alpha$. Consequently, the operation $\bar{\circ}$ is well-defined. Next we show that $G/\bar{A}$ is a BCC-algebra with respect to the operation $\bar{\circ}$. Let $[x]^\alpha, [y]^\alpha, [z]^\alpha \in G/\bar{A}$. Then

$$([x]^\alpha \circ [y]^\alpha) \circ ([z]^\alpha \circ [w]^\alpha) = ([x \cdot y]^\alpha \circ [z \cdot w]^\alpha) \circ ([x]^\alpha \circ [z]^\alpha)$$

$$= ([x \cdot y]^\alpha \circ [z \cdot w]^\alpha) \circ ([x \cdot z]^\alpha)$$

$$= ([x \cdot y] \cdot (z \cdot w))^\alpha \circ ([x \cdot z]^\alpha)$$

$$= ([x \cdot y] \cdot (z \cdot w))^\alpha \circ ([x \cdot z]^\alpha)$$

which shows that (C1) is true. Similarly, we obtain (C2) and (C3). Suppose that $[x]^\alpha \circ [y]^\alpha = [0]^\alpha$ and $[y]^\alpha \circ [x]^\alpha = [0]^\alpha$. Then $[x \cdot y]^\alpha = [0]^\alpha = [y \cdot x]^\alpha$, which implies that $\bar{A}(x \cdot y) = \bar{A}(y \cdot x) > \alpha$ and $\bar{A}(y \cdot x) = \bar{A}(y \cdot x) > \alpha$. Hence $(x, y) \in \mathcal{R}_\alpha$, and so $[x]^\alpha \circ [y]^\alpha = [0]^\alpha$. Therefore we have the following theorem.

**Theorem 3.4.** If $\bar{A}$ is a fuzzy BCC-ideal of $G$ and $\alpha \in [0, 1)$, then $G/\bar{A}, \bar{\circ}, [0]^\alpha$ is a BCC-algebra.

Using a BCC-ideal, Dudek and Zhang gave a congruence relation on $G$ as follows: Let $J$ be a BCC-ideal of $G$ and let $x, y \in G$. The relation $\sim$ on $G$ defined by

$$x \sim y \quad \text{if and only if} \quad x \cdot y \in J \quad \text{and} \quad y \cdot x \in J$$

is a congruence on $G$ (see [6]). We denote the equivalence class containing $x$ by $[[x]]_J$, i.e.,

$$[[x]]_J := \{y \in G \mid x \sim y\}.$$

Note that $x \sim y$ if and only if $[[x]]_J = [[y]]_J$. Denote the set of all equivalence classes of $G$ by $G/J$, i.e., $G/J := \{[[x]]_J \mid x \in G\}$. Then $(G/J, \bar{\circ}, [[0]]_J)$ is a BCC-algebra.
Let $f$ be a mapping defined on $G$. If $\mathcal{B}$ is a fuzzy set in $f(G)$, then the fuzzy set $f^{-1}(\mathcal{B}) := \bar{B} \circ f$ in $G$, i.e., the fuzzy set defined by $f^{-1}(\bar{B})(x) = \bar{B}(f(x))$ for all $x \in G$, is called the \textit{preimage} of $\mathcal{B}$ under $f$.

**Lemma 3.5.** Let $f : G \rightarrow H$ be an onto homomorphism of BCC-algebras. If $\mathcal{B}$ is a fuzzy BCC-ideal of $H$, then $f^{-1}(\mathcal{B})$ is a fuzzy BCC-ideal of $G$.

**Proof.** Assume that $\mathcal{B}$ is a fuzzy BCC-ideal of $H$. Taking “min” instead of a $t$-norm “$T$” in [4, Proposition 3], we know that $f^{-1}(\mathcal{B})$ is a fuzzy BCC-ideal of $G$. \hfill $\square$

**Theorem 3.6.** Let $f : G \rightarrow H$ be an onto homomorphism of BCC-algebras. If $\mathcal{B}$ is a fuzzy BCC-ideal of $H$, then $G / f^{-1}(\mathcal{B})$ is isomorphic to $H / \mathcal{B}$.

**Proof.** Let $\alpha \in [0, 1)$. Define a mapping $\mathcal{h} : G / f^{-1}(\mathcal{B}) \rightarrow H / \mathcal{B}$ by

$$h([x]_{\alpha}^{f^{-1}(\mathcal{B})}) = [f(x)]_{\alpha}^{\mathcal{B}}, \quad \forall [x]_{\alpha}^{f^{-1}(\mathcal{B})} \in G / f^{-1}(\mathcal{B}).$$

Assume that $[x]_{\alpha}^{f^{-1}(\mathcal{B})} = [y]_{\alpha}^{f^{-1}(\mathcal{B})}$. Then $(x, y) \in R_{f^{-1}(\mathcal{B}), \alpha}$, and so

$$\mathcal{B}([f(x) * f(y)]) = \mathcal{B}([f(x) * f(y)]) = f^{-1}(\mathcal{B})(x * y) > \alpha$$

and

$$\mathcal{B}([f(y) * f(x)]) = \mathcal{B}([f(y) * f(x)]) = f^{-1}(\mathcal{B})(y * x) > \alpha.$$ 

It follows that $(f(x), f(y)) \in R_{f^{-1}(\mathcal{B}), \alpha}$ so that $[f(x)]_{\alpha}^{\mathcal{B}} = [f(y)]_{\alpha}^{\mathcal{B}}$. Hence $\mathcal{h}$ is well-defined. We claim that $\mathcal{h}$ is one-one. For any $[x]_{\alpha}^{f^{-1}(\mathcal{B})}, [y]_{\alpha}^{f^{-1}(\mathcal{B})} \in G / f^{-1}(\mathcal{B})$, if $\mathcal{h}([x]_{\alpha}^{f^{-1}(\mathcal{B})}) = \mathcal{h}([y]_{\alpha}^{f^{-1}(\mathcal{B})})$ then $[f(x)]_{\alpha}^{\mathcal{B}} = [f(y)]_{\alpha}^{\mathcal{B}}$ and hence $([f(x), f(y)]) \in R_{f^{-1}(\mathcal{B}), \alpha}$. Thus

$$f^{-1}(\mathcal{B})(x * y) = \mathcal{B}([f(x) * f(y)]) = \mathcal{B}([f(x) * f(y)]) > \alpha$$

and

$$f^{-1}(\mathcal{B})(y * x) = \mathcal{B}([f(y) * f(x)]) = \mathcal{B}([f(y) * f(x)]) > \alpha.$$ 

Therefore $(x, y) \in R_{f^{-1}(\mathcal{B}), \alpha}$, that is, $[x]_{\alpha}^{f^{-1}(\mathcal{B})} = [y]_{\alpha}^{f^{-1}(\mathcal{B})}$. Obviously, $\mathcal{h}$ is onto. Finally, we show that $\mathcal{h}$ is a homomorphism. Let $[x]_{\alpha}^{f^{-1}(\mathcal{B})}, [y]_{\alpha}^{f^{-1}(\mathcal{B})} \in G / f^{-1}(\mathcal{B})$. Then

$$\mathcal{h}([x]_{\alpha}^{f^{-1}(\mathcal{B})} \circ [y]_{\alpha}^{f^{-1}(\mathcal{B})}) = \mathcal{h}([x * y]_{\alpha}^{f^{-1}(\mathcal{B})}) = [f(x) * f(y)]_{\alpha}^{\mathcal{B}} = [f(x)]_{\alpha}^{\mathcal{B}} \circ [f(y)]_{\alpha}^{\mathcal{B}} = \mathcal{h}([x]_{\alpha}^{f^{-1}(\mathcal{B})}) \circ \mathcal{h}([y]_{\alpha}^{f^{-1}(\mathcal{B})}).$$ 

This proves the theorem. \hfill $\square$

Given a fuzzy BCC-ideal of $G$ and $\alpha \in [0, 1)$, the BCC-homomorphism $\pi : G \rightarrow G / \mathcal{A}$, $x \mapsto [x]_{\alpha}^{\mathcal{A}}$, is called the \textit{natural (or canonical) homomorphism} of $G$ onto $G / \mathcal{A}$. In the above Theorem 3.6, if we define canonical homomorphisms $p : G \rightarrow G / \mathcal{A}$ and $q : H \rightarrow H / \mathcal{B}$ then it is easy to show that $\mathcal{h} \circ p = q \circ f$, i.e., the following diagram commutes:

\[ \begin{array}{ccc}
G & \xrightarrow{f} & H \\
\downarrow p & & \downarrow q \\
G / \mathcal{A} & \xrightarrow{\mathcal{h}} & H / \mathcal{B} \\
\end{array} \]
The fundamental homomorphism theorem for BCC-algebras is well-known, i.e., if $f : G \to H$ is an onto homomorphism of BCC-algebras, then $G / \ker f \cong H$. Given BCC-algebras $G_1$ and $G_2$ define a binary operation "\oplus" on $G_1 \times G_2$ by

$$(x_1, x_2) \oplus (y_1, y_2) := (x_1 \circ y_1, x_2 \circ y_2), \forall (x_1, x_2), (y_1, y_2) \in G_1 \times G_2.$$ 

Then it can be easily seen that $(G_1 \times G_2 ; \circ, (0, 0))$ is a BCC-algebra. Now we discuss a fuzzy BCC-ideal in a BCC-algebra $G_1 \times G_2$.

**Proposition 3.7.** Let $\mathcal{A}$ and $\mathcal{B}$ be fuzzy BCC-ideals of BCC-algebras $G_1$ and $G_2$ respectively. Define a mapping $\mathcal{A} \times \mathcal{B} : G_1 \times G_2 \to [0, 1]$ by

$$(\mathcal{A} \times \mathcal{B})(x, y) = \min\{\mathcal{A}(x), \mathcal{B}(y)\}, \forall (x, y) \in G_1 \times G_2.$$ 

Then $\mathcal{A} \times \mathcal{B}$ is a fuzzy BCC-ideal of $G_1 \times G_2$.

**Proof.** For any $(x, y) \in G_1 \times G_2$ we have

$$(\mathcal{A} \times \mathcal{B})(0, 0) = \min\{\mathcal{A}(0), \mathcal{B}(0)\} \geq \min\{\mathcal{A}(x), \mathcal{B}(y)\} = (\mathcal{A} \times \mathcal{B})(x, y).$$

Let $(x_1, x_2), (y_1, y_2), (a_1, a_2) \in G_1 \times G_2$. Then

$$(\mathcal{A} \times \mathcal{B})(((x_1, x_2) \circ (a_1, a_2)) \circ (y_1, y_2))$$

$$= (\mathcal{A} \times \mathcal{B})((x_1 * a_1) \ast (x_2 * a_2) \ast y_1, (x_2 * a_2) \ast y_2)$$

$$= \min\{\mathcal{A}((x_1 * a_1) \ast y_1), \mathcal{B}((x_2 \ast a_2) \ast y_2)\}$$

and $(\mathcal{A} \times \mathcal{B})(a_1, a_2) = \min\{\mathcal{A}(a_1), \mathcal{B}(a_2)\}$. Hence

$$(\mathcal{A} \times \mathcal{B})((x_1, x_2) \circ (y_1, y_2))$$

$$= (\mathcal{A} \times \mathcal{B})(x_1 * y_1, x_2 * y_2) = \min\{\mathcal{A}(x_1), \mathcal{B}(x_2)\}$$

$$\geq \min\{\min\{\mathcal{A}((x_1 * a_1) \ast y_1), \mathcal{A}(a_1)\}, \min\{\mathcal{B}((x_2 \ast a_2) \ast y_2), \mathcal{B}(a_2)\}\}$$

$$= \min\{\min\{\mathcal{A}((x_1 * a_1) \ast y_1), \mathcal{B}((x_2 \ast a_2) \ast y_2)\}, \min\{\mathcal{A}(a_1), \mathcal{B}(a_2)\}\}$$

$$= \min\{(\mathcal{A} \times \mathcal{B})(((x_1, x_2) \circ (a_1, a_2)) \circ (y_1, y_2)), (\mathcal{A} \times \mathcal{B})(a_1, a_2)\}.$$ 

This shows that $\mathcal{A} \times \mathcal{B}$ is a fuzzy BCC-ideal of $G_1 \times G_2$. □

**Theorem 3.8.** If $\mathcal{A}$ and $\mathcal{B}$ are fuzzy BCC-ideals of BCC-algebras $G$ and $H$, respectively, then $\frac{G \times H}{\mathcal{A} \times \mathcal{B}} \cong G/\mathcal{A} \times H/\mathcal{B}$. 

**Proof.** Let $\alpha \in [0, 1]$. If we define $\Psi : G \times H \to G/\mathcal{A} \times H/\mathcal{B}$ by $\Psi(x, y) = ([x]_\alpha^A, [y]_\alpha^B)$, then it is easy to verify that $\Psi$ is an onto homomorphism. By the fundamental homomorphism theorem, we obtain $\frac{G \times H}{\mathcal{A} \times \mathcal{B}} \cong G/\mathcal{A} \times H/\mathcal{B}$. We now claim that $||((x, y))||_{\mathcal{K}er \Psi} = [((x, y))]_{\mathcal{A} \times \mathcal{B}}$. Indeed,
and

\[(a, b) \in [(x, y)]^A_{\alpha} \times B^B_{\alpha} \]
\[\Leftrightarrow ((a, b), (x, y)) \in \mathbb{R}_A^{A \times B} \]
\[\Leftrightarrow (\tilde{A} \times \tilde{B})((a, b) \circ (x, y)) > \alpha, (\tilde{A} \times \tilde{B})((x, y) \circ (a, b)) > \alpha \]
\[\Leftrightarrow (\tilde{A} \times \tilde{B})(x \circ a, y \circ b) > \alpha, (\tilde{A} \times \tilde{B})(y \circ a, y \circ b) > \alpha \]
\[\Leftrightarrow \min\{\tilde{A}(x \circ a), \tilde{B}(y \circ b)\} > \alpha, \min\{\tilde{A}(x \circ a), \tilde{B}(y \circ b)\} > \alpha \]
\[\Leftrightarrow (a, x) \in \mathbb{R}_A^{A}, (b, y) \in \mathbb{R}_B^{B}, \]

which shows that \(\|(x, y)\|_{\ker \Psi} = [(x, y)]^A_{\alpha} \times B^B_{\alpha}\). Hence

\[\frac{G \times H}{\tilde{A} \times \tilde{B}} = \frac{G \times H}{\ker \Psi} \cong \frac{G}{\tilde{A}} \times \frac{H}{B}, \]

proving the proof.

**Theorem 3.9.** Let \(\tilde{A}\) be a fuzzy BCC-ideal of \(G\) and let \(\alpha \in [0, 1)\). If \(J^*\) is a BCC-ideal of \(G/\tilde{A}\), then there exists a BCC-ideal

\[J := \cup\{[x]_{\alpha}^A | [x]_{\alpha}^A \in J^*\} \]

in \(G\) such that \(J/\tilde{A} = J^*\).

**Proof.** If \(J^*\) is a BCC-ideal of \(G/\tilde{A}\), then \([0]_{\alpha}^A \in J^*\), and so 0 \(\in J\). Let \(x, y, z \in G\) be such that \(y \in J\) and \((x \circ y) \circ z \in J\). Then \(y \in [x]_{\alpha}^A\) and \((x \circ y) \circ z \in [x]_{\alpha}^A\) for some \([a]_{\alpha}^A, [b]_{\alpha}^B \in J^*\).

It follows that \([a]_{\alpha}^A = [a]_{\alpha}^A\) and

\[\frac{[b]_{\alpha}^B = [(x \circ y) \circ z]_{\alpha}^A = ([x]_{\alpha}^A \circ [y]_{\alpha}^A) \circ [z]_{\alpha}^A = ([x]_{\alpha}^A \circ [a]_{\alpha}^A) \circ [z]_{\alpha}^A \]

so that \([x \circ z]_{\alpha}^A = [x]_{\alpha}^A \circ [z]_{\alpha}^A \in J^*\) since \(J^*\) is a BCC-ideal. Thus \(x \circ z \in J\), and so \(J\) is a BCC-ideal of \(G\). Moreover,

\[J/\tilde{A} = \{[u]_{\alpha}^A | u \in J\} \]
\[= \{[u]_{\alpha}^A | \exists [x]_{\alpha}^A \in J^* \text{ such that } u \in [x]_{\alpha}^A\} \]
\[= \{[u]_{\alpha}^A | \exists [x]_{\alpha}^A \in J^* \text{ such that } [u]_{\alpha}^A = [x]_{\alpha}^A\} \]
\[= \{[u]_{\alpha}^A | [u]_{\alpha}^A \in J^*\} = J^*, \]

proving the proof.

**Theorem 3.10.** Let \(\tilde{A}\) be a fuzzy BCC-ideal of \(G\). If \(J\) is a BCC-ideal of \(G\) such that \(J/\tilde{A}\) is a BCC-ideal of \(G/\tilde{A}\), then \(\frac{G/\tilde{A}}{J/\tilde{A}} \cong \frac{G}{J}\).

**Proof.** Define \(\phi : \frac{G/\tilde{A}}{J/\tilde{A}} \rightarrow G/J\) by \(\phi([x]_{\alpha}^A, J/\tilde{A}) = [x]_{\alpha}^A\) for all \([x]_{\alpha}^A, J/\tilde{A} \in G/\tilde{A}\). Suppose that \(\|\|_{\tilde{A}} /J/\tilde{A} = [y]_{\alpha}^B, J/\tilde{A} \in \frac{G/\tilde{A}}{J/\tilde{A}}\). Then \([x]_{\alpha}^A \sim [y]_{\alpha}^B\), and so \([x \circ y]_{\alpha}^A = [x]_{\alpha}^A \circ [y]_{\alpha}^B \in J/\tilde{A}\) and \([y \circ x]_{\alpha}^A = [y]_{\alpha}^B \circ [x]_{\alpha}^A \in J/\tilde{A}\). This means that \(x \circ y \in J\) and \(y \circ x \in J\), i.e., \(x \sim y\). Thus

\[\phi([x]_{\alpha}^A, J/\tilde{A}) = [x]_{\alpha}^A \]
\[= [b]_{\alpha}^B \]
\[= \phi([y]_{\alpha}^B, J/\tilde{A}) = [y]_{\alpha}^B \]

and so \(\phi\) is well defined. For every \([x]_{\alpha}^A, J/\tilde{A}, [y]_{\alpha}^B, J/\tilde{A} \in G/\tilde{A}\), we have

\[\phi([x]_{\alpha}^A, J/\tilde{A} \ast [y]_{\alpha}^B, J/\tilde{A}) = \phi([x]_{\alpha}^A \circ [y]_{\alpha}^B, J/\tilde{A})\]
\[= \phi([x \circ y]_{\alpha}^A, J/\tilde{A}) = [x \circ y]_{\alpha}^A \]
\[= \phi([y \circ x]_{\alpha}^B, J/\tilde{A}) = [y \circ x]_{\alpha}^B \]

which proves that \(\phi\) is well defined and \(\phi\) is a homomorphism.
Hence $\phi$ is a homomorphism. Obviously, $\phi$ is onto. Finally, we show that $\phi$ is one-one. If $\phi([x]_J^A) = \phi([y]_J^A)$, then $[x]_J^A = [y]_J^A$ and hence $x \sim y$. If $[a]_J^A \in [x]_J^A$, then $[a]_J^A \sim [x]_J^A$ and hence $[a \circ x]_J^A = [a]_J^A \circ [x]_J^A \in J/\tilde{A}$ and $[x \circ a]_J^A = [x]_J^A \circ [a]_J^A \in J/\tilde{A}$. It follows that $a \circ x, x \circ a \in J$, i.e., $a \sim x$ so that $a \sim y$. Hence $[a]_J^A \in [y]_J^A$, which shows that $[x]_J^A \subseteq [y]_J^A$. Similarly, we obtain $[y]_J^A \subseteq [x]_J^A$. Therefore $G/\tilde{A} \cong G/J$, proving the proof.

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