GENERALIZED K-ASSOCIATIVE BCI-ALGEBRAS

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Abstract. In this paper, we discuss the BCI-algebras satisfying \((x \ast y) \ast z^k \leq x \ast (y \ast z)\),
where \(k\) is a fixed positive integer, and give some properties of such algebras.

1. Introduction and Preliminaries In [1], Q.P. Hu and K. Iséki discussed the BCI-algebras satisfying \((x \ast y) \ast z = x \ast (y \ast z)\), which is called an associative BCI-algebra. Moreover, it’s proved that \(X\) is associative if and only if \(0 \ast x = x\) for all \(x\) in \(X\). In [2], W. Huang discussed the BCI-algebras satisfying \(0 \ast x^k = x\), which is called a \(K\)-associative BCI-algebra. Moreover, it’s proved that \(X\) is \(K\)-associative if and only if \((x \ast y) \ast z^k = x \ast (y \ast z)\). In [3], C.C. Xie investigated the BCI-algebras satisfying \(x \ast (y \ast z) \leq x \ast (y \ast z)\), which is called a quasi-associative BCI-algebra. In this paper, we discuss the BCI-algebras satisfying \((x \ast y) \ast z^k \leq x \ast (y \ast z)\), which is called a generalized \(K\)-associative BCI-algebra.

For any elements \(x, y\) in a BCI-algebra \(X\), we use \(x \ast y^n\) denotes the element \((\cdots (x \ast y) \ast y) \ast y \cdots) \ast y\), where \(y\) occurs \(n\) times.

A BCI-algebra is an algebra \((X; \ast, 0)\) of type \((2, 0)\) with the following conditions.

1. \((x \ast y) \ast (x \ast z) = (x \ast y) \ast (z \ast y) = 0\)
2. \((x \ast (x \ast y)) \ast y = 0\)
3. \(x \ast x = 0\)
4. \(x \ast y = 0 = y \ast x\) implies \(x = y\).

For a BCI-algebra \(X, P(X) = \{x \in X \mid 0 \ast x = 0\}\) is called \(p\)-radical of \(X\). If \(P(X) = 0\), then we call \(X\) is a \(p\)-semisimple BCI-algebra. For any positive integer \(k\), put \(N_k(X) = \{x \in X \mid 0 \ast x^k = 0\}\).

Definition 1.1 A BCI-algebra \(X\) is called a generalized \(K\)-associative if \((x \ast y) \ast z^k \leq x \ast (y \ast z)\) for any \(x, y, z \in X\).

It’s clear that if \(X\) is a generalized \(K\)-associative BCI-algebra then \(0 \ast x^{k+1} = 0\). In fact, let \(x = 0\) and \(y = z = x\), we have \(0 \ast x^{k+1} \leq 0 \ast (x \ast x) = 0\), that is, \(0 \ast x^{k+1} = 0\).

Lemma 1.2 ([2]) Let \(X\) be a BCI-algebra and \(P(X)\) the \(p\)-radical of \(X\). Then \(X = N^{k+1}(X)\) for some positive integer \(k\) if and only if \(X/P(X)\) is \(K\)-associative.

Lemma 1.3 ([2]) Let \(X\) be a BCI-algebra and \(k\) a positive integer, then the following conditions are equivalent:

1. \(0 \ast x = 0 \ast (0 \ast x^k)\)
2. \(0 \ast (0 \ast x) = (0 \ast x^k)\)
3. \(x \in N_{k+1}(X)\)

Lemma 1.4 ([2]) Let \(X\) be a BCI-algebra and \(k\) a positive integer, then the following conditions are equivalent:

1. \(X\) is \(K\)-associative
2. \((x \ast y) \ast z^k = x \ast (y \ast z)\) for all \(x, y\) and \(z\) in \(X\).
Example 1.5 (i) Any K-associative $BCI$-algebra is generalized K-associative.
(ii) Let $X = \{0, a, b\}$ and $s$ be given the table

<table>
<thead>
<tr>
<th>$s$</th>
<th>0</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$b$</td>
</tr>
<tr>
<td>$a$</td>
<td>0</td>
<td>0</td>
<td>$b$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $X$ is generalized K-associative, but not K-associative.

2. Main Results

Theorem 2.1 Let $X$ be a $BCI$-algebra and $P(X)$ the p-radical of $X$, then the following conditions are equivalent: (i) $X$ is generalized K-associative.
(ii) $0 * x^k = 0 * (0 * x)$ for any $x \in X$.
(iii) $X/P(X)$ is K-associative.

Proof. (i) implies (ii) Assume $X$ is generalized K-associative, we have $0 * 0 * z^k \leq 0 * (z * z)$, that is, $0 * z^k \leq 0 * (0 * z)$. On the other hand, we have $(0 * (0 * z)) * (0 * z^k) = 0 * ((0 * z) * z^k) = 0 * (0 * z^{k+1}) = 0$.
(ii) implies (i). Assume that $0 * x^k = 0 * (0 * x)$ holds for any $x \in X$, we have

$$
((0 * y) * z^k) * (0 * (y * z)) = ((0 * y) * (0 * (y * z))) * z^k \\
\leq ((0 * z) * y) * z^k \\
= ((0 * y) * z) * z^k \\
= 0 * z^{k+1} \\
= 0
$$

that is, $(x * y) * z^k \leq x * (y * z)$ for any $x, y, z \in X$. Therefore $X$ is generalized K-associative.

(iii) implies (ii). Assume that $X/P(X)$ is K-associative, we have $X = N^{k+1}(X)$ by Lemma 1.3. Hence $X/P(X)$ is K-associative by Lemma 1.2.

Finally, (ii) implies (iii). Assume that $X/P(X)$ is K-associative, we have $X = N^{k+1}(X)$ by Lemma 1.2, and that $0 * (0 * x) = 0 * x^k$ for any $x \in X$ by Lemma 1.3.

Theorem 2.2 Every generalized K-associative $BCI$-algebra contains a K-associative $BCI$-algebra $A(X)$ such that $X/P(X) \cong A(X)$.

Proof. Put $A(X) = \{x \in X \mid 0 * x^k = x\}$, it’s a K-associative subalgebra. Define a homomorphism by

$$
\Phi : A(X) \to X/P(X)
$$

Let $(X) = C_0$ for some $x \in A(X)$, this means $C_x = C_0$ and $x \in P(X)$. Hence $x = 0 * x^k = 0$, and $\Phi$ is monic. From Theorem 2.1, we know that $X/P(X)$ is K-associative, therefore $C_x = C_0 * C_x^k = C_{0 * x^k}$ holds for each $x$ in $A(X)$. Which shows that is epimorphism, because $0 * x^k \in A(X)$. This completes the proof.

Definition 2.3 An ideal $I$ is called generalized K-associative if for each $x$ in $I$, we have $0 * x^k = 0 * (0 * x)$. 
Theorem 2.4 Every $BCI$-algebras contains a maximal generalized K-associative ideal, which is also a subalgebra.

Proof. Put $Q(X) = \{ x \in X \mid 0 \ast x^k = 0 \ast (0 \ast x) \}$, then it’s a subalgebra. In fact, assume $x, y \in Q(X)$, then $0 \ast x^k = 0 \ast (0 \ast x)$ and $0 \ast y^k = 0 \ast (0 \ast y)$. Hence $0 \ast (x \ast y)^k = (0 \ast x^k) \ast (0 \ast y^k) = (0 \ast (0 \ast x)) \ast (0 \ast (0 \ast y)) = 0 \ast ((0 \ast x) \ast (0 \ast y)) = 0 \ast (0 \ast (x \ast y))$ and that $x \ast y \in Q(X)$. Now we show that it’s also an ideal of $X$. Assume $y, x \ast y \in Q(X)$, then $0 \ast y^{k+1} = 0$ and $0 \ast (x \ast y)^{k+1} = 0$. Hence $0 \ast x^k \ast (0 \ast y^k) = 0 \ast (x \ast y)^k \leq x \ast y$ and $(0 \ast x^k) \ast (x \ast y) \leq 0 \ast y^k$. That is, $(0 \ast (x \ast y)) \ast x^k \leq 0 \ast y^k$. Hence $0 \ast x^k = ((0 \ast (x \ast y)) \ast x^k) \ast (0 \ast (x \ast y)) \leq (0 \ast y^k) \ast (0 \ast (x \ast y)) \leq (x \ast y) \ast y^k$. Therefore $0 \ast x^k \ast x = ((x \ast y) \ast y^k) \ast x = ((x \ast x) \ast y) \ast y^k = 0 \ast y^{k+1} = 0$. This implies that $x \in Q(X)$. The proof is completed.

References

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