ON THE NUMBER OF THE NON-EQUIVALENT 1-REGULAR SPANNING SUBGRAPHS OF THE COMPLETE GRAPHS OF EVEN ORDER

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ABSTRACT. The Dihedral group $D_n$ acts on the complete graph $K_n$ naturally. This action of $D_n$ induces the action on the set of the 1-regular spanning subgraphs of the complete graph $K_n$ of even order $n$. In this paper we calculate the number of the equivalence classes of the 1-regular spanning subgraphs of the complete graph $K_n$ of even order $n$ by this action by using Burnside’s Lemma. This problem was presented by Dr. Shun-ichiro Koh who is a physicist of Kochi University. Also we calculate the number of the equivalence classes of the maximal matchings of the complete graph $K_n$ with odd order $n$ by the group action of the Dihedral group $D_n$.

Let $n$ be even and be greater than or equal to 2. Let $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ be the vertices of the complete graph $K_n$. The action to $K_n$ of the Dihedral group $D_n = \{\rho_0, \rho_1, \ldots, \rho_{n-1}, \sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$ is defined by

\[ \rho_i(v_k) = v_{(k+i) \mod n} \text{ for } 0 \leq i \leq n - 1, \quad 0 \leq k \leq n - 1 \]

\[ \sigma_i(v_k) = v_{(n+i-k) \mod n} \text{ for } 0 \leq i \leq n - 1, \quad 0 \leq k \leq n - 1 \]

Let $X_n$ be the set of the 1-regular spanning subgraphs of $K_n$. Then the above action induces the action on $X_n$ of the Dihedral group $D_n$.

The equivalence classes of $X_4$ are given with the next figure.

The equivalence classes of $X_6$ are given with the next figure.

The equivalence classes of $X_8$ are given with the next figure.

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We calculate the number of the equivalence classes by this group action. This problem was presented by Dr. Shun-ichiro Koh who is a physicist of Kochi University. These computations can be done by using Burnside’s lemma.

Definition 1. Let $P$ be a nonempty collection of permutations on the same finite set of objects $Y$ such that $P$ is a group. Then the mathematical structure $[P : Y]$ is a permutation group.

Definition 2. Let $P = [P : Y]$ be a permutation group, and let $\pi \in P$. The fixed-point set of the permutation $\pi$ is the subset $\text{Fix}(\pi) = \{ y \in Y | \pi(y) = y \}$.

Definition 3. Let $P = [P : Y]$ be a permutation group. The orbit of an object $y \in Y$ is the set $\{ \pi(y) | \pi \in P \}$ of all the objects onto which $y$ is permuted.

Theorem 1. (Burnside’s lemma) Let $P = [P : Y]$ be a permutation group with $n$ orbits. Then

$$ n = \frac{1}{|P|} \sum_{\pi \in P} |\text{Fix}(\pi)| $$

Notation 1. Let $(2k + 1)!! = \prod_{d=0}^{k} (2d + 1)$ for $k \geq 0$ and $(-1)!! = 1$.

Our main theorem is the following:

Theorem 2. The number of the non-equivalent 1-regular spanning subgraphs of the complete graph $K_n$ of even order $n$ is

$$ \frac{1}{2n} \left\{ \sum_{i=0}^{n-1} R_i^n + \frac{n}{2} (S_n + S_{n-2}) \right\} $$

Here $R_i^n$ is given by

1. in the case $(n, i) = 2d+1$:
   $$ \sum_{k=0}^{d} \frac{(2d + 1)}{2k + 1} \times (2d - 2k - 1)!! \times \left( \frac{n}{2d + 1} \right)^{d-k} $$

2. in the case $(n, i) = 2d$:
   if $n/2d \equiv 1 \pmod{2}$ then
   $$ (2d - 1)!! \times \left( \frac{n}{2d} \right)^d $$
   if $n/2d \equiv 0 \pmod{2}$ then
   $$ \sum_{k=0}^{d} \frac{(2d)}{2k} \times (2d - 2k - 1)!! \times \left( \frac{n}{2d} \right)^{d-k} $$

And $S_n$ is given by the following recursive formula:

$$ S_0 = 1, S_2 = 1, S_n = S_{n-2} + (n-2)S_{n-4} \text{ for } n \geq 4 $$

We must determine the numbers of the fixed points of each permutation $\rho_i$ and $\sigma_i$ to prove the Theorem by using Burnside’s Lemma.
Lemma 1. The number of the 1-regular spanning subgraphs of $K_n$ is $(n-1)!!$. This is the number of the fixed points of $\rho_i$.

Proof. We prove this lemma by the induction on $n$. The number of the 1-regular spanning subgraphs of $K_2$ is one. We suppose that the number of the 1-regular spanning subgraphs of $K_{n-2}$ is $(n-3)!!$. For each edge $(v_0, v_i)$ of $K_n$, $1 \leq i \leq n-1$, there are $(n-3)!!$ 1-regular spanning subgraphs of $K_n - \{v_0, v_i\}$. Then totally there are $(n-1)!!$ 1-regular spanning subgraphs of $K_n$. □

Remark 1. It is easily checked that $P_0^n$ is equal to $(n-1)!!$.

Lemma 2. If $(n,i)=1$ then the number of the fixed points of $\rho_i$ is one.

Proof. If $H = \{v \in v_{n/2+\alpha} \mid 0 \leq \alpha \leq n/2-1\}$ then $H$ is a 1-regular spanning subgraph of $K_n$ and $\rho_i(H) = H$. Conversely, let $H$ be a 1-regular spanning subgraph of $K_n$ which is fixed by $\rho_i$ and let $\nu \nu_{n/2}$ be an edge of $H$. Since $(n,i)=1$, there is an integer $\alpha$ such that $\alpha \equiv m \mod n$. Then $\rho_i^\alpha(\nu_0) = \nu_m$ and $\rho_i^\alpha(\nu_m) = \nu_{(m+i) \mod n}$. Since $\rho_i(H) = H$, we have $\nu_0 \nu_m = \nu_{(m+i) \mod n}$. Then we have $m + \alpha \equiv 0 \mod n$ and $2m \equiv 0 \mod n$ and therefore $m = n/2$ and $\nu_0 \nu_{n/2} \in H$. Since $\{\rho_i^{\alpha}(0) \mid 0 \leq \alpha \leq n-1\} = \{0, 1, 2, \ldots, n-1\}$, $H$ is uniquely determined by $\nu_0 \nu_{n/2}$ and $H = \{v \in v_{n/2+\alpha} \mid 0 \leq \alpha \leq n/2-1\}$. Then the number of the fixed points of $\rho_i$ is one. □

Notation 2. Let $M_n$ be the 1-regular spanning subgraph $\{v \in v_{n/2+\alpha} \mid 0 \leq \alpha \leq n/2-1\}$ of $K_n$.

Lemma 3. If $(n,i)=2$ and $n \equiv 2 \mod 4$ then the number of the fixed points of $\rho_i$ is $n/2$ and if $(n,i)=2$ and $n \equiv 0 \mod 4$ then the number of the fixed points of $\rho_i$ is $n/2+1$.

Proof. Since $(n,i)=2$, the equation $x^2 \equiv m \mod n$ has a solution if and only if $m$ is even. Then if $V_0 = \{v_0, v_2, v_4, \ldots, v_{n-2}\}$ and $V_1 = \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ then $\rho_i(V_0) = V_0$ and $\rho_i(V_1) = V_1$. Let $H$ be a 1-regular spanning subgraph of $K_n$ such that $\rho_i(H) = H$ and let $\nu_0 \nu_m \in H$. If $m$ is even then the edge $\nu_0 \nu_m$ induces a 1-regular spanning subgraph of $K_{n/2}$ that is fixed by $\rho_i$. Since $(n/2, i/2)=1$, the subgraph is uniquely determined by Lemma 2. Similarly, the induced subgraph $H[V_1]$ is also unique 1-regular spanning subgraph of $K_{n/2}$ that is fixed by $\rho_i$. Then we have that $H = M_n$. Let $m$ be odd. Since $\rho_i(V_0) = V_0$ and $\rho_i(V_1) = V_1$, edge $\nu_0 \nu_m$ determines unique 1-regular spanning subgraph $H = \{v \in v_{(m+i) \mod n} \mid 0 \leq \alpha \leq n/2-1\}$.

Therefore if $n \equiv 2 \mod 4$ then there are $n/2$ 1-regular spanning subgraph of $K_n$, which are fixed by $\rho_i$ and if $n \equiv 0 \mod 4$ then there are $n/2+1$ 1-regular spanning subgraph of $K_n$ which are fixed by $\rho_i$. We have the results. □

Lemma 4. The number of the way of dividing $2m$ objects into $m$ sets which contain two objects is $(2m-1)!!$.

Proof. This is easily verified by the induction on $m$ and this number is essentially same the number given in Lemma 1. □

Lemma 5. If $(n,i)=2d+1$ then the number of the fixed points of $\rho_i$ is

$$\sum_{k=0}^{d} \binom{2d+1}{2k+1} \times (2d-2k-1)!! \times \left(\frac{n}{2d+1}\right)^{d-k}$$
Proof. Let \( V_0 = \{ v_0, v_{2d+1}, v_{4d+2}, \ldots, v_{n-2d-1} \} \), \( V_1 = \{ v_1, v_{2d+2}, v_{4d+3}, \ldots, v_{n-2d} \} \), \( V_2 = \{ v_2, v_{2d+3}, v_{4d+4}, \ldots, v_{n-2d+1} \} \). \( V_3 = \{ v_3, v_{2d+4}, \ldots, v_{n-2d+2} \} \), \( V_d = \{ v_{2d}, v_{4d}, v_{6d}, \ldots, v_{n-1} \} \).

Since \((n, i) = 2d + 1\), the equation \( x_i \equiv m \) (mod \( n \)) has a solution if and only if \( 2d + 1 \) divides \( m \). Then we have \( \rho_i(V_k) = V_k \) for \( 0 \leq k \leq 2d \). Let \( H \) be a 1-regular spanning subgraph of \( K_n \) which is fixed by \( \rho_i \) and let \( v_a, v_b \) be an edge of \( H \). If \( v_a \in V_k \) and \( v_b \in V_k \) then the induced subgraph \( H[V_k] \) is a 1-regular spanning subgraph of \( K_n / (2d+1) \) which is fixed by \( \rho_{i/(2d+1)} \) and it is unique 1-regular spanning subgraph \( M_{n / (2d+1)} \) by Lemma 2. If \( v_a \in V_{k_1} \) and \( v_b \in V_{k_2} \) then the induced subgraph \( H[V_{k_1} \cup V_{k_2}] \) is a 1-regular spanning subgraph of \( K_{2n / (2d+1)} \) which is fixed by \( \rho_{i/(2d+1)} \). Since \( (2n / (2d+1), i / (2d+1)) = 2 \) and \( 2n / (2d+1) \equiv 0 \) (mod 4), the number of the 1-regular spanning subgraphs of \( K_{2n / (2d+1)} \) which is fixed by \( \rho_{i/(2d+1)} \) is \( n / (2d+1) + 1 \) by Lemma 3 and one 1-regular spanning subgraph among these subgraphs is \( M_{2n / (2d+1)} \). We calculate the number of the case that \( 2k + 1 \) sets of vertices make 1-regular spanning graph \( M_{n / (2d+1)} \) and the remaining \( 2(d - k) \) sets of vertices make 1-regular spanning subgraph with pair. There are \( \binom{2d+1}{k} \times (2d - 2k - 1) \)!! combinations of the sets of vertices like these by Lemma 4. Then, if \( k < d \) then the number of the 1-regular spanning subgraphs fixed by \( \rho_i \) which are not \( M_n \) is

\[
\frac{(2d+1)}{(2k+1)} \times (2d - 2k - 1)!! \times \left( \frac{n}{2d+1} \right)^{d-k}
\]

If \( k = d \) then the number of the 1-regular spanning subgraphs fixed by \( \rho_i \) is one and this subgraph is \( M_n \). Therefore the total number of the 1-regular spanning subgraphs fixed by \( \rho_i \) is given by

\[
\sum_{k=0}^{d} \frac{(2d+1)}{(2k+1)} \times (2d - 2k - 1)!! \times \left( \frac{n}{2d+1} \right)^{d-k}
\]

We have the results. \( \square \)

Lemma 6. If \( (n, i) = 2d \) and \( n / (2d) \equiv 1 \) (mod 2) then the number of the fixed points of \( \rho_i \) is

\[
(2d - 1)!! \times \left( \frac{n}{2d} \right)^d
\]

and if \( (n, i) = 2d \) and \( n / (2d) \equiv 0 \) (mod 2) then the number of the fixed points of \( \rho_i \) is

\[
\sum_{k=0}^{d} \frac{(2d)}{(2k)} \times (2d - 2k - 1)!! \times \left( \frac{n}{2d} \right)^{d-k}
\]

Proof. Let \( V_0 = \{ v_0, v_{2d}, v_{4d}, \ldots, v_{n-2d} \} \), \( V_1 = \{ v_1, v_{2d+1}, v_{4d+1}, \ldots, v_{n-2d+1} \} \), \( V_2 = \{ v_2, v_{2d+2}, v_{4d+2}, \ldots, v_{n-2d+2} \} \), \( V_{d-1} = \{ v_{2d-1}, v_{4d-1}, v_{6d-1}, \ldots, v_{n-1} \} \). Since \( (n, i) = 2d \), the equation \( x_i \equiv m \) (mod \( n \)) has a solution if and only if \( 2d \) divides \( m \). Then \( \rho_i(V_k) = V_k \) for \( 0 \leq k \leq 2d - 1 \).

Let \( n / (2d) \) be odd. Since \( |V_k| = n / (2d) \) is odd, \( H[V_k] \) is not 1-regular spanning subgraph of \( K_{n / (2d)} \) for all \( k \). Accordingly, two vertices of each edge of \( H \) are contained in two subsets of vertices. If \( v_a \in V_k \) and \( v_b \in V_k \) for an edge \( v_a, v_b \) of \( H \) then the induced subgraph \( H[V_k \cup V_{k'}] \) is a 1-regular spanning subgraph of \( K_{n / (2d)} \) which is fixed by \( \rho_{i / (2d)} \). Since \( n / (2d) \equiv 2 \) (mod 4), the number of such 1-regular spanning subgraphs of \( K_{n / (2d)} \) which is fixed by \( \rho_{i / (2d)} \) is \( n / (2d) \). Since the number of the pairings of \( V_0, V_1, \ldots, V_{2d-1} \) is \( (2d - 1)!! \), the total number of the 1-regular spanning subgraphs of \( K_n \) which is fixed by \( \rho_{i / (2d)} \) is

\[
(2d - 1)!! \times \left( \frac{n}{2d} \right)^d
\]
Next let \( n/(2d) \) be even. Since \( |V_k| = n/(2d) \) is even, if there is some edge \( v_a v_\beta \in H \) such that \( v_a \) and \( v_\beta \) are both contained in some \( V_k \) then the induce subgraph \( H[V_k] \) is a 1-regular spanning subgraph of \( K_{n/2d} \) fixed by \( \rho_{n/(2d)} \). By the essentially same augments as above, in this case, we have that the number of the 1-regular spanning subgraphs of \( K_n \) which is fixed by \( \rho_k \) is

\[
\sum_{k=0}^{d} \binom{2d}{2k} \times (2d - 2k - 1)! \times \left( \frac{n}{2d} \right)^{d-k}
\]

We have the results.

Lemma 7. The number of the fixed points of \( \sigma_0 \) is equal to the number of the fixed points of \( \sigma_{2d} \) for all \( 1 \leq d \leq n/2 - 1 \).

Proof. Let \( H \) be a 1-regular spanning subgraph of \( K_n \) fixed by \( \sigma_0 \). Then it is easily verified that \( \rho_{d}(H) \) is a 1-regular spanning subgraph of \( K_n \) fixed by \( \sigma_{2d} \). Conversely, if \( H \) is a 1-regular spanning subgraph of \( K_n \) fixed by \( \sigma_{2d} \) then \( \rho_{d}^{-1}(H) \) is a 1-regular spanning subgraph of \( K_n \) fixed by \( \sigma_0 \). Then we have the results.

Similarly, we have the next Lemma.

Lemma 8. The number of the fixed points of \( \sigma_1 \) is equal to the number of the fixed points of \( \sigma_{2d+1} \) for all \( 1 \leq d \leq n/2 - 1 \).

Lemma 9. The number of the fixed points of \( \sigma_0 \) is equal to the number of the 1-regular spanning subgraphs of \( K_{n-2} \) fixed by \( \sigma_1 \).

Proof. Let \( H \) be a 1-regular spanning subgraph of \( K_n \) fixed by \( \sigma_0 \) and \( v_0v_m \in H \). Since \( \sigma_0(v_0) = v_0 \), \( \sigma(v_m) \) must be \( v_m \). Since \( \sigma(v_m) = v_{(m+0-m) \mod n} \), \( m \) must be \( n/2 \). We remove two vertices \( v_0 \) and \( v_{n/2} \) from \( H \) and change the labels of the vertices of \( H \) from \( v_1, v_2, \ldots, v_{n/2-1}, v_0, v_1, \ldots, v_{n/2-2} \) and from \( v_{n/2+1}, v_{n/2+2}, \ldots, v_{n-1} \) to \( v_{n/2-1}, v_{n/2}, \ldots, v_n \). Let \( H' \) be the resulting graph. Since \( \sigma_0(H) = H \), we have \( \sigma_{n-3}(H') = H' \). Conversely, let \( H' \) be a 1-regular spanning subgraph of \( K_{n-2} \) fixed by \( \sigma_{n-3} \). We change the labels of the vertices of \( H' \) from \( v_0, v_1, \ldots, v_{n/2-2} \) to \( v_1, v_2, \ldots, v_{n/2-1} \) and from \( v_{n/2+1}, v_{n/2+2}, \ldots, v_{n-1} \) to \( v_{n/2+1}, v_{n/2+2}, \ldots, v_{n-1} \) and add the edge \( v_0v_{n/2} \) to it. Let \( H \) be the resulting graph. \( H \) is a 1-regular spanning subgraph of \( K_n \) fixed by \( \sigma_0 \). This correspondence is one to one correspondence between the set of the 1-regular spanning subgraphs of \( K_n \) fixed by \( \sigma_0 \) and the set of the 1-regular spanning subgraphs of \( K_{n-2} \) fixed by \( \sigma_{n-3} \). Then we have the results by Lemma 8.

Lemma 10. Let \( S_n \) be the number of the fixed points of \( \sigma_1 \) for \( X_n \). Then we have

\[
S_4 = 3, S_6 = 7 \quad \text{and} \quad S_n = S_{n-2} + (n - 2)S_{n-4} \quad \text{for all} \ n \geq 8.
\]

Proof. By the direct computation, we can easily checked that \( S_4 = 3 \) and \( S_6 = 7 \). We study two kinds of constitutions that compose 1-regular spanning subgraphs of \( K_n \) fixed by \( \sigma_1 \) inductively.

The first method is the following:

Let \( H \) be a 1-regular spanning subgraph of \( K_{n-2} \) fixed by \( \sigma_1 \). We change the labels of vertices of \( H \) from \( v_0 \) to \( v_{n-1} \) and from \( v_1, v_2, \ldots, v_{n-3} \) to \( v_3, v_4, \ldots, v_{n-2} \) and add an edge \( v_0v_1 \) to it. Let \( H_0 \) be the resulting graph. Then \( H_0 \) is a 1-regular spanning subgraph of \( K_n \) such that \( \sigma_1(H_0) = H_0 \). We change the labels of vertices of \( H \) from \( v_{n/2}, v_{n/2+1}, \ldots, v_{n-3} \)
to \(v_{n/2+2}, v_{n/2+3}, \ldots, v_{n-1}\) and add an edge \(v_{n/2}v_{n/2+1}\) to it. Let \(H_1\) be the resulting graph. Then \(H_1\) is a 1-regular spanning subgraph of \(K_n\) such that \(\sigma_1(H_1) = H_1\).

The second method is the following:

Let \(H\) be a 1-regular spanning subgraph of \(K_{n-4}\) fixed by \(\sigma_1\). We change the labels of the vertices of \(H\) from \(v_0, v_1, v_2, \ldots, v_{n-5}\) to \(v_1, v_2, \ldots, v_{n-5}\). Let \(H_0\) be the graph which is added edges \(v_0v_2\) and \(v_0v_{n-1}\) to it and \(H_1\) be the graph which is added edges \(v_0v_2\) and \(v_1v_{n-1}\) to it. Then \(H_0\) and \(H_1\) are 1-regular spanning subgraphs of \(K_n\) fixed by \(\sigma_1\). For each \(1 \leq i \leq n/2 - 2\), we change the labels of the vertices of \(H\) from \(v_0, v_3, \ldots, v_{n/2-1}, v_{n/2}, \ldots, v_n\) to \(v_0, v_3, \ldots, v_{n/2-1}, v_{n/2}, \ldots, v_n\). Let \(H_0'\) be the graph which is added two edges \(v_1v_{n/2}\) and \(v_0v_{n/2+1}\) and \(H_1'\) be the graph which is added two edges \(v_0v_{n/2}\) and \(v_1v_{n/2+1}\). For each \(1 \leq i \leq n/2 - 2\), we change the labels of the vertices of \(H\) from \(v_{i+1}, v_{i+2}, \ldots, v_{n/2-1}, v_{n/2-1}, v_{n/2-2}\) to \(v_{i+1}, v_{i+2}, \ldots, v_{n/2-1}, v_{n/2}, \ldots, v_n\). Let \(H_2'\) be the graph which is added two edges \(v_{i+1}v_{n/2+1}\) and \(v_0v_{n/2+1}\). Then \(H_2'\) and \(H_{2+i}\) are 1-regular spanning subgraphs of \(K_n\) fixed by \(\sigma_1\). By these constructions, we can construct \(2S_{n-2} + 2 \times 2 \times (n/2 - 1) \times S_{n-4}\) 1-regular spanning subgraphs of \(K_n\) fixed by \(\sigma_1\). Clearly there are doubling two pieces of each. Also, it is clear to be able to compose all the 1-regular spanning subgraphs of \(K_n\) fixed by \(\sigma_1\) by these methods. Then the number of the 1-regular spanning subgraphs of \(K_n\) fixed by \(\sigma_1\) is given by \(S_{n-2} + (n - 2)S_{n-4}\). We have the results.

**Remark 2.** Let \(S_0 = 1\) and \(S_2 = 1\). Then we have \(S_n = S_{n-2} + (n - 2)S_{n-4}\) for \(n \geq 4\).

Then we completely proved Theorem 2.

**Remark 3.** We calculated the non-equivalent 1-regular spanning subgraphs of \(K_n, n \leq 12\) by computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:

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Next let \(n\) be odd and be greater than or equal to 3. Let \(\{v_0, v_1, v_2, \ldots, v_{n-1}\}\) be the vertices of the complete graph \(K_n\). The action to \(K_n\) of the Dihedral group \(D_n = \{\rho_0, \rho_1, \ldots, \rho_{n-1}, \sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}\) is defined by

\[
\rho_i(v_k) = v_{(k+i) \mod n} \quad \text{for} \quad 0 \leq i \leq n - 1, 0 \leq k \leq n - 1
\]

\[
\sigma_i(v_k) = v_{(n+2i-k) \mod n} \quad \text{for} \quad 0 \leq i \leq n - 1, 0 \leq k \leq n - 1
\]

Let \(Y_n\) be the set of the maximal matchings of \(K_n\). Then the above action induces the action on \(Y_n\) of the Dihedral group \(D_n\). We calculate the number of the equivalence classes by this group action.
Theorem 3. The number of the non-equivalent maximal matchings of the complete graph \( K_n \) with odd order \( n \) is

\[
\frac{1}{2n}(n!! + nS_{n-1})
\]

Here \( S_n \) is given in Lemma 10.

Proof. This Theorem is also proved by Burnside’s Lemma. To construct a maximal matching we choose an isolated vertex and then choose \((n-1)/2\) pairings of resulting \( n-1 \) vertices. There are \( n \times (n-2)!! \) combinations like these by Lemma 4. Then the number of the maximal matchings of the complete graph \( K_n \) is \( n!! \) and this number is the number of the fixed points of \( \rho_0 \) by \( \rho_0 \). Since there is only one isolated vertex, \( \rho_i, 1 \leq i \leq n-1 \), fixes no maximal matchings of the complete graph \( K_n \). Let \( i \) be greater than 0 and less than \( n \) and \( H \) be a maximal matching of the complete graph \( K_n \) such that \( \sigma_i(H) = H \). Since \( \sigma_i(v_i) = v_i, v_i \) is an isolated vertex of \( H \). If \( i = 0 \) then we remove the vertex \( v_0 \) from \( H \) and change the labels of the vertices of \( H \) from \( v_1,v_2, \cdots,v_{(n-1)/2} \) to \( v_0,v_1,\cdots,v_{(n-3)/2} \). Let \( H_0 \) be the resulting graph. Then \( H_0 \) is a 1-regular spanning subgraph of \( K_{n-1} \) such that \( \sigma_{n-2}(H_0) = H_0 \). By this construction, we can construct an one to one correspondence between the set of the maximal matchings of the complete graph \( K_n \) such that \( \sigma_0(H) = H \) and the set of the 1-regular spanning subgraph of \( K_{n-1} \) such that \( \sigma_{n-2}(H_0) = H_0 \). If \( 1 \leq i \leq n-1 \) then we remove the vertex \( v_i \) from \( H \) and change the labels of the vertices of \( H \) from \( v_{i+1},v_{i+2},\cdots,v_{n-1} \) to \( v_i,v_{i+1},\cdots,v_{n-2} \). Let \( H_i \) be the resulting graph. Then \( H_i \) is a 1-regular spanning subgraph of \( K_{n-1} \) such that \( \sigma_{2i-1}(mod \ n)(H_i) = H_i \). By this construction, we can construct an one to one correspondence between the set of the maximal matchings of the complete graph \( K_n \) such that \( \sigma_i(H) = H \) and the set of the 1-regular spanning subgraph of \( K_{n-1} \) such that \( \sigma_{2i-1}(mod \ n)(H_i) = H_i \). Then the number of the fixed points of \( \sigma_i \) is \( S_{n-1} \). Then we have the results. □

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