CLOSURE-PRESERVING SUM THEOREMS

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Abstract.

Let $X$ be a space with a closure-preserving cover $\mathcal{F}$ consisting of countably compact closed subsets. In this paper we prove the following: (1) if $X$ is normal and each $F \in \mathcal{F}$ is weakly infinite-dimensional, then $X$ is weakly infinite-dimensional; (2) if $X$ is collectionwise normal and each $F \in \mathcal{F}$ is a $C$-space, then $X$ is a $C$-space.

1 Introduction

We assume that all spaces are normal. In this paper we study sum theorems for infinite-dimensional spaces. Usual and undefined terms can be found in [2].

A space $X$ is weakly infinite-dimensional if for every countable collection $\{ (A_i, B_i) : i \in \mathbb{N} \}$ of pairs of disjoint closed subsets of $X$ there exists a partition $L_i$ in $X$ between $A_i$ and $B_i$ for each $i \in \mathbb{N}$ such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$. If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of a space $X$, then $\mathcal{A} \subset \mathcal{B}$ means that $\mathcal{A}$ is a refinement of $\mathcal{B}$, i.e., for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$. Notice that $\mathcal{A}$ need not be a cover even if $\mathcal{B}$ is a cover. A space $X$ is a $C$-space if for every countable collection $\{ G_i : i \in \mathbb{N} \}$ of open covers of $X$ there exists a countable collection $\{ H_i : i \in \mathbb{N} \}$ of collections of pairwise disjoint open subsets of $X$ such that $H_i < G_i$ and $\bigcup_{i=1}^{\infty} H_i = X$. A collection $\{ A_s : s \in S \}$ of subsets of a space $X$ is closure-preserving if $\text{Cl}(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} \text{Cl} A_s$ for every $S' \subset S$. A collection $\{ A_s : s \in S \}$ of subsets of a space $X$ is hereditarily closure-preserving if every collection $\{ B_s : s \in S \}$, where $B_s \subset A_s$ for every $s \in S$, is closure-preserving. Let us note that every locally finite collection is hereditarily closure-preserving and that every hereditarily closure-preserving collection is closure-preserving.

Let $X$ be a space with a closed cover $\mathcal{F}$ consisting of weakly infinite-dimensional subsets. Hadzilovic [3] proved that $X$ is weakly infinite-dimensional provided that $X$ is countably paracompact and $\mathcal{F}$ is locally finite. Polkowski [5] proved that $X$ is weakly infinite-dimensional provided that $X$ is hereditarily normal and $\mathcal{F}$ is hereditarily closure-preserving. By using the same method of Polkowski, we can easily show that Hadzilovic’s result above holds under the assumption that $\mathcal{F}$ is hereditarily closure-preserving.

Addis and Gresham [1] proved that if a paracompact and hereditarily collectionwise normal space $X$ can be represented as the union of a locally finite collection $\mathcal{F}$ of closed $C$-spaces, then $X$ is a $C$-space. Komoda [4] proved this result under the assumption that $X$ is either paracompact or hereditarily collectionwise normal and $\mathcal{F}$ is hereditarily closure-preserving.

On the other hand Telgársky and Yajima [6] proved that if a space $X$ has a closure-preserving closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is countably compact and $\dim F \leq n$, then $\dim X \leq n$. The purpose of this paper is to give analogous results for weakly infinite-dimensional spaces and for $C$-spaces.

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2 The Main Theorems

The following lemma which was proved by Telgársky and Yajima [6] plays an important role in the proofs of our main theorems.

2.1. Lemma (Telgársky and Yajima [6]). Let $X$ be a space with a closure-preserving cover $\mathcal{F}$ consisting of countably compact closed sets. Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of collections of subsets of $X$ and $\{X_i : i \in \mathbb{N}\}$ a sequence of closed subsets of $X$ satisfying the following conditions:

- $X_{i+1} \subset X_i$, $\mathcal{F}_i$ is a maximal disjoint subcollection of $\mathcal{F}|X_i = \{F \cap X_i : F \in \mathcal{F}\}$, and
- $X_{i+1} \cap \bigcup \mathcal{F}_i = \emptyset$.

Then we have $\bigcap_{i=1}^{\infty} X_i = \emptyset$.

Since every $F_\sigma$-subset of a normal space is also normal, one can easily prove the following lemma.

2.2. Lemma. Let $F$ be a closed subspace of a space $X$ and $Z$ a zero-set in $X$. Let $A$ and $B$ be disjoint closed subsets of $X$ such that $Z \cap A = Z \cap B$. If $Z \cap F$ is a partition in $F$ between $A \cap F$ and $B \cap F$, then there exists a partition $L$ in $X$ between $A$ and $B$ such that $L \cap F \subset Z$.

2.3. Theorem. If a space $X$ has a closure-preserving closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is countably compact and weakly infinite-dimensional, then $X$ is weakly infinite-dimensional.

Proof. Let $\{(A_{ij}, B_{ij}) : i, j \in \mathbb{N}\}$ be a countable collection of pairs of disjoint closed subsets of $X$. For every $i \in \mathbb{N}$, inductively, we shall construct a closed subset $X_i$, a collection $\mathcal{F}_i$ and a closed subset $L_{ij}$ for each $j \in \mathbb{N}$ satisfying the following conditions:

- $\mathcal{F}_i$ is a maximal disjoint subcollection of $\mathcal{F}|X_i$, and
- $L_{ij}$ is a partition in $X$ between $A_{ij}$ and $B_{ij}$ for each $j \in \mathbb{N}$ such that $\bigcup \mathcal{F}_i \cap \bigcup_{j=1}^{\infty} L_{ij} = \emptyset$.

First we set $X_1 = X$. Assume that a closed subset $X_i$ has been constructed. By Zorn’s Lemma, take a maximal disjoint subcollection $\mathcal{F}_i$ of $\mathcal{F}|X_i$. For every $F \cap X_i \in \mathcal{F}_i$, where $F \in \mathcal{F}$, there exists $\ell(F) \in \mathbb{N}$ and a partition $T_{ij}(F)$ between $A_{ij} \cap F \cap X_i$ and $B_{ij} \cap F \cap X_i$ for each $j \in \mathbb{N}$ such that $\bigcup_{j=1}^{\ell} T_{ij}(F) = \emptyset$, because $F \cap X_i$ is countably compact and weakly infinite-dimensional. For every $\ell \in \mathbb{N}$ let us set

$$\mathcal{F}_\ell = \{F \cap X_i \in \mathcal{F}_i : \ell(F) = \ell\}, \quad K_\ell = \bigcup \mathcal{F}_\ell,$$

and

$$T_{ij}(F) = \bigcup \{T_{ij}(F) : F \cap X_i \in \mathcal{F}_\ell\} \text{ for each } j \in \mathbb{N}.$$

Since $\mathcal{F}_\ell$ is a closure-preserving and pairwise disjoint, $\mathcal{F}_\ell$ is discrete in $X$. Thus the set $T_{ij}(F)$ is a partition in $K_\ell$ between $A_{ij} \cap K_\ell$ and $B_{ij} \cap K_\ell$. Since $\{K_\ell : \ell \in \mathbb{N}\}$ is discrete in $X$, there exists a discrete collection $\{U_\ell : \ell \in \mathbb{N}\}$ of open subsets of $X$ such that $K_\ell \subset U_\ell$. Since $\bigcap_{j=1}^{\ell} T_{ij}(F) = \emptyset$, take a zero-set $Z_{ij}$ in $X$ such that $T_{ij}(F) \subset Z_{ij}$, and let $Z_{ij} \cap Z_{ij} = \emptyset$ for all $i, j \in \mathbb{N}$. Let us set $Z_i = \bigcup \{Z_{ij} : j \in \mathbb{N}\}$ and $K_i = \bigcup \{K_\ell : \ell \in \mathbb{N}\}$. Since $\{U_\ell : \ell \in \mathbb{N}\}$ is discrete in $X$, $Z_i$ is a zero-set in $X$ and $Z_i \cap K_i$ is a partition in $K_i$ between $A_{ij} \cap K_i$ and $B_{ij} \cap K_i$. The Lemma 2.2, take a partition $L_{ij}$ in $X$ between $A_{ij}$ and $B_{ij}$ such that $L_{ij} \cap K_i \subset Z_i$. Then we have $\bigcup \mathcal{F}_i \cap \bigcup_{j=1}^{\infty} L_{ij} = K_i \cap \bigcup_{j=1}^{\infty} L_{ij} \subset K_i \cap \bigcup_{j=1}^{\infty} Z_{ij} = \emptyset$.

Finally, we put $X_{i+1} = \bigcup_{j=1}^{\infty} L_{ij} \cap X_i$.

By Lemma 2.1, $\bigcap_{i=1}^{\infty} X_i = \emptyset$, therefore we have $\bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} L_{ij} = \bigcap_{i=1}^{\infty} X_i = \emptyset$. This implies that the space $X$ is weakly infinite-dimensional. Theorem 2.3 has been proved.

Next we consider a sum theorem for $C$-spaces.
2.4. Lemma. Let $F$ be a closed subset of a collectionwise normal space $X$ and $\{G_i : i \in \mathbb{N}\}$ be a collection of open covers of $X$. If $F$ is a countably paracompact $C$-space, then there exists a discrete collection $H_i$ of open subsets of $X$ for each $i \in \mathbb{N}$ such that $H_i < G_i$ and $F \subseteq \bigcup_{i=1}^{\infty} H_i$.

Proof. Take a collection $U_i$ of pairwise disjoint open subsets of $F$ for each $i \in \mathbb{N}$ such that $U_i < G_i/F$ and $\bigcup_{i=1}^{\infty} U_i = F$. For every $U \in U_i$ take $G(U) \in G_i$ with $U \subseteq G(U)$. We set $U_i = \bigcup U_i$ for every $i \in \mathbb{N}$. Obviously, $\{U_i : i \in \mathbb{N}\}$ is a countable open cover of $F$. Thus there exists an open cover $\{V_i : i \in \mathbb{N}\}$ of $F$ with $\text{Cl}(U_i) \subseteq U_i$. Let us set $U' = U \cap V_i$ for every $U \in U_i$. The collection $\{\text{Cl}_X U' : U \in U_i\}$ is discrete in $X$. Hence we can take a discrete collection $\{H(U) : U \in U_i\}$ of open subsets of $X$ such that $\text{Cl}_X U' \subseteq H(U) \subseteq G(U)$. Let us set $H_i = \{H(U) : U \in U_i\}$. Obviously, we have $H_i < G_i$ and $F \subseteq \bigcup_{i=1}^{\infty} H_i$.

2.5. Theorem. If a collectionwise normal space $X$ has a closure-preserving closed cover $F$ such that each $F \in F$ is a countably compact $C$-space, then $X$ is a $C$-space.

Proof. Let $\{G_i : i \in \mathbb{N}\}$ be a countable collection of open covers of $X$. For every $i \in \mathbb{N}$, inductively, we shall construct a closed subset $X_i$, a collection $F_i$ and a collection $H_{ij}$ for each $j \in \mathbb{N}$ satisfying the following conditions:

- $F_i$ is a maximal disjoint subcollection of $F|X_i$, and
- $H_{ij}$ is a collection of pairwise disjoint open subsets of $X$ for each $j \in \mathbb{N}$ such that $H_{ij} < G_i$ and $\bigcup F_i \subseteq \bigcup_{j=1}^{\infty} H_{ij}$.

First we set $X_1 = X$. Assume that a closed subset $X_i$ has been constructed. By Zorn’s Lemma, take a maximal disjoint subcollection $F_i$ of $F|X_i$. Since $F_i$ is closure-preserving and pairwise disjoint, $F_i$ is discrete in $X$. Since $F \cap X_i \in F_i$ is a countably compact $C$-space, $\bigcup F_i$ is a countably paracompact $C$-space. By Lemma 2.4, there exists a collection $H_{ij}$ of pairwise disjoint open subsets of $X$ such that $H_{ij} < G_i$ and $\bigcup F_i \subseteq \bigcup_{j=1}^{\infty} H_{ij}$. Finally, we put $X_{i+1} = X_i - \bigcup_{j=1}^{\infty} H_{ij}$. By Lemma 2.1, $\bigcap_{i=1}^{\infty} X_i = \emptyset$, therefore we have $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} H_{ij} = X$. This implies that the space $X$ is a $C$-space.

2.6. Problem. Does Theorem 2.5 hold under the assumption that $X$ is normal?

References


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