INFINITESIMAL CONFORMAL TRANSFORMATIONS ON TANGENT BUNDLES WITH THE LIFT METRIC I+II

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Abstract. Let $M$ be a complete, simply connected Riemannian manifold with positive constant scalar curvature, and $TM$ its tangent bundle with the lift metric I+II. If $TM$ admits an essential infinitesimal conformal transformation, then $M$ is isometric to the standard sphere. Furthermore if $M$ is compact, then the assumption “essential” is reduced to “non-homothetic”.

1. Introduction

Let $(M, g)$ be a Riemannian manifold and $\nabla$ its Riemannian connection. A transformation $f$ of $M$ is called a projective transformation if it preserves the geodesics, where each geodesic should be confounded with a subset of $M$ by neglecting its affine parameter. Furthermore, $f$ is called an affine transformation if it preserves the connection $\nabla$. We then remark that an affine transformation may be characterized as a projective transformation which preserves the affine parameter together with geodesics.

Let $V$ be a vector field on $M$, and let us consider a local one-parameter group $\{f_t\}$ of local transformations of $M$ generated by $V$. Then $V$ is called an infinitesimal conformal transformation if each $f_t$ is a local conformal transformation. $V$ is called an infinitesimal projective transformation if each $f_t$ is a local projective transformation. Similarly $V$ is called an infinitesimal affine transformation if each $f_t$ is a local affine transformation. Clearly an infinitesimal affine transformation is an infinitesimal projective transformation. The converse is not true in general. Indeed the standard sphere $S^n(c)$ with the the radius $\frac{1}{\sqrt{c}}$, which is a space of positive constant curvature $c$, admits a non-affine infinitesimal projective transformation.

As a converse problem, the following conjecture is known.

Conjecture A. Let $M$ be a complete, simply connected Riemannian manifold with positive constant scalar curvature. Assume that $M$ admits a non-affine infinitesimal projective transformation. Then $M$ is isometric to the standard sphere!

Let $TM$ be the tangent bundle over $M$. Then we can consider some lift metrics on $TM$, for example, the complete lift metric, the lift metric I+II, etc.

Recently one of the authors proved the following

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Theorem A ([Y3]). Let \((M,g)\) be a complete, simply connected Riemannian manifold with positive constant scalar curvature and \(TM\) its tangent bundle with the complete lift metric. If \(TM\) admits an essential infinitesimal conformal transformation, then \(M\) is isometric to the standard sphere.

In this paper, we prove the following

Theorem 1. Let \((M,g)\) be a complete, simply connected Riemannian manifold with positive constant scalar curvature and \(TM\) its tangent bundle with the lift metric \(I+II\). If \(TM\) admits an essential infinitesimal conformal transformation, then \(M\) is isometric to the standard sphere.

Theorem 2. Let \((M,g)\) be a compact, simply connected Riemannian manifold with positive constant scalar curvature and \(TM\) its tangent bundle with the lift metric \(I+II\). If \(TM\) admits a non-homothetic infinitesimal conformal transformation, then \(M\) is isometric to the standard sphere.

Therefore we have the following new conjecture.

Conjecture B. Let \((M,g)\) be a complete, simply connected Riemannian manifold with positive constant scalar curvature and \(TM\) its tangent bundle with the lift metric \(I+II\). Assume that \(TM\) admits a non-homothetic infinitesimal conformal transformation. Then is \(M\) isometric to the standard sphere? 

In the present paper everything will be always discussed in the \(C^\infty\)-category, and Riemannian manifolds will be assumed to be connected and dimension \(n > 1\).

2. Preliminaries

Let \((M,g)\) be a Riemannian manifold and \(\nabla\) its Riemannian connection. Let \(V\) be a vector field on \(M\). It is well-known that \(V\) is an infinitesimal conformal transformation if and only if there exists a function \(\psi\) on \(M\) satisfying

\[L_V g = \psi g,\]

where \(L_V\) is the Lie derivation with respect to \(V\). In this case, \(\psi\) is called the associated function of \(V\). Especially, if \(\psi\) is constant, then \(V\) is called an infinitesimal homothetic transformation. Furthermore, \(V\) is an infinitesimal isometry if and only if \(\psi\) is zero constant. A vector field \(V\) on \(M\) is an infinitesimal projective transformation if and only if there exists a 1-form \(\Omega\) on \(M\) satisfying

\[(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X\]

for any \(X, Y \in T^1(M)\). In this case, \(\Omega\) is called the associated 1-form of \(V\).

We have the following Tanno’s Theorem ([O], [T]):

Lemma 1. Let \((M,g)\) be a complete, simply connected Riemannian manifold. In order that \(M\) admits a non-constant scalar function \(f\) on \(M\) satisfying

\[\nabla_k \nabla_j \nabla_i f + c(2g_k \nabla_i f + g_m \nabla_j f + g_m \nabla_k f) = 0\]

for some positive constant \(c\), it is necessary and sufficient that \(M\) is isometric to the standard sphere of radius \(\frac{1}{\sqrt{c}}\).

One of the authors proved the following
Lemma 2 ([Y1]). Let $(M, g)$ be a compact, simply connected Riemannian manifold with constant scalar curvature $S$. If $M$ admits an non-affine infinitesimal projective transformation, then $S$ is positive and $M$ is isometric to the standard sphere.

Let $\Gamma_{ji}^b$ be the coefficients of $\nabla$, i.e., $\nabla_{\partial_j} \partial_i =: \Gamma_{ji}^a \partial_a$, where $\partial_b := \frac{\partial}{\partial x^b}$ and $(x^b)$ is the local coordinates of $M$. We define a local frame $\{E_i, E_i\}$ of $TM$ as follows:

$$E_i := \partial_i - y^b \Gamma_{ib}^a \partial_a \quad \text{and} \quad E_i := \partial_i,$$

where $(x^b, y^b)$ is the induced coordinates of $TM$ and $\partial_i := \frac{\partial}{\partial y^i}$. $\{E_i, E_i\}$ is called the adapted frame of $TM$. Then $\{dx^b, \delta y^b\}$ is the dual frame of $\{E_i, E_i\}$, where $\delta y^b := dy^b + y^b \Gamma_{ib}^a dx^a$.

By straightforward calculations, we have the following

Lemma 3. The Lie brackets of the adapted frame of $TM$ satisfy the following identities:

1. $[E_j, E_i] = y^b K_{ij}^b E_a$,
2. $[E_j, E_i] = \Gamma_{ja}^c E_a$,
3. $[E_j, E_i] = 0$,

where $K = (K_{kji}^b)$ denotes the Riemannian curvature tensor of $M$ defined by $K_{kji}^b := \partial_b \Gamma_{ji}^k - \partial_j \Gamma_{ki}^b + \Gamma_{ja}^c \Gamma_{ki}^b - \Gamma_{ki}^c \Gamma_{ja}^b$.

Lemma 4. Let $\nabla$ be a vector field on $TM$. Then

1. $L_{\nabla} E_i = [\nabla, E_i] = -(E_i \nabla^a) E_a + (\nabla^b K_{ib}^a - \nabla^b \Gamma_{bi}^a - E_a \nabla^a) E_a$,
2. $L_{\nabla} E_i = [\nabla, E_i] = -(\partial_i \nabla^a) E_a + (\nabla^b \Gamma_{bi}^a - \partial_i \nabla^a) E_a$,
3. $L_{\nabla} dx^b = (E_a \nabla^a) dx^b + (\partial_i \nabla^b) \delta y^a$,
4. $L_{\nabla} \delta y^b = \{y^b \nabla^a K_{bac}^{b} + \nabla^b \Gamma_{ba}^{c} - E_a \nabla^a \nabla^b \} dx^a - (\nabla^b \Gamma_{ba}^{c} - \partial_a \nabla^b) \delta y^a$,

where $(\nabla^b, \partial_i)$ = $\nabla^a E_a + \nabla^a E_a := \nabla$.

We denote by $T_r(M)$ the set of all tensor fields of type $(r, s)$ on $M$. Similarly, we denote by $T^r_s(TM)$ the corresponding set on $TM$.

3. Infinitesimal conformal transformations on $TM$

Let $(M, g)$ be a Riemannian manifold and $TM$ its tangent bundle. The lift metric $I+H$ is defined by $\bar{g} = g_{ab} dx^a dx^b + 2g_{ab} dx^b \delta y^a$. A vector field $\nabla$ on $TM$ is an infinitesimal conformal transformation if and only if there exists a function $\rho$ on $TM$ such that

$$L_{\nabla} \bar{g} = 2 \rho \bar{g}.$$
Proposition. Let \((M, g)\) be a Riemannian manifold and \(TM\) its tangent bundle with the lift metric \(I+II\). Then \(\tilde{V}\) is an infinitesimal conformal transformation with the associated function \(\tilde{\rho}\) on \(TM\) if and only if there exist \(\psi \in T^0_0(M)\), \(B = (B^h)\), \(C = (C^h)\), \(\Phi = (\Phi^h) \in T^1_0(M)\) and \(A = (A_i^h) \in T^1_1(M)\) satisfying

\[
\begin{align*}
(3.1) & \quad (\tilde{V}^h, \tilde{V}^b) = (B^h + y^a A_i^a, C^h - y^a (A_i^a + g^b \nabla B^a) + (2\psi + y^a \Phi^a)y^b), \\
(3.2) & \quad \tilde{\rho} = \psi + y^a \Phi^a, \\
(3.3) & \quad A_{ij} + A_{ji} = 0, \\
(3.4) & \quad \nabla_j A_i^h = \Phi^h \delta_j^i - \Phi^b g_{ji}, \\
(3.5) & \quad L_{B+C} g_{ji} = \nabla_j C_i + \nabla_i C_j + \nabla_j B_i + \nabla_i B_j = 2\psi g_{ji}, \\
(3.6) & \quad L_B \Gamma_{ji}^h = \nabla_j \nabla_i B^h + K_{aji}^h B^a = \Psi_j^i \delta_j^i + \Psi_i \delta_j^i - g_{ji} \Phi^h, \\
(3.7) & \quad \nabla_j \Phi_i^a + \nabla_i \Phi_j^a = 0, \\
(3.8) & \quad K_{akji} A_i^a = -g_{kj} \nabla_j \Phi_i^a + g_{kj} \nabla_j \Phi_k^a,
\end{align*}
\]

where \((\tilde{V}^h, \tilde{V}^b) := V^a E_a + \tilde{\rho} E^a = \tilde{V}, \psi := \partial_i \psi, \text{ and } A_{ij} := A_i^a g_{ai} \text{ etc.}

\[
\text{Proof.} \quad \text{Here we prove only the necessary condition of Proposition because it is easy to prove the sufficient condition.}
\]

Let \(\tilde{V}\) be an infinitesimal conformal transformation with the associated function \(\tilde{\rho}\) on \(TM\). Here we have

\[
\begin{align*}
(L_{\tilde{V}} g) &= \{2g_{ij}(y^b \tilde{V}^b K_{bic}^a + \tilde{V}^b \Gamma_{bic}^a + E_i \tilde{V}^a) + \tilde{V}^a \partial_a g_{ij} + 2g_{ai} E_j \tilde{V}^a \} dx^i dx^j \\
&\quad + 2\{ \tilde{V}^a \partial_a g_{ij} + g_{ai} E_j \tilde{V}^a - g_{aj} (\tilde{V}^b \Gamma_{ib}^a - \partial_i \tilde{V}^a) + g_{aj} \partial_i \tilde{V}^a \} dx^i \delta y^j + 2g_{ai} (\partial_j \tilde{V}^a) \delta y^j \delta y^i.
\end{align*}
\]

From \((L_{\tilde{V}} g) = 2\tilde{\rho} \tilde{g} = 2\tilde{\rho} g_{ij} dx^i dx^j + 4\tilde{\rho} g_{ij} dx^i \delta y^j,\) we obtain

\[
(3.9) \quad g_{ai} \partial_j \tilde{V}^a + g_{aj} \partial_i \tilde{V}^a = 0,
\]

\[
(3.10) \quad \tilde{V}^a \partial_a g_{ij} + g_{ai} E_j \tilde{V}^a - g_{aj} (\tilde{V}^b \Gamma_{ib}^a - \partial_i \tilde{V}^a - \partial_i \tilde{V}^a) = 2\tilde{\rho} g_{ij}
\]

and

\[
(3.11) \quad \tilde{V}^a \partial_a g_{ij} + g_{aj} (y^b \tilde{V}^b K_{bic}^a + \tilde{V}^b \Gamma_{bic}^a + E_i \tilde{V}^a) \\
+ g_{aj} (y^b \tilde{V}^b K_{bic}^a + \tilde{V}^b \Gamma_{bic}^a + E_i \tilde{V}^a + E_j \tilde{V}^a) \\
= 2\tilde{\rho} g_{ji}.
\]

From (3.9) there exist \(B = (B^h) \in T^1_0(M)\) and \(A = (A_i^h) \in T^1_1(M)\) satisfying

\[
(3.12) \quad \tilde{V}^b = B^h + y^a A_i^a \quad \text{and} \quad A_{ji} + A_{ij} = 0.
\]

Substituting (3.12) into (3.10), we have

\[
(3.13) \quad \nabla_j B_i - A_{ji} + g_{aj} \partial_i \tilde{V}^a + y^a \nabla_j A_{ai} = 2\tilde{\rho} g_{ji}.
\]
Operating $\partial_k$ to (3.13), we have
\[ \nabla_j A_{ki} + g_{aj} \partial_k \nabla^a = 2 g_{ji} \partial_k \bar{\rho}, \]
from which, changing the roles of $k$ and $i$ and comparing these equations, we get
\[ (3.14) \quad \nabla_j A_{ki} = g_{ji} \partial_k \bar{\rho} - g_{kj} \partial_i \bar{\rho}. \]
Here we put $\Phi_i := \frac{1}{n-1} \nabla_a A^a_i$. Transacting (3.14) by $g^{ij}$, we obtain
\[ (3.15) \quad \partial_k \bar{\rho} = \Phi_k, \]
from which
\[ (3.16) \quad \nabla_k A_{ji} = \Phi_j g_{ki} - \Phi_i g_{kj} \]
and
\[ (3.17) \quad \bar{\rho} = \psi + y^a \Phi_a, \]
where $\psi$ is a function on $M$. Substituting (3.16) and (3.17) into (3.13), we have
\[ (3.18) \quad \bar{\nabla}^a = C^b - y^a (A^b_a + g^{bb} \nabla_b B_a) + (2 \psi + y^a \Phi_a) y^b, \]
where $C = (C^b)$ is a vector field on $M$.

Substituting (3.12), (3.16), (3.17) and (3.18) into (3.11), we have
\[ (3.19) \quad L_{B+C} g_{ji} = \nabla_j C_i + \nabla_i C_j + \nabla_j B_i + \nabla_i B_j = 2 \psi g_{ji}, \]
\[ (3.20) \quad L_B \Gamma^{ji}_{k} = \nabla_j \nabla_i B^b + K_{a}^{\hat{j}hi} B^a = \Psi_j \delta^i_k + \Psi_i \delta^j_k - g_{ji} \phi_k \]
and
\[ K_{a}^{\hat{i}jk} A^a_k + K_{a}^{\hat{j}ih} A^a_j + K_{a}^{\hat{k}ij} A^a_i + K_{a}^{\hat{j}ih} A^a_j = g_{kj} \nabla_i \phi_k + g_{kj} \nabla_i \phi_j + g_{ki} \nabla_j \phi_i + g_{ki} \nabla_i \phi_k. \]
Transacting (3.21) by $g^{kb}$, we get
\[ (3.22) \quad \nabla_j \phi_i + \nabla_i \phi_j = 0. \]

Lastly we pove
\[ (3.23) \quad K_{a}^{\hat{i}jk} A^a_k = -g_{kj} \nabla_i \phi_k + g_{ki} \nabla_j \phi_k. \]
In fact, we put $P_{k}^{ji} := K_{a}^{\hat{i}jk} A^a_k + g_{kj} \nabla_i \phi_k - g_{ki} \nabla_j \phi_k$. Then (3.21) is rewritten as follows:
\[ (3.24) \quad P_{k}^{ji} + P_{k}^{ij} + P_{j}^{kikh} + P_{j}^{kikh} = 0. \]
By virtue of the first Bianchi identity, we have
\[ (3.25) \quad P_{k}^{ji} + P_{j}^{ik} + P_{i}^{kj} = 0. \]
On the other hand, applying the Ricci identity to (3.4), we get
\[ (3.26) \quad P_{k}^{ji} + P_{k}^{ij} = 0. \]
Using (3.24), (3.25) and (3.26), we obtain $P_{k}^{ji} = 0$, i.e., (3.23). This completes the proof of the necessary condition.
Q.E.D.
Corollary 1. Let \((M, g)\) be a Riemannian manifold and \(TM\) its tangent bundle with the lift metric \(I+II\). Then \(\tilde{V}\) is an inessential infinitesimal conformal transformation with the associated function \(\psi\) on \(TM\) if and only if there exist \(B = (B^b), C = (C^b) \in \mathcal{T}^1_i(M)\) and \(A = (A^b_a) \in \mathcal{T}^1_i(M)\) satisfying

\[
\begin{align*}
(1) & \quad (\tilde{V}^h_a, \tilde{V}^b) = (B^h + y^a A^h_a, C^b - y^a (A^h_a + g^{bh} \nabla_b B^a) + 2 \psi y^b), \\
(2) & \quad A_{ji} + A_{ij} = 0, \\
(3) & \quad \nabla_j A^h_a = 0, \\
(4) & \quad L_{B^h} g_{ij} = \nabla_j C_i + \nabla_i C_j + \nabla_j B_i + \nabla_i B_j = 2 \psi g_{ij}, \\
(5) & \quad L_{B^h} \Gamma_{ji}^h = \nabla_j \nabla_i B^h + K_{a^i j}^h B^a = \Psi_j \delta^h_i + \Psi_i \delta^h_j, \\
(6) & \quad K_{akji}^h A^h_a = 0, \\
\end{align*}
\]

where \((\tilde{V}^h, \tilde{V}^b) := \tilde{V}^a E_a + \tilde{V}^b E_b \equiv \tilde{V}\) and \(\Psi_i := \nabla_i \psi = \partial_i \psi\).

4. Proofs of Theorems

Proof of Theorem 1.

Let \((M, g)\) be a complete, simply connected Riemannian manifold with positive constant scalar curvature \(S\), and \(TM\) its tangent bundle with the lift metric \(I+II\): \(\tilde{g} := g_{ab} dx^a dx^b + 2 \bar{g}_{a b} dy^a\). Assume that \(TM\) admits an essential infinitesimal conformal transformation \(\tilde{V}\).

Operating \(\nabla_i\) to (3.8) and using (3.4), we have

\[(\nabla_i K_{akji})^a A^a_i = -K_{ikji}^b \Phi^b + g_{kb} K_{akji}^b \Phi^b - g_{kb} \nabla_i \nabla_j \Phi^b + g_{kb} \nabla_i \nabla_j \Phi^b.\]

Transvecting (4.1) by \(g^b\) and using the Ricci identity, we get

\[
(\nabla_b K_{akji})^a A^a_i = (n - 1) K_{akji}^b \Phi^b - k_{kj} \nabla_i \nabla_j \Phi^a + g_{ki} \nabla_i \nabla_j \Phi^a
= (n - 1) K_{akji}^b \Phi^b - k_{kj} R_{ai} \Phi^a + g_{ki} R_{a j} \Phi^a,
\]

where \(R = (R_{ji})\) is the Ricci tensor of \(M\).

On the other hand, using (3.7) and (3.8), we have

\[K_{bji}^a A^a_i = \nabla_j \Phi^a.\]

Operating \(\nabla_h\) to (4.3), and using (3.4) and the first Bianchi identity, we find

\[(\nabla_k K_{hji}^a) A^a_i = -K_{bji}^a \nabla_k A^a_i + \nabla_k \nabla_j \Phi^a
= \nabla_k \nabla_j \Phi^a + K_{akji}^a \Phi^a,
\]

from which, we obtain

\[
(\nabla_b K_{akji})^a A^a_i
= (\nabla_i K_{akji} - \nabla_j K_{akbi})^a A^a_i
= \nabla_i \nabla_j \Phi^a + K_{a^i j}^a \Phi^a - \nabla_j \nabla_i \Phi^a + K_{ajki}^a \Phi^a
= -(K_{akji} + K_{ajki} + K_{akij}) \Phi^a = 0.
\]
Therefore (4.2) is reduced to
\[(n-1) K_{\alpha j i} \Phi^\alpha = (g_{k j} R_{\alpha i} - g_{k i} R_{\alpha j}) \Phi^\alpha.\]  
(4.5)

Transvecting (3.8) by $g^{k j}$, we have
\[R_{\alpha i} A^\alpha_h = (n - 1) \nabla_h \Phi_i.\]  
(4.6)

Operating $\nabla_k$ to (4.6) and using (3.4), we have
\[(n - 1) \nabla_k \nabla_j \Phi_i = (\nabla_k R_{\alpha i}) A_j^\alpha + R_{k i} \Phi_j - g_{k j} R_{\alpha i} \Phi^\alpha,\]  
(4.7)

from which, since the scalar curvature $S$ of $M$ is constant, we obtain
\[n R_{\alpha i} \Phi^\alpha = S \Phi_i.\]  
(4.8)

From (3.4) and (4.6), we have
\[R_{\alpha j} A^\alpha_i + R_{\alpha i} A^\alpha_j = 0.\]  
(4.9)

Operating $\nabla_k$ to (4.9) and using (3.4), we get
\[Q_{k j i} = -Q_{k i j},\]  
(4.10)

where $Q_{k j i} := (\nabla_k R_{\alpha j}) A^\alpha_i + R_{k j} \Phi_i - g_{k j} R_{\alpha i} \Phi^\alpha$. Transvecting (4.1) by $g^k$ and using the second Bianchi identity and the Ricci identity,
\[\nabla_j R_{\alpha i} - \nabla_i R_{\alpha j} A^\alpha_h = 0.\]  
(4.11)

From the definition of $Q = (Q_{k j i})$ and (4.11), we get
\[Q_{k j i} = Q_{j k i}.\]  
(4.12)

Therefore, using (4.10) and (4.12), we obtain $Q_{k j i} = 0$, i.e.,
\[(\nabla_k R_{\alpha j}) A^\alpha_i = -R_{k j} \Phi_i + g_{k j} R_{\alpha i} \Phi^\alpha.\]  
(4.13)

Substituting (4.13) into (4.7) and using (4.5), we obtain
\[L_\Phi \Gamma^h_{j i} = \nabla_j \nabla_i \Phi^h + K \Gamma^h_{j i} \Phi^\alpha = 0.\]  
(4.14)

Here we put $f := \frac{n(n-1)}{2S} \Phi^\alpha$. First we assume that $f$ is non-constant. Transvecting (4.5) by $\Phi^i$ and using (4.8), we obtain
\[F_i = \nabla_i f = A^\alpha_i \Phi^\alpha.\]  
(4.15)

where $F_i := \nabla_i f = \partial_i f$. Operating $\nabla_j$ to (4.15) and using (3.4), we have
\[\nabla_j F_i = A^\alpha_i \nabla_j \Phi^\alpha + \Phi_j \Phi_i - (\Phi^\alpha \Phi^\alpha) g_{ji}.\]  
(4.16)
Operating $\nabla_k$ to (4.16), and using (3.4), (4.5) and (4.14), we obtain
\[
\nabla_k \nabla_j F_i = A_i^a \nabla_k \nabla_j \Phi_a - \frac{S}{n(n-1)} F_j g_{ki} + \Phi_j \nabla_k \Phi_i - \frac{2S}{n(n-1)} F_k g_{ji} \\
= -A_i^a K_{bkji} \Phi^b + \Phi_j \nabla_k \Phi_i - \frac{S}{n(n-1)} (2F_{jkji} + F_j g_{ki}) \\
= -\frac{S}{n(n-1)} (2F_k g_{ji} + F_j g_{ki} + F_i g_{kj}).
\]
(4.17)

Therefore, by virtue of Lemma 1, $M$ is isometric to the standard sphere.

Next, we assume that $f$ is constant. Then $f$ is non-zero because the infinitesimal conformal transformation $\bar{V}$ is essential. From (4.15) we have
\[
A_i^a \Phi_a = 0.
\]
(4.18)

Transvecting (4.5) by $A^i_b$, and using (3.8), (4.6) and (4.8), we have
\[
(R_{ah} A_k^a - \frac{S}{n} A_{kh}) \Phi_j = 0,
\]
from which, because $f$ is non-zero constant, we get
\[
R_{ah} A_k^a = \frac{S}{n} A_{kh}.
\]
(4.19)

Substituting (4.6) and (4.19) into (3.8),
\[
K_{bkji} A_k^a = -g_{kj} \nabla_i \Phi_b + g_{ki} \nabla_j \Phi_b \\
= \frac{1}{n-1} (g_{kj} R_{ai} - g_{ki} R_{aj}) A_k^a \\
= \frac{S}{n(n-1)} (g_{ki} A_j b - g_{kj} A_i b).
\]
(4.20)

Operating $\nabla_i$ to (4.20), and using (3.4), (4.5) and (4.8), we get
\[
(\nabla_i K_{bkji}) A_k^a + K_{ikji} \Phi_b = \frac{S}{n(n-1)} (g_{ik} g_{kj} - g_{kj} g_{ki}) \Phi_i.
\]
(4.21)

Because $f$ is non-zero constant, transvecting (4.21) by $\Phi_b$ and using (4.18), we obtain
\[
K_{ikji} = \frac{S}{n(n-1)} (g_{ik} g_{kj} - g_{kj} g_{ki}).
\]
(4.22)

Thus $M$ is a space of positive constant curvature. Therefore $M$ is isometric to the standard sphere.

Q.E.D.

Proof of Theorem 2.

Let $\bar{V}$ be a non-homothetic infinitesimal conformal transformation on $TM$. If $\bar{V}$ is essential, then $M$ is isometric to the standard sphere by virtue of Theorem 1. If $\bar{V}$ is inessential, then there exists a non-affine infinitesimal projective transformation $B$ on $M$ by virtue of Corollary 1 (5). In fact, the associated function $\psi$ of $\bar{V}$ is non-constant, i.e., the associated 1-form $\Psi = (\Psi_i)$ of $B$ is non-zero, because $\bar{V}$ is non-homothetic. Therefore, by virtue of Lemma 2, $M$ is also isometric to the standard sphere.

Q.E.D.
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