PROPERTY(δ) AND NORMALITY OF PRODUCT SPACES

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Abstract. In this paper we shall prove that under the assumption of some conditions, the property (δ) of \( X \times Y \) implies the normality of \( X \times Y \).

1. Introduction.

Throughout this paper we assume that each space is a Hausdorff space and each map is continuous and onto. \( \mathbb{N} \) denotes the set of positive integers. We consider the following question.

Question. Let \( X \) be a space and \( f : Y \to Z \) be a closed map. Suppose \( X \times Y \) is normal. Is \( X \times Z \) normal?

Concerning this, the followings are well known.
(1) In case \( X \) is compact, the above question solved affirmatively by M. E. Rudin [11].
(2) In case \( X \) is a metric space, the above question solved affirmatively by M. E. Rudin and M. Starbird [12].
In this paper we shall obtain some results in connection with the above question.
Refering to the equivalence of normality and countable paracompactness, many results are known. Let us remember the following.
(3) (Hoshina [4]). Suppose \( X \) is a paracompact \( \sigma \)-space and \( Y \) a normal P-space. Then \( X \times Y \) is normal if and only if it is countably paracompact.
(4) (Bešlagić and Chiba [1]). Suppose \( X \) is a first countable paracompact P-space and \( Y \) the closed image of a normal M-space. Then \( X \times Y \) is normal if and only if it is countably paracompact.

In the above theorem (3), Hoshina [4] essentially proved that \( X \times Y \) has property (δ) if and only if it is normal if and only if it is countably paracompact.

In this note, we shall prove that in the above theorem (4), similar result holds.

2. Preliminaries.

Definition 1. A space \( X \) has property \( (\delta) \) [5, p. 145] if, for any open subset \( U_n, n \in \mathbb{N} \), and any closed subset \( B \) such that \( \bigcap_n \overline{U_n} \cap B = \emptyset \) and \( \bigcap_n \overline{U_n} \) and \( B \) are separated by open subsets of \( X \). A subset \( A \) of a space \( X \) is a regular \( G_\delta \)-set if \( A \) is written as \( A = \bigcap_n \overline{U_n} = \bigcap_n \overline{U_n} \) with some open subsets \( U_n \) of \( X \). According to Mack [7], \( X \) is \( \delta \)-normal if, for every pair of disjoint closed subsets one of which is a regular \( G_\delta \)-set, there are disjoint open subsets containing them.

Normality implies property (δ) and property (δ) implies \( \delta \)-normality. By Mack [7], countable paracompactness implies \( \delta \)-normality. Hoshina [5, 2.5, Lemma] improved this result to: countable paracompactness implies property (δ).

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Definition 2. A space X is collectionwise Hausdorff if each discrete collection of points \( \{x_\lambda | \lambda \in \Lambda \} \) is separated by open sets, i.e., there exists a pairwise disjoint collection \( \{U_\lambda | \lambda \in \Lambda \} \) of open sets in X such that \( x_\lambda \in U_\lambda \) for each \( \lambda \in \Lambda \).

Definition 3. A space X is perfect if each closed set is a \( G_\delta \)-set. A space X is perfectly normal if X is normal and perfect.

Definition 4. ([8]). A space X is a \( P \)-space if for any set \( \Omega \) and for any collection \( \{G(\alpha_1, ..., \alpha_n) | \alpha_i \in \Omega, i = 1, 2, ..., n \in N \} \) of open subsets of X such that \( G(\alpha_1, ..., \alpha_n) \subseteq G(\alpha_1, ..., \alpha_n, \alpha_{n+1}) \) for \( \alpha_i \in \Omega, i = 1, 2, ..., n, n+1 \), there exists a collection \( \{F(\alpha_1, ..., \alpha_n) | \alpha_i \in \Omega, i = 1, 2, ..., n \in N \} \) of closed subsets of X such that the conditions (1), (2) below are satisfied:

1. \( F(\alpha_1, ..., \alpha_n) \subseteq G(\alpha_1, ..., \alpha_n) \) for \( \alpha_i \in \Omega, i = 1, 2, ..., n \);
2. \( X = \bigcup_{\alpha \in \Omega} F(\alpha) \) for any sequence \( (\alpha_n) \) such that \( X = \bigcup_{\alpha \in \Omega} F(\alpha) \).

Definition 5. ([8]). A space X is an M-space if there is a sequence \( \{U_m | m \in N \} \) of locally finite open covers of X satisfying the following conditions: For every \( x \in X \) and \( \{K_m | m \in N \} \) a decreasing sequence of nonempty closed subsets of X with \( K_m \subseteq \bigcap \{U \in U_m | x \in U \} \) for \( m \in N \), we have that \( \bigcap_{m \in N} K_m \neq \emptyset \).

Definition 6. ([9],[10]). A space X is an \( \sigma \)-space if it has a \( \sigma \)-locally finite net.

Fact 1. ([3]). Let X be a normal M-space and Y a first countable paracompact \( P \)-space. Then \( X \times Y \) is collectionwise normal.

Fact 2. (1) ([2]). If X is a regular \( \sigma \)-space and \( f : X \rightarrow Y \) is a closed map, then \( Y = \bigcup_{n=0}^\infty Y_n, Y_n \) is closed discrete in Y for each \( n \geq 1 \), for each \( y \in Y_0, f^{-1}(y) \) is compact and \( Y_0 \cap (\bigcup_{n=1}^\infty Y_n) = \emptyset \).

(2) ([6]). If X is a normal M-space and \( f : X \rightarrow Y \) is a closed map, then \( Y = \bigcup_{n=0}^\infty Y_n, Y_n \) is closed discrete in Y for each \( n \geq 1 \), for each \( y \in Y_0, f^{-1}(y) \) is countably compact and \( Y_0 \cap (\bigcup_{n=1}^\infty Y_n) = \emptyset \).

Fact 3. Let A and B be disjoint closed subsets in X. If there are open sets \( U_n \) and \( V_n, n = 0, 1, 2, ... \) such that \( A \subset \bigcup_{n} U_n, B \subset \bigcup_{n} V_n \) and \( \overline{U_n} \cap \overline{V_n} = \emptyset \) for each n, then A and B are separated by open sets in X. In fact, \( U = \bigcup_{n=0}^\infty (U_n \setminus \bigcup_{j \leq n} \overline{U_j}) \) and \( V = \bigcup_{n=0}^\infty (V_n \setminus \bigcup_{j \leq n} \overline{U_j}) \) are disjoint open sets in X such that \( U \supset A \) and \( V \supset B \).

3. THEOREMS

Theorem 1. Let X be a first countable space and \( f : Y \rightarrow Z \) be a closed map which satisfies the condition: \( Z = \bigcup_{n=0}^\infty Z_i, Z_i \) is closed discrete in Z for \( i > 0 \), for each \( z \in Z_0, f^{-1}(z) \) is countably compact and \( Z_0 \cap (\bigcup_{n=1}^\infty Z_i) = \emptyset \). Suppose X \( \times Y \) is normal and Z is collectionwise Hausdorff and perfectly normal. If X \( \times Z \) has property (\( \delta \)), then X \( \times Z \) is normal.

Proof. (1). Let A and B be disjoint closed subsets in X \( \times Z \) such that A \( \subset X \times Z_0 \). Then there are disjoint open sets U and V in X \( \times Z \) such that \( A \subset U, B \subset V \).

(Proof). Put \( g = 1_X \times f : X \times Y \rightarrow X \times Z \) and let \( C = g^{-1}(A), D = g^{-1}(B) \). Then C and D are disjoint closed subsets in X \( \times Y \). Since X \( \times Y \) is normal, there are disjoint open sets G and H in X \( \times Y \) such that C \( \subset G \) and D \( \subset H \).

For each \( a = (x, z) \in A, g^{-1}(a) = \{x \} \times f^{-1}(z) \subset G \). Since \( f^{-1}(z) \) is countably compact and X is first countable, there is an open set \( V_\alpha \) in X and an open set \( W_\alpha \) in Y such that \( \{x \} \times f^{-1}(z) \subset V_\alpha \times W_\alpha \subset G \). Put \( W_0 = Z \setminus f(Y \setminus W_\alpha) \). Then \( W_0 \) is open in Z and \( a \in V_\alpha \times W_\alpha \) and \( f^{-1}(W_\alpha) \subset W_\alpha \). Put \( U = \bigcup \{V_\alpha \times W_\alpha, a \in A \} \). Then U is open in X \( \times Z \) such that \( U \supset A \) and \( g^{-1}(U) \subset G \).
Similarly we can define an open set $V$ such that $V \supset B$ and $g^{-1}(V) \subset H$. Then it is obvious that $U \cap V = \emptyset$.

(II). Let $A$ and $B$ are disjoint closed subsets in $X \times Z$ such that $A \subset X \times Z$ for some $n > 0$. Then there is an open set $U$ in $X \times Z$ such that $A \subset U, \overline{U} \cap B = \emptyset$.

(Proof). Put $B_n = B \cap (X \times Z_n)$. Since $Z$ is collectionwise Hausdorff, there is a pairwise disjoint collection $\mathcal{U} = \{U(z) : z \in Z_n\}$ of open sets in $Z$ such that $z \in U(z)$ for each $z \in Z_n$. For each $z \in Z_n$, put $A_z = \{x \in X | (x, z) \in A\}$ and $B_z = \{x \in X | (x, z) \in B\}$. Then $A_z$ and $B_z$ are disjoint closed subsets in $X$. Since $X$ is normal, there are disjoint open sets $G_z$ and $H_z$ in $X$ such that $A_z \subset G_z$ and $B_z \subset H_z$.

For each $z \in Z_n$, there are open sets $K_z$ and $L_z$ in $X \times Z$ such that $A_z \times \{z\} \subset K_z \subset G_z \times U(z)$ and $K_z \cap B = \emptyset$, $B_z \times \{z\} \subset L_z \subset H_z \times U(z)$ and $L_z \cap A = \emptyset$.

Put $K = \bigcup\{K_z : z \in Z_n\}$ and $L = \bigcup\{L_z : z \in Z_n\}$. Then $K$ and $L$ are disjoint open sets in $X \times Z$ such that $A \subset K$ and $B \subset L$.

Put $B = B \times L$. Then $B$ and $X \times Z$ are disjoint subsets in $X \times Z$. Since $Z$ is perfectly normal, $Z_n$ is a regular $G_\delta$-set. Therefore $X \times Z_n$ is a regular $G_\delta$-set of $X \times Z$.

Since $X \times Z$ has property $(\delta)$, there are open sets $M_n$ and $M'_n$ such that $A \subset M_n$, $X \times Z_n \subset M'_n$, and $M_n \cap M'_n = \emptyset$. Then $A \subset \cap_{n=1}^\infty M_n$, $(\cap_{n=1}^\infty M_n) \cap (\cup_{n=1}^\infty M'_n) = \emptyset$.

Put $B_0 = B \setminus \cup_{n=1}^\infty M'_n$. Then $B_0$ is a closed set of $X \times Z$ such that $B_0 \subset X \times Z_0$. Since $A$ and $B_0$ are disjoint closed sets in $X \times Z$ such that $A \subset M_0$ and $B_0 \subset M_0'$, then $A \subset \cap_{n=1}^\infty M_n$ and $(\cap_{n=1}^\infty M_n) \cap \cup_{n=1}^\infty M'_n = \emptyset$.

(IV). Now we finish the proof. Let $A$ and $B$ are disjoint closed sets in $X \times Z$. Put $A_n = A \cap (X \times Z_n)$ and $B_n = B \cap (X \times Z_n)$ for each $n > 0$. By (II), there are open sets $U_n, n = 1, 2, \ldots$ such that $A_n \subset U_n, \overline{U_n} \cap B = \emptyset$. Put $A_0 = A \setminus \cup_{n=1}^\infty U_n$. Then, by (III), there is an open set $U_0$ in $X \times Z$ such that $A_0 \subset U_0, U_0 \cap B = \emptyset$. Then, $U_n$ are open sets in $X \times Z$ such that $A \subset \cup_{n=1}^\infty U_n, \overline{U_n} \cap B = \emptyset$ for each $n \geq 0$. Similarly there are open sets $V_n, n = 0, 1, 2, \ldots$ such that $B \subset \cup_{n=1}^\infty V_n, \overline{V_n} \cap A = \emptyset$ for each $n$. Therefore, by Fact 3, $A$ and $B$ are separated by open sets.

By the similar proof of Theorem 1, we obtain

**Theorem 2.** Let $X$ be a space and $f : Y \to Z$ be a closed map which satisfy the condition: $Z = \bigcup_{i=0}^\infty Z_i$, $Z_i$ is closed, discrete in $Z$ for $i > 0$, for each $z \in Z_0, f^{-1}(z)$ is compact and $Z_0 \cap (\bigcup_{i=0}^\infty Z_i) = \emptyset$. Suppose $X \times Y$ is normal, $Z$ is collectionwise Hausdorff and perfectly normal. If $X \times Z$ has property $(\delta)$, then $X \times Z$ is normal.

**Corollary 1.** Let $X$ be a space and $f : Y \to Z$ be a closed map. Suppose $X \times Y$ is normal, $Y$ is a $\sigma$-space and $Z$ is a collectionwise Hausdorff space. If $X \times Z$ has property $(\delta)$, then $X \times Z$ is normal.

**Proof.** By Fact 2 (1), $Z = \bigcup_{i=0}^\infty Z_i$, $Z_i$ is closed, discrete in $Z$ for $i > 0$, for each $z \in Z_0, f^{-1}(z)$ is compact and $Z_0 \cap (\bigcup_{i=1}^\infty Z_i) = \emptyset$. By [10, Th. 2.8], $Z$ is perfectly normal. Therefore, Corollary 1 follows from Theorem 2.
Corollary 2. Let $X$ be a space and $f : Y \to Z$ be a closed map. Suppose $X \times Y$ is normal and $Y$ is paracompact $\sigma$-space. If $X \times Z$ has property $(\delta)$, then $X \times Z$ is normal.

Theorem 3. Suppose $X$ is a first countable paracompact $P$-space and $Y$ the closed image of a normal $M$-space. Then the following are equivalent.

1) $X \times Y$ has property $(\delta)$.
2) $X \times Y$ is normal.
3) $X \times Y$ is countably paracompact.

It is sufficient to prove the following.

Proposition. Suppose $X$ is a first countable paracompact $P$-space and $Y$ the closed image of a normal $M$-space. If $X \times Y$ has property $(\delta)$, then $X \times Y$ is normal.

Proof. Let $Z$ be a normal $M$-space and $f : Z \to Y$ a closed map. Then $X \times Z$ is normal by Fact 1. Moreover, by Fact 2 (2), $Y = \bigcup_{i=0}^{\infty} Y_i$ is normal, discrete for each $i \neq 0$ and $f^{-1}(y)$ is countably compact for each $y \in Y_0$ and $Y_0 \cap (\bigcup_{i=0}^{\infty} Y_i) = \emptyset$.

(1) Let $A$ and $B$ are disjoint closed subsets in $X \times Y$ such that $A \subset X \times Y_0$. Then there are disjoint open sets $U$ and $V$ in $X \times Y$ such that $A \subset U$ and $B \subset V$.

Proof is quite similar to that of (1) in Theorem 1.

(II) Let $A$ and $B$ are disjoint closed subsets in $X \times Y$ such that $A \subset X \times Y_n$ for some $n \neq 0$. Then $A$ and $B$ are separated by open sets in $X \times Y$.

(Proof) Let $A_y = \{ x \in X | (x, y) \in A \}$ for each $y \in Y_n$. Since $Y_n$ is discrete and $Y$ is collectionwise normal, there is a discrete collection $\forall \{ U(y) \} y \in Y_n \}$ of open sets in $Y$ such that $y \in U(y)$ for each $y$. Then ($\forall y \in Y_n$, there is an open set $G_y$ in $X \times Y$ such that $A_y \cap \{ y \} \subset G_y \subset X \times U(y)$ and $G_y \cap B = \emptyset$.

(Proof of ($\forall$)). For each $x \in X$, there are open sets $V(x)$ in $X$ and $W_x$ in $Y$ such that $x \in V(x)$, $y \in W_x \subset U(y)$ and $\overline{V(x)} \times \overline{W_x} \cap B = \emptyset$. Then $\forall \{ V(x) | x \in A \} \cup \{ X - A \}$ is an open cover of $X$. Since $X$ is paracompact, there is a locally finite open cover $\forall = \{ U \} \cup \{ V \}$ of $X$ such that $V \subset \forall (x)$ and $V \subset X - A$. Put $G_y = \forall \{ V \times W_x | x \in A \}$. Then $G_y$ has the desired properties.

Let $G = \bigcup \{ G_y | y \in Y_n \}$. Then $G$ is open in $X \times Y$ such that $G \supset A$ and $\overline{G} \cap B = \emptyset$.

(III) Now we finish the proof. Let $A$ and $B$ are disjoint closed subsets in $X \times Y$. Put $A_n = A \cap (X \times Y_n)$ for each $n \neq 0$. Then, by (II), there are open sets $G_n$ in $X \times Y$ such that $A_n \subset G_n$ and $\overline{G_n} \cap B = \emptyset$. Put $A_0 = A \setminus \bigcup_{n=1}^{\infty} G_n$. Then, for each $n \neq 0$, $A_0$ and $X \times Y_n$ are disjoint closed subsets in $X \times Y$. Therefore, by (II), there are open sets $M_n$, $L_n$ in $X \times Y$ such that $A_0 \subset M_n$, $X \times Y_n \subset L_n$ and $M_n \cap L_n = \emptyset$. Hence $\overline{M_n} \cap L_n = \emptyset$. Put $B_0 = B \setminus \bigcup_{n=1}^{\infty} L_n$. Then $A_0$ and $B_0$ are disjoint closed subsets in $X \times Y$ such that $A_0 \subset M_0$ and $B_0 \subset L_0$. Then $A_0 \subset \bigcap_{n=0}^{\infty} M_n$ and $(\bigcap_{n=0}^{\infty} M_n) \cap (\bigcap_{n=0}^{\infty} L_n) = \emptyset$. Hence $\bigcap_{n=0}^{\infty} M_n \subset B_0$. Since $X \times Y$ has property $(\delta)$, there is an open set $G_0$ in $X \times Y$ such that $\bigcap_{n=0}^{\infty} M_n \subset G_0$ and $\overline{G_0} \cap B = \emptyset$. Thus $A_0 \subset G_0$ and $\overline{G_0} \cap B = \emptyset$. Therefore $A \subset \bigcup_{n=0}^{\infty} G_n$ and $\overline{G_n} \cap B = \emptyset$ for each $n \geq 0$.

Similarly we can choose open sets $H_n$ in $X \times Y$ such that $B \subset \bigcup_{n=0}^{\infty} H_n$ and $\overline{H_n} \cap B = \emptyset$ for each $n \geq 0$. Hence $A$ and $B$ are separated by open sets in $X \times Y$ by Fact 3.

References


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