ON FIBERING CERTAIN 3-MANIFOLDS OVER THE CIRCLE

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Abstract. The fibration of a certain 3-dimensional manifold over the circle is studied to generalize celebrated Tollefson's theorem. It is proved that if the 3-dimensional manifold admits the proper \( k \)-cyclic action, then it can be fibered over the circle. In addition, the fibration of the orbit space over the circle is obtained.

1 Introduction. In this paper, we study the fibration of certain 3-dimensional manifolds over the circle \( S^1 \). In [2], J. L. Tollefson proved that if the 3-manifold \( M^3 \) admits the proper \( \mathbb{Z}_k \)-action for some prime number \( k \) and \( H_1(M^3/\mathbb{Z}_k;\mathbb{Z}) \) is \( k \)-torsion free, then \( M^3 \) can be fibered over the circle \( S^1 \). The main goal of the present work is to relax the conditions on \( M^3 \) and \( k \) in Tollefson's theorem mentioned above. In addition we obtain the fibered the orbit space \( M^3/\mathbb{Z}_k \) over \( S^1 \).

The contents of the present paper are as follows. In \( \S 2 \), we describe preliminary materials, and state Theorem 1 which is the one of our main results concerned with a simple criteria for the infinite cyclic covering of given CW-complex to be dominated by a finite CW-complex. In \( \S 3 \), we prove several lemmas which are necessary to prove Theorem 1. \( \S 4 \) is devoted to the proof of Theorem 1. In \( \S 5 \), applying Theorem 1, we clarify the condition for the existence of the fibered map \( M^3 \to S^1 \). In \( \S 6 \), applying the result in \( \S 5 \), we consider the fibered of the orbit 3-manifold \( M^3/\mathbb{Z}_k \).

2 Preliminaries and results. Let \( X \) be the topological space. Suppose that \( g : X \to S^1 \) is the continuous map, and \( P_k : S^1 \to S^1 \) is the standard \( k \)-fold covering map defined by \( P_k(t) = t^k \), where \( S^1 \) is the circle. \( W_k \) denotes the \( k \)-fold covering of \( X \) induced by the map \( g \), that is,

\[
W_k = \{(x, t) \in X \times S^1 \mid g(x) = P_k(t)\}
\]

Then we have the commutative diagram

\[
\begin{array}{ccc}
W_k & \xrightarrow{\rho_2} & S^1 \\
\downarrow{\rho_1} & & \downarrow{P_k} \\
X & \xrightarrow{g} & S^1,
\end{array}
\]

where \( \rho_1(x, t) = x \) and \( \rho_2(x, t) = t \). We say that \( W_k \) admits the proper free \( \mathbb{Z}_k \)-action if a generator of the covering \( \mathbb{Z}_k \)-action on \( W_k \) is homotopic to the identity \( \text{Id}_{W_k} \).

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Similarly to the above, we have the infinite cyclic covering $\tilde{X}$ of $X$ induced via $g$ by the commutative diagram
\[
\begin{array}{c}
\tilde{X} \\ \downarrow \quad \downarrow \operatorname{ex}
\end{array}
\begin{array}{c}
\mathbb{R} \\ \operatorname{ex} \\ X \\ \downarrow
\end{array}
\begin{array}{c}
S^1,
\end{array}
\]
where $\operatorname{ex} : \mathbb{R} \to S^1$ is the universal covering map defined by $\operatorname{ex}(s) = \exp(2\pi is)$.

We say that $Y$ is dominated by $Z$ if there exists a continuous maps $\phi : Y \to Z$ and $\psi : Z \to Y$ such that $\psi \circ \phi : Y \to Y$ is homotopic to the identity $\operatorname{Id}_{Y}$.

Now we can state the first main result of the present paper.

**Theorem 1.** Let $X$ be an arewise connected finite CW-complex and $g : X \to S^1$ be a continuous map such that $g_* : \pi_1(X) \rightarrow \pi_1(S^1)$ is an epimorphism. If $W_k$ admits a proper free $\mathbb{Z}_k$-action for some integer $k \geq 2$, then $\tilde{X}$ is dominated by a finite CW-complex.

**3 Lemmas.** In this section we prove five lemmas which are necessary to show Theorem 1.

Firstly we define the following two spaces:
\[
X[m,n] = \{ (x,s) \in X \times [m,n] \mid g(x) = \exp(2\pi is) \},
\]
\[
X[m,n]_* = X[m,n]/(x,m) \sim (x,n),
\]
that is, $X[m,n]_*$ is a space in which $(x,m)$ is identified with $(x,n)$ for every $x \in X$, where $m,n \in \mathbb{Z}$ and $m < n$. Then we have the following.

**Lemma 1.** $X[0,k]_*$ is homeomorphic to $W_k$ defined by $(1)$;
\[
X[0,k]_* \simeq W_k.
\]

**Proof.** Let us consider the maps
\[
\phi_1 : W_k \longrightarrow X[0,k]_*,
\]
and
\[
\psi_1 : X[0,k]_* \longrightarrow W_k
\]
defined by
\[
\phi_1(x,t) = (x,ks)
\]
for $t = \exp(2\pi is) \in S^1$, $(0 \leq s < 1)$, and
\[
\psi_1(x,s) = (x, \exp\left( \frac{2\pi is}{k} \right)), \quad (0 \leq s < k).
\]

It is easy to see that the maps $\phi_1$ and $\psi_1$ are well defined and continuous. Moreover we have
\[
\psi_1 \circ \phi_1(x,t) = \psi_1(x,ks) = (x, \exp(2\pi is)) = (x,t),
\]
where $t = \exp(2\pi is) \in S^1$, $(0 \leq s < 1)$, and
\[
\phi_1 \circ \psi_1(x,s) = \phi_1(x, \exp\left( \frac{2\pi is}{k} \right)) = (x, s)
\]
for $0 \leq s < 1$. Hence $\phi_1$ is the homeomorphism. □

Let $h$ be a generator of the proper free $\mathbb{Z}_k$-action on $W_k$, that is, $h$ is homotopic to $\operatorname{Id}_{W_k}$. Let
\[
H : W_k \times I \longrightarrow W_k
\]
be a homotopy from $\operatorname{Id}_{W_k}$ to $h$, where $I = [0,1]$. 

Put
\[ H(x, t, j) = (h_j^{(1)}(x, t), h_j^{(2)}(x, t)), \quad (x, t) \in W_k, \quad j \in I, \]
then we have
\[ (3) \quad h_j^{(1)}(x, t) = x, \quad h_0^{(2)}(x, t) = t, \quad h_1^{(1)}(x, t) = x, \quad \{h_1^{(2)}(x, t)\}^k = t^k. \]

Let us consider
\[ F = P_k \circ \rho_2 \circ H : W_k \times I \longrightarrow S^1, \]
where \( P_k \) is the standard \( k \)-fold covering map of \( S^1 \) and \( \rho_1(x, t) = x, \rho_2(x, t) = t. \)

We have
\[ (4) \quad F(x, t, j) = P_k \circ \rho_2 \circ H(x, t, j) = P_k \circ \rho_2(h_j^{(1)}(x, t), h_j^{(2)}(x, t)) = P_k(h_j^{(2)}(x, t)) = h_j^{(2)}(x, t)^k = g(h_j^{(1)}(x, t)). \]

From (3) and (4), we have
\[
\begin{align*}
F(x, t, 0) &= (h_0^{(2)}(x, t))^k = t^k \\
F(x, t, 1) &= (h_1^{(2)}(x, t))^k = t^k.
\end{align*}
\]

Hence, if we fix \((x, t) \in W_k,\)
\[ F(x, t, s) : I/\{0, 1\} \simeq S^1 \longrightarrow S^1 \]
defines the map from \( S^1 \) to \( S^1. \) On the one hand, note that \( W_k \) is connected. Thus \( F(x, t, j) \)
takes on only one non-zero degree \( c \) to every \((x, t) \in W_k.\) Next we have

**Lemma 2.** Define \( \tilde{h} : \overline{X} \rightarrow \overline{X} \) by
\[ \tilde{h}(x, ks + km) = (x, ks + km + c), \]
where \( 0 \leq s < 1, \ m \in \mathbb{Z}, \ c \) is the degree of \( F(x, t, j) \) mentioned above, and \( k \) is given in Theorem 1. Then the identity map \( I_{\overline{X}} \) is homotopic to \( \tilde{h}. \)

**Proof.** Let us consider the map
\[ \rho : \overline{X} \ni (x, ks + km) \longmapsto (x, ks) \in \overline{X}[0, k], \quad (0 \leq s < 1, \ m \in \mathbb{Z}). \]
By the definition of the space \( \overline{X}[0, k], \) it is easy to see that the map \( \rho \) is well defined and continuous. Next we show that there exists a map \( \tilde{F} \) such that the following diagram is commutative.
We have immediately
\[
F \circ (\phi_1^{-1} \times \text{Id}) \circ (\rho \times \text{Id})(x, ks + km, j) = F \circ (\phi_1^{-1} \times \text{Id})(x, k, s, j) = F(x, \exp(2\pi is), j) = (h_j^{(2)}(x, \exp(2\pi is))^k).
\]
In particular, for \( j = 0 \), we have
\[
F \circ (\phi_1^{-1} \times \text{Id}) \circ (\rho \times \text{Id})(x, ks + km, 0) = (h_0^{(2)}(x, \exp(2\pi is))^k = (\exp(2\pi is))^k = \exp(2\pi ik) = \exp(2\pi i(k s + km)).
\]
Let us define the map \( \bar{F}_0 : \mathfrak{X} \times \{0\} \to \mathbb{R} \) by
\[
\bar{F}_0(x, ks + km, 0) = ks + km, \quad 0 \leq s < 1, \quad m \in \mathbb{Z},
\]
then \( \bar{F}_0 \) is the lifting of \( F \circ (\phi_1^{-1} \times \text{Id}) \circ (\rho \times \text{Id}) \mid \mathfrak{X} \times \{0\} \). Hence, by the covering homotopy property, there exists a continuous map \( \bar{F} : \mathfrak{X} \times I \to \mathbb{R} \) which is the extension of \( \bar{F}_0 \) such that the above diagram is commutative. We have immediately
\[
\exp(2\pi i(\bar{F}(x, ks + km, j))) = h_j^{(2)}(x, \exp(2\pi is))^k,
\]
\[
\bar{F}(x, ks + km, 0) = ks + km,
\]
\[
\bar{F}(x, ks + km, 1) = ks + km + c.
\]
Let us define the map \( \bar{H} : \mathfrak{X} \times I \to \mathfrak{X} \) by
\[
\bar{H}(x, ks + km, j) = (h_j^{(1)}(x, \exp(2\pi is)), \bar{F}(x, ks + km, j)).
\]
Then the map \( \bar{H} \) turns out to be the homotopy from the identity \( \text{Id}_\mathfrak{X} \) to \( \bar{h} \) by the following three facts (i), (ii) and (iii):
(i) \( \bar{H} \) is well defined, since
\[
(h_j^{(1)}(x, \exp(2\pi is)), \bar{F}(x, ks + km, j)) \in \mathfrak{X}
\]
holds by
\[
g(h_j^{(1)}(x, \exp(2\pi is)) = h_j^{(2)}(x, \exp(2\pi is))^k = \exp(2\pi i\bar{F}(x, ks + km, j)).
\]
(ii) \( \bar{H}(x, ks + km, 0) = (h_0^{(1)}(x, \exp(2\pi is)), \bar{F}(x, ks + km, 0)) = (x, ks + km).
\]
(iii) \( \bar{H}(x, ks + km, 1) = (h_1^{(1)}(x, \exp(2\pi is)), \bar{F}(x, ks + km, 1)) = (x, ks + km + c).
\]
This completes the proof. \( \Box \)

**Remark.** If \( c < 0 \), we define \( \hat{h} : \mathfrak{X} \to \mathfrak{X} \) by \( \hat{h}(x, s) = (x, s - c) \). Then it follows that \( \text{Id}_\mathfrak{X} \) is homotopic to \( \hat{h} \). Hence, in what follows, we assume that the degree \( c \) is a positive integer.
For the degree \( c \), we consider the two spaces \( W_c \) and \( \overline{W}_c \), where \( \overline{W}_c \) is the infinite cyclic covering of \( W_c \) induced by the following \( \overline{\gamma} \):

\[
\begin{array}{ccc}
W_c & \xrightarrow{\overline{\gamma}} & S^1 \\
\downarrow & & \downarrow \overline{p}_c \\
X & \xrightarrow{\gamma} & S^1.
\end{array}
\]

Next we have the following.

Lemma 3. \( X \) is homeomorphic to \( \overline{W}_c \).

Proof. Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
\overline{W}_c & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \text{ex} \\
W_c & \xrightarrow{\overline{\gamma}} & S^1 \\
\downarrow & & \downarrow \overline{p}_c \\
X & \xrightarrow{\gamma} & S^1.
\end{array}
\]

We have

\[
\overline{W}_c = \{(x, t, s) \in W_c \mid \overline{\gamma}(x, t) = \exp(2\pi is) \}\n\]

\[
= \{(x, t, s) \in X \times S^1 \times \mathbb{R} \mid g(x) = t', t = \exp(2\pi is) \}.
\]

Hence \( \overline{W}_c \) is homeomorphic to the space

\[
\{(x, s) \in X \times \mathbb{R} \mid g(x) = \exp(2\pi is) \}.
\]

On the other hand, we have

\[
X = \{(x, s) \in X \times \mathbb{R} \mid g(x) = \exp(2\pi is) \}
\]

Let us define the maps \( \phi_2 \) and \( \psi_2 \) by \( \phi_2(x, s) = (x, s/c) \) and \( \psi_2(x, s) = (x, cs) \) respectively. It is easily seen that the maps \( \phi_2 \) and \( \psi_2 \) are well-defined and continuous. Moreover, it follows that \( \psi_2 \circ \phi_2(x, s) = (x, s) \) and \( \phi_2 \circ \psi_2(x, s) = (x, s) \). Hence \( \phi_2 : X \to \overline{W}_c \) is the homeomorphism. \( \square \)

Next we have the following.

Lemma 4. The map \( \tau : \overline{W}_c \to \overline{W}_c \) defined by \( \tau(x, s) = (x, s + 1) \) is homotopic to \( \text{Id}(\overline{W}_c) \).

Proof. Let us define the map \( \bar{L} \) by the commutative diagram

\[
\begin{array}{ccc}
\overline{X} \times I & \xrightarrow{\overline{H}} & \overline{X} \\
\phi_2 \times \text{Id} \downarrow & & \downarrow \phi_2 \\
\overline{W}_c \times I & \xrightarrow{\bar{L}} & \overline{W}_c.
\end{array}
\]

We have

\[
\bar{L}(x, s, 0) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})^{-1}(x, s, 0)
\]

\[
= \phi_2 \circ \overline{H}(x, cs, 0)
\]

\[
= \phi_2(x, cs) = (x, s),
\]

\[
\bar{L}(x, s, 1) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 1)
\]

\[
= \phi_2 \circ \overline{H}(x, s, 1 + 1)
\]

\[
= \phi_2(x, s) = (x, s),
\]

\[
\bar{L}(x, s, 0) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 0)
\]

\[
= \phi_2 \circ \overline{H}(x, cs, 0)
\]

\[
= \phi_2(x, cs) = (x, s),
\]

\[
\bar{L}(x, s, 1) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 1)
\]

\[
= \phi_2 \circ \overline{H}(x, s, 1 + 1)
\]

\[
= \phi_2(x, s) = (x, s),
\]

\[
\bar{L}(x, s, 0) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 0)
\]

\[
= \phi_2 \circ \overline{H}(x, cs, 0)
\]

\[
= \phi_2(x, cs) = (x, s),
\]

\[
\bar{L}(x, s, 1) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 1)
\]

\[
= \phi_2 \circ \overline{H}(x, s, 1 + 1)
\]

\[
= \phi_2(x, s) = (x, s),
\]

\[
\bar{L}(x, s, 0) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 0)
\]

\[
= \phi_2 \circ \overline{H}(x, cs, 0)
\]

\[
= \phi_2(x, cs) = (x, s),
\]

\[
\bar{L}(x, s, 1) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 1)
\]

\[
= \phi_2 \circ \overline{H}(x, s, 1 + 1)
\]

\[
= \phi_2(x, s) = (x, s),
\]

\[
\bar{L}(x, s, 0) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 0)
\]

\[
= \phi_2 \circ \overline{H}(x, cs, 0)
\]

\[
= \phi_2(x, cs) = (x, s),
\]

\[
\bar{L}(x, s, 1) = \phi_2 \circ \overline{H} \circ (\phi_2 \times \text{Id})(x, s, 1)
\]

\[
= \phi_2 \circ \overline{H}(x, s, 1 + 1)
\]

\[
= \phi_2(x, s) = (x, s),
\]
and
\[ L(x, s, 1) = \phi_2 \circ \widetilde{H} \circ (\phi_2 \times \text{Id})^{-1}(x, s, 1) \]
\[ = \phi_2 \circ \widetilde{H}(x, cs, 1) \]
\[ = \phi_2(x, cs + c) = (x, s + 1). \]

Hence the map $L$ is the homotopy from $\text{Id}_{\overline{W}_c}$ to $\tau$. \hfill \square

Next we have the following.

**Lemma 5.** $\overline{W}_c$ is dominated by $\overline{W}_c \times \mathbb{R}$.\hfill \[\square\]

**Proof.** Put
\[ \overline{W}_c \times \mathbb{R} = \overline{W}_c \times \mathbb{R}/(x, s, t) \sim (x, s + m, t - m), \quad m \in \mathbb{Z}. \]

Now let us define the map $\phi : \overline{W}_c \to \overline{W}_c \times \mathbb{R}$ by $\phi(x, s) = [x, s, 0] \in \overline{W}_c \times \mathbb{R}$. Then $\phi$ is well-defined continuous map. In addition, let us define the map $\psi' : \overline{W}_c \times \mathbb{R} \to \overline{W}_c$ by
\[ \psi'(x, s, t) = \overline{L}(x, s + m, r) \]
if $t = m + r$ for $0 \leq r < 1$ and $m \in \mathbb{Z}$. Then $\psi'$ is well-defined. If $\psi'$ is continuous at $(x, s, 0) \in \overline{W}_c \times \mathbb{R}$, then $\psi'$ turns out to be continuous at every point $(x, s, t) \in \overline{W}_c \times \mathbb{R}$ by the definition of $\psi'$. Therefore it suffices to show that $\psi'$ is continuous at $(x, s, 0)$. Now suppose that $(x_a, s_a, \epsilon_a)$ tends to $(x, s, 0)$ in $\overline{W}_c \times \mathbb{R}$. We have
\[ \psi'(x_a, s_a, \epsilon_a) = \overline{L}(x_a, s_a, \epsilon_a) \to \overline{L}(x, s, 0) = (x, s) \]
as $\epsilon_a \downarrow 0$. On the other hand, we have
\[ \psi'(x_a, s_a, \epsilon_a) = \overline{L}(x_a, s_a, \epsilon_a) \]
\[ = \overline{L}(x_a, s_a - 1, \epsilon_a + 1) \]
\[ = \overline{L}(x, s - 1, 1) = (x, s) \]
as $\epsilon_a \uparrow 0$. Since
\[ \psi'(x, s, 0) = \overline{L}(x, s, 0) = (x, s), \]
it is shown that $\psi'$ is continuous at $(x, s, 0)$. Hence the map $\psi'$ is continuous. Since
\[ \psi'(x, s, t) = \psi'(x, s + m, s - m) \quad m \in \mathbb{Z} \]
by the definition, $\psi'$ can be extended to the continuous map $\psi : \overline{W}_c \times \mathbb{R} \to \overline{W}_c$. Since
\[ \psi \circ \phi(x, s) = \psi(x, s, 0) = \overline{L}(x, s, 0) = (x, s), \]
$\psi \circ \phi$ is the identity map of $\overline{W}_c$. This completes the proof. \hfill \square

4 Proof of Theorem 1. By using the above five lemmas, we can prove Theorem 1 immediately.

In fact, first of all, we have
\[ \mathcal{X} \simeq \overline{W}_c \]
by lemma 3. Next let us regard the infinite covering $\overline{W}_c$ as the principal $\mathbb{Z}$-bundle. Then there exists the associated principal $\mathbb{R}$-bundle
\[ \mathbb{R} \quad \longrightarrow \quad \overline{W}_c \times \mathbb{R} \quad \longrightarrow \quad W_c. \]

On the other hand, by lemma 5, $W_c$ turns out to be dominated by $\overline{W}_c \times \mathbb{R}$. Since $\mathbb{R}$ is $\infty$-connected and $W_c$ is the CW-complex, the associated principal $\mathbb{R}$-bundle is trivial. Hence
\(W_e \times \mathbb{R}\) is homotopy equivalent to \(W_e\). Hence \(X\) is dominated by the finite CW-complex \(W_e\). This completes the proof of Theorem 1.

5 Fibering 3-manifold. In this section, we clarify the condition for the existence of the fibering map \(M^3 \to S^1\) for the 3-manifold \(M^3\).

Let us define two classes \(C_0\) and \(C_1\) as follows. \((X,g) \in C_0\) if and only if \(X\) is the arcwise connected finite CW-complex and \(g : X \to S^1\) is the continuous map such that \(g_* : \pi_1(X) \to \pi_1(S^1)\) is the epimorphism. On the other hand, \((X,g) \in C_1\) if and only if \((X,g) \in C_0\) and there exists an integer \(k \geq 2\) such that a generator of covering \(\mathbb{Z}_k\)-action \(h\) is homotopic to the identity map of \(W_k\), where \(W_k\) is the \(k\)-fold covering over \(S^1\) which is induced from the standard \(k\)-fold covering over \(S^1\) via \(g\).

Here we briefly mention the celebrated theorem due to Stallings concerning with the fibration of 3-manifold; Suppose that the topological 3-manifold \(M^3\) is compact and irreducible. If there exists the exact sequence

\[
0 \longrightarrow G \longrightarrow \pi_1(M^3) \overset{\phi}{\longrightarrow} \pi_1(S^1) \longrightarrow 0
\]

such that \(G\) is finitely generated and \(G \neq \mathbb{Z}/2\mathbb{Z}\). Then there exists the fibering map \(g : M^3 \to S^1\) such that \(g_* = \phi\), and the fiber \(T\) is the connected 2-manifold with \(\pi_1(T) \cong G\).

Then we have the following

**Theorem 2.** Let the topological 3-manifold \(M^3\) be connected, compact and irreducible. Suppose that there exists \(g : M^3 \to S^1\) such that \((M^3,g) \in C_1\). Moreover assume that \(H_1(M^3;\mathbb{Z})\) has no element of order 2. Then there exists the fibering map \(M^3 \to S^1\) which is homotopic to \(g\).

**Proof.** By the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\overline{M^3} & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \text{ex} \\
M^3 & \longrightarrow & S^1,
\end{array}
\]

where \(\overline{M^3}\) is the infinite cyclic covering of \(M^3\) induced via \(g\), we obtain the exact sequences

\[
0 \longrightarrow \pi_1(\mathbb{R}) \longrightarrow \pi_1(S^1) \longrightarrow \mathbb{Z} \longrightarrow 0
\]

\[
0 \longrightarrow \pi_1(M^3) \longrightarrow \pi_1(M^3) \longrightarrow \mathbb{Z} \longrightarrow 0,
\]

where \(g_*\) is the epimorphism. Then we have the exact sequence

\[
0 \longrightarrow \pi_1(M^3) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(S^1) \longrightarrow 0.
\]

Here \(\pi_1(M^3)\) corresponds to the group \(G\) in the fibering theorem due to Stallings mentioned above. Hence it suffices for the proof to show that \(\pi_1(M^3)\) is finitely generated and \(\pi_1(M^3) \neq \mathbb{Z}/2\mathbb{Z}\). By triangulating \(M^3\) as a finite complex, \(M^3\) turns out to be dominated by the finite CW-complex from Theorem 1. By \([3]\), \(\pi_1(M^3)\) is finitely generated. On the other hand, since \(H_1(M^3;\mathbb{Z})\) is assumed to have no elements of order 2, we can conclude that \(\pi_1(M^3)\) has no elements of order 2 from Hurewicz homomorphism. Since \(\pi_1(M^3) \to \pi_1(M^3)\) is the monomorphism, \(\pi_1(M^3)\) has also no elements of order 2. Thus we have shown that
\( \pi_1(\overline{M^2}) \) satisfies the condition of the fibering theorem due to Stallings. This completes the proof. \( \square \)

6 Fibering the orbit 3-manifold. In this section, applying Theorem 2, we consider the fibration of the orbit 3-manifold \( M/\mathbb{Z}_k \). We have the following.

**Theorem 3.** Let the topological 3-manifold \( M^3 \) be connected, compact and irreducible. Suppose that there exists the free \( \mathbb{Z}_k \)-action \((k \geq 2)\) on \( M^3 \) such that there exists a generator \( h \) of the \( \mathbb{Z}_k \)-action which is homotopic to \( \text{Id}_{M^3} \). If \( H_1(M^3/\mathbb{Z}_k;\mathbb{Z}) \) has no elements of order 2 and order \( k \), then both \( M^3/\mathbb{Z}_k \) and \( M^3 \) can be fibered over the circle.

For the proof of Theorem 3, it is necessary to show the following algebraic fact.

**Lemma 6.** Let \( F \) be the finitely generated free module and \( p : F \to \mathbb{Z}_k \) be the epimorphism. Then there exists \( v \in F \) such that \( p(v) \in \mathbb{Z}_k \) is the generator of \( \mathbb{Z}_k \) and there exists the basis \( B \) of \( F \) such that \( v \in B \) and \( B \setminus \{v\} \subset p^{-1}(0) \).

Note that if we fix the isomorphism \( F \cong \mathbb{Z}^l \), Lemma 6 is equivalent to the following Lemma 7 which is almost obvious.

**Lemma 7.** Let \( \{e_1, e_2, \ldots, e_l\} \) be the standard basis of \( \mathbb{Z}^l \). If the greatest common measure of integers \( m_1, m_2, \ldots, m_l \) is 1, then there exists the basis of \( \mathbb{Z}^l \) which contains \( \sum_{j=1}^l m_j e_j \).

Now we can prove Theorem 3.

**Proof.** Since \( M^3 \) is compact and irreducible, \( M^3/\mathbb{Z}_k \) is also compact and irreducible. Let us consider the following diagram.

\[
\begin{array}{ccc}
\mathbb{Z}_k & \longrightarrow & \mathbb{Z}_k \\
\downarrow & & \downarrow \\
M^3 & \longrightarrow & E \\
\downarrow & & \downarrow \\
M^3/\mathbb{Z}_k & \longrightarrow & K(\mathbb{Z}_k, 1),
\end{array}
\]

where \( \mathbb{Z}_k \to E \to K(\mathbb{Z}_k, 1) \) is the universal \( \mathbb{Z}_k \)-bundle. Hence there exists the bundle map \( f : M^3/\mathbb{Z}_k \to K(\mathbb{Z}_k, 1) \). Then we obtain the exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi_1(E) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0 \\
\uparrow & & \uparrow f_* & & \uparrow & & \uparrow & & \uparrow f_* \\
0 & \longrightarrow & \pi_1(M^3) & \longrightarrow & \pi_1(M^3/\mathbb{Z}_k) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0.
\end{array}
\]

By this diagram, it turns out that \( f_* : \pi_1(M^3/\mathbb{Z}_k) \to \mathbb{Z}_k \) is the epimorphism. Since \( \mathbb{Z}_k \) is the abelian group, \( f_* \) can be factored as \( f_* = \theta \cdot \beta \), where \( \theta \) and \( \beta \) are defined by

\[
\begin{array}{ccc}
\pi_1(M^3/\mathbb{Z}_k) & \longrightarrow & \mathbb{Z}_k \\
\beta & \downarrow & \uparrow \theta \\
H_1(M^3/\mathbb{Z}_k; \mathbb{Z}) & \longrightarrow & H_1(M^3/\mathbb{Z}_k; \mathbb{Z}).
\end{array}
\]

\( \square \)
where $\beta$ is Hurewicz homomorphism. Note that $\beta$ and $\theta$ are the epimorphisms, and $H_1(M^3/\mathbb{Z}_k;\mathbb{Z})$ is $k$-torsion free. By lemma 6 and lemma 7, we obtain the following diagram from the above one.

$$
\begin{array}{ccc}
\pi_1(M^3/\mathbb{Z}_k) & \xrightarrow{f_*} & \mathbb{Z}_k \\
\downarrow & & \downarrow \text{mod } k\text{-reduction} \\
F \equiv \bigoplus \mathbb{Z} & \xrightarrow{\text{Projection}} & \mathbb{Z}.
\end{array}
$$

Hence we have the following diagram.

$$
\begin{array}{ccc}
\pi_1(M^3/\mathbb{Z}_k) & \xrightarrow{f_*} & \mathbb{Z}_k \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{\text{Id}} & \mathbb{Z}.
\end{array}
$$

Since $S^1$ and $K(\mathbb{Z}_k,1)$ are Eilenberg-MacLane spaces, there exist the bundle maps

$$
\begin{array}{ccc}
M^3 & \xrightarrow{p} & S^1 \\
\downarrow & & \downarrow \\
M^3/\mathbb{Z}_k & \xrightarrow{g} & S^1 \rightarrow K(\mathbb{Z}_k,1),
\end{array}
$$

where $P_k$ is the standard $k$-fold covering and $p$ is induced from $P_k$. Hence $(M^3/\mathbb{Z}_k,\mathbb{Z})$ belongs to the class $C_1$ and $H_1(M^3/\mathbb{Z}_k;\mathbb{Z})$ has no elements of order 2 by the assumption. Therefore, by Theorem 2, there exists the fibering map which is homotopic to $g$. This completes the proof. \qed

References


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