STRICT CONVERGENCE OF ITERATIVE SEQUENCES FOR ASYMPTOTICALLY NONEXPANSIVE Mappings IN BANACH SPACES

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Abstract. In this paper, we deal with an iteration process for an asymptotically nonexpansive mapping and prove a strong convergence theorem for the mapping in Banach spaces, which is a generalization of the recent result of Shioji and Takahashi [12].

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E and let T be a mapping of C into itself. Then, we denote by F(T) the set of fixed points of T. A mapping T of C into itself is said to be nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \) for every \( x, y \in C \) and a mapping T of C into itself is said to be asymptotically nonexpansive with Lipschitz constants \( \{k_n\} \) if \( \lim_{n \to \infty} k_n = 1 \) and \( \|T^n x - T^n y\| \leq k_n \|x - y\| \) for every \( x, y \in C \) (see [3]).

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself. Let \( x_0 \in C \). Halpern [4] and Reich [9] considered the following iteration process:

\[
(1) \quad x_0 \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n 
\]

for each \( n = 0, 1, 2, \ldots \), where \( \{\alpha_n\} \) is a sequence in \( [0, 1] \). Wittmann [15] showed that \( \{x_n\} \) defined by (1) converges strongly to the element of F(T) which is nearest to \( x \) if \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=0}^{\infty} \alpha_n = \infty \), \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \) and \( T \neq \emptyset \). Shioji and Takahashi [10] extended the result of Wittmann [15] to a Banach space.

Let T be an asymptotically nonexpansive mapping of a nonempty bounded closed convex subset C of H and let \( x \in C \). Using the concept of mean, Shimizu and Takahashi [13] studied the strong convergence of the following iteration process for an asymptotically nonexpansive mapping:

\[
(2) \quad x_0 \in C, \quad x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n 
\]

for sufficient large integer n, where \( \{\alpha_n\} \) is a sequence in \( [0, 1] \). Shioji and Takahashi [11] extended the result of [13] to a Banach space. Further, Shioji and Takahashi [12] proved the following theorem by using the results of [11] (see also [14]): Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a nonempty bounded closed convex subset of E. Let T be an asymptotically nonexpansive mapping on C with Lipschitz constants \( \{k_n\} \). Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq 

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\[ \alpha_n \leq 1, \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \left( (1 - \alpha_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} b_j \right)^2 - 1 \right) < \infty. \]

Let \( x \in C \) and let \( \{x_n\} \) be the sequence defined by

\[ x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \]

for each \( n = 0, 1, 2, \ldots \). Then, \( \{x_n\} \) converges strongly to \( P x \), where \( P \) is the sunny nonexpansive retraction from \( C \) onto \( F(T) \). Mann [6] introduced the following iteration process for approximating fixed points of a nonexpansive mapping \( T \) on a nonempty closed convex subset \( C \) in a Hilbert space:

\[ x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \]

for each \( n = 0, 1, 2, \ldots, \) where \( \{\alpha_n\} \) is a sequence in \([0, 1]\). Later, Reich [8] studied the sequence defined by (4) in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained a weak convergence theorem (see also [1]).

In this paper, we introduce an iteration process for mappings of \( C \) into itself by using the ideas of [1, 6, 12]. We prove a strong convergence theorem for an asymptotically nonexpansive mapping, which is a generalization of the result of Shiqi and Takahashi [12].

2. Preliminaries

Throughout this paper, \( E \) is a real Banach space and \( E^* \) is the dual space of \( E \). We write \( x_n \rightharpoonup x \) (or \( \lim_{n \to \infty} x_n = x \)) to indicate that the sequence \( \{x_n\} \) of vectors converges strongly to \( x \). We also denote by \( \langle y, x^* \rangle \) the value of \( x^* \in E^* \) at \( y \in E \). We denote by \( \mathbb{N} \) the set of all nonnegative integers. We also denote \( \max \{a, 0\} \) by \( (a)_+ \) for a real number \( a \).

A Banach space \( E \) is said to be strictly convex if \( \|x + y\|/2 < 1 \) for \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y \). In a strictly convex Banach space, we have that if \( \|x\| = \|y\| = \| (1 - \lambda) x + \lambda y \| \) for \( x, y \in E \) and \( \lambda \in (0, 1) \) then \( x = y \). For every \( \varepsilon \) with \( 0 \leq \varepsilon \leq 2 \), we define the modulus \( \delta(\varepsilon) \) of convexity of \( E \) by

\[ \delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} \mid \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}. \]

A Banach space \( E \) is said to be uniformly convex if \( \delta(\varepsilon) > 0 \) for every \( \varepsilon > 0 \). If \( E \) is uniformly convex, then for \( r, \varepsilon \) with \( r \geq \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that

\[ \left\| \frac{x + y}{2} \right\| \leq r \left( 1 - \delta \left( \frac{\varepsilon}{r} \right) \right) \]

for every \( x, y \in E \) with \( \|x\| \leq r, \|y\| \leq r \) and \( \|x - y\| \geq \varepsilon \). It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

The multi-valued mapping \( J \) from \( E \) into \( E^* \) defined by

\[ J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \} \quad \text{for every} \quad x \in E \]

is called the duality mapping of \( E \). From the Hahn-Banach theorem, we see that \( J(x) \neq \emptyset \) for all \( x \in E \). A Banach space \( E \) is said to be smooth if the limit

\[ \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \]

exists for each \( x \) and \( y \) in \( S_1 \), where \( S_1 = \{ u \in E : \|u\| = 1 \} \). The norm of \( E \) is said to be uniformly Gâteaux differentiable if for each \( y \) in \( S_1 \), the limit is attained uniformly for \( x \) in \( S_1 \). We know that if \( E \) is smooth then the duality mapping is single-valued and norm to weak-star continuous and that if the norm of \( E \) is uniformly Gâteaux differentiable then
the duality mapping is single-valued and norm to weak-star uniformly continuous on each bounded subset of $E$.

Let $C$ be a nonempty convex subset of $E$ and let $K$ be a nonempty subset of $C$. A mapping $P$ of $C$ onto $K$ is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. A mapping $P$ of $C$ onto $K$ is said to be a retraction if $Px = x$ for each $x \in K$. We know from [2, 7] that if $E$ is smooth, then a retraction $P$ of $C$ onto $K$ is sunny and nonexpansive if and only if

$$\langle x - Px, J(y - Px) \rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in K.$$ 

Hence, there is at most one sunny nonexpansive retraction of $C$ onto $K$. If there is a sunny nonexpansive retraction of $C$ onto $K$, $K$ is said to be a sunny nonexpansive retract of $C$.

The following proposition related to the existence of sunny nonexpansive retractions was proved in [11].

**Proposition 2.1.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $T$ be an asymptotically nonexpansive mapping on $C$ with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Then, $F(T)$ is a sunny nonexpansive retract of $C$.

### 3. Lemmas

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be a mapping of $C$ into itself. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $0 \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$, and let $x \in C$. Now consider the following iteration process:

\[
\begin{align*}
  x_0 & \in C \\
  x_{n+1} & = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j y_n, \\
  y_n & = \beta_n x_n + (1 - \beta_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n
\end{align*}
\]

for each $n \in \mathbb{N}$. Especially, if $\beta_n = 1$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is written by (3). We prove a strong convergence theorem for an asymptotically nonexpansive mapping $T$ on $C$ with Lipschitz constants $\{k_n\}$, which is a generalization of the result of Shiqi and Takahashi [12]. Without loss of generality, we may assume $k_n \geq 1$ for each $n \in \mathbb{N}$. Since $k_n \geq 1$ for each $n \in \mathbb{N}$, we obtain the following lemmas.

**Lemma 3.1.** Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be an asymptotically nonexpansive mapping on $C$ with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers such that $0 \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$ and

\[
\sum_{n=0}^{\infty} (1 - \alpha_n)(M_n - 1) < \infty,
\]

where $M_n = \left(\frac{1}{n+1} \sum_{j=0}^{n} k_j\right) \left(\beta_n + (1 - \beta_n) \left(\frac{1}{n+1} \sum_{j=0}^{n} k_j\right)\right)$. Let $x \in C$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by (5). Then, $\{x_n\}$ and $\{y_n\}$ are bounded. Further, $\{T^j x_n\}$ and $\{T^j y_n\}$ are bounded for each $j \in \mathbb{N}$.

**Proof.** Let $K_0 = \sup_{x \in X} k_n$. We obtain

\[
1 \leq \beta_n + (1 - \beta_n) \left(\frac{1}{n+1} \sum_{j=0}^{n} k_j\right) \leq K_0
\]
for each $n \in \mathbb{N}$. Set $M_n = \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \left( \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \right)$. Then, we obtain $1 \leq M_n \leq K_0^2$ for each $n \in \mathbb{N}$. Let $z \in F(S)$. Then, it follows from (5) that

\[
\| y_n - z \| = \left\| \beta_n (x_n - z) + (1 - \beta_n) \frac{1}{n+1} \sum_{j=0}^{n} (T^j x_n - z) \right\|
\leq \beta_n \| x_n - z \| + (1 - \beta_n) \frac{1}{n+1} \sum_{j=0}^{n} \| T^j x_n - z \|
\]

(7)

\[
\leq \left( \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \right) \| x_n - z \|
\]

(8)

\[
\leq K_0 \| x_n - z \|
\]

for each $n \in \mathbb{N}$. By (5), we also obtain

\[
\| x_{n+1} - z \| = \left\| \alpha_n (x - z) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} (T^j y_n - z) \right\|
\leq \alpha_n \| x - z \| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} \| T^j y_n - z \|
\]

(9)

\[
\leq \alpha_n \| x - z \| + (1 - \alpha_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \| y_n - z \|
\]

(10)

\[
\leq \| x - z \| + K_0 \| y_n - z \|
\]

for each $n \in \mathbb{N}$. Since $F(T) \neq \emptyset$, from (8) and (10), we see that $\{x_n\}$ is bounded if and only if $\{y_n\}$ is bounded.

By (7) and (9), for each $n \in \mathbb{N}$, we have

\[
\| x_{n+1} - z \|
\leq \alpha_n \| x - z \| + (1 - \alpha_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \left( \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \right) \| x_n - z \|
\]

(11)

\[
= \alpha_n \| x - z \| + (1 - \alpha_n) M_n \| x_n - z \|.
\]

Set $h_n = ((1 - \alpha_n) M_n - 1)_+$. Since $h_n = ((1 - \alpha_n) M_n - 1)_+ \leq (1 - \alpha_n) (M_n - 1)$ for each $n \in \mathbb{N}$, we obtain

\[
\sum_{n=0}^{\infty} h_n < \infty
\]
by (6). By (11), for each $n \in \mathbb{N}$, we have
\[
\|x_{n+1} - z\|
\leq (1 - (1 - \alpha_n)) \|x - z\| + (1 - \alpha_n) M_n \|x_n - z\|
\leq \{1 + (1 - \alpha_n)(M_n - 1) - (1 - \alpha_n) M_n(1 - \alpha_n)\} \|x - z\|
\leq (1 + (1 - \alpha_n)(M_n - 1) + (1 - \alpha_n) M_n(1 - \alpha_n) - (1 - \alpha_n) M_n(1 - \alpha_n) M_n(1 - \alpha_n - 1)) \|x_n - z\|
\leq 1 + (1 - \alpha_n)(M_n - 1) + \sum_{i=1}^{n-1} (1 - \alpha_i)(M_i - 1) \prod_{j=i+1}^{n} (1 - \alpha_j) M_j]
\leq \left\{1 + (1 - \alpha_n)(M_n - 1) + \sum_{i=1}^{n-1} (1 - \alpha_i)(M_i - 1) \prod_{j=i+1}^{n} (1 + h_j)\right\} \|x - z\|
+ \prod_{j=0}^{n} (1 + h_j) \|x_0 - z\|
\leq \prod_{j=0}^{n} (1 + h_j) \left\{1 + \sum_{i=1}^{n} (1 - \alpha_i)(M_i - 1) \|x - z\| + \|x_0 - z\|\right\}
\leq \exp \left(\sum_{j=0}^{\infty} h_j\right) \left\{1 + \sum_{i=1}^{\infty} (1 - \alpha_i)(M_i - 1) \|x - z\| + \|x_0 - z\|\right\}.
\]
(13)

Hence by (6) and (12), we obtain that \(\{\|x_n - z\|\}\) is bounded. Therefore, \(\{x_n\}\) and \(\{y_n\}\) are bounded.

Let \(L_0 = \sup_{n} \{\|x_n - z\|\}\). Then, it follows from (8) that
\[
\|T^j x_n - z\| \leq k_j \|x_n - z\| \leq K_0 L_0
\]
and
\[
\|T^j y_n - z\| \leq k_j \|y_n - z\| \leq K_0 \cdot K_0 L_0 = K_0^2 L_0
\]
for each \(j, n \in \mathbb{N}\). Hence, \(\{T^j x_n\}\) and \(\{T^j y_n\}\) are also bounded for each \(j \in \mathbb{N}\). \(\square\)

Lemma 3.2 and Proposition 3.3 were proved by Shioji and Takahashi [11].

**Lemma 3.2.** Let \(C\) be a nonempty closed convex subset of a uniformly convex Banach space \(E\) and let \(T\) be an asymptotically nonexpansive mapping on \(C\) with Lipschitz constants \(\{k_n\}\) such that \(F(T) \neq \emptyset\). Then, for each \(r > 0\),
\[
\lim_{m \to \infty} \lim_{n \to \infty} \sup_{y \in C \cap B_r} \left\|\frac{1}{n+1} \sum_{j=0}^{n} T^j y - T^m \left(\frac{1}{n+1} \sum_{j=0}^{n} T^j y\right)\right\| = 0,
\]
where \(B_r = \{z \in E : \|z\| \leq r\}\).
Proposition 3.3. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable. Let $T$ be an asymptotically non-expansive mapping on $C$ with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$ and let $P$ be the sunny nonexpansive retraction from $C$ onto $F(T)$. Let $\{d_n\}$ be a sequence of real numbers such that $0 < d_n \leq 1$, $\lim_{n \to \infty} d_n = 0$ and
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} k_j - 1 < 1.
\]
Let $x \in C$ and let $z_n$ be the unique point of $C$ which satisfies
\[
z_n = d_n x + (1 - d_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j z_n
\]
for $n \geq n_0$, where $n_0$ is a sufficiently large integer. Then, $\{z_n\}$ converges strongly to $P(x)$.

Remark 3.4. The inequality
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} k_j - 1 < 1
\]
yields
\[(1 - d_n) \cdot \frac{1}{n+1} \sum_{j=0}^{n} k_j < 1
\]
for all sufficiently large integer $n$. So for such $n$, there exists a unique point $z_n$ of $C$ satisfying $z_n = d_n x + (1 - d_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j z_n$, since the mapping $T_n$ from $C$ into itself defined by
\[T_n u = d_n x + (1 - d_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j u
\]
is a contraction, that is,
\[\|T_n u - T_n v\| \leq (1 - d_n) \cdot \frac{1}{n+1} \sum_{j=0}^{n} k_j \|u - v\|
\]
for each $u, v \in C$.

4. Strong Convergence Theorems

Our main result is the following, which is a generalization of Shioji and Takahashi’s result [12]:

Theorem 4.1. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $T$ be an asymptotically nonexpansive mapping on $C$ with Lipschitz constants $\{k_n\}$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $0 \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$,
\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty
\]
and
\[
\sum_{n=0}^{\infty} (1 - \alpha_n)(M_n - 1) < \infty,
\]
where...
where \( M_n = \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \left( \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \right) \). Let \( x \in C \) and let \( \{x_n\} \) be the sequence defined by

\[
\begin{cases}
\quad x_0 \in C \\
\quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j y_n, \\
\quad y_n = \beta_n x_n + (1 - \beta_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n
\end{cases}
\]

(16)

for each \( n \in \mathbb{N} \). Then, \( \{x_n\} \) converges strongly to \( Px \), where \( P \) is the sunny nonexpansive retraction from \( C \) onto \( F(T) \).

**Proof.** Set \( M_n = \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \left( \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \right) \) and set \( K_0 = \sup_n k_n \).

Since \( F(T) \neq \emptyset \) and \( \sum_{n=0}^{\infty} (1 - \alpha_n) (M_n - 1) < \infty \), from Lemma 3.1, we see that \( \{x_n\}, \{y_n\}, \{T^j x_n\} \) and \( \{T^j y_n\} \) are bounded for each \( j \in \mathbb{N} \).

Since \( \lim_{n \to \infty} k_n \leq 1 \), we can choose a sequence \( \{d_n\} \) of real numbers such that \( d_n > 0, \lim_{n \to \infty} d_n = 0 \),

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} k_j - 1 < 1
\]

(17)

and

\[
\left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right)^2 \leq 1 + d_n^2
\]

(18)

for each \( n \in \mathbb{N} \) (see also [13]). By the reason in Remark 3.4, there exists the unique point \( z_m \) of \( C \) satisfying \( z_m = d_m x + (1 - d_m) \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m \) for all sufficiently large integer \( m \).

Without loss of generality, we may assume that \( d_m \leq 1/2 \) for all \( m \in \mathbb{N} \) and \( z_m \) is defined for all \( m \in \mathbb{N} \). We know that \( \{z_n\} \) converges strongly to \( Px \) by Proposition 3.3. From (16),
for each \( m, n \in \mathbb{N} \), we have

\[
\frac{1}{m+1} \sum_{j=0}^{m} T_j x_{n+1} - x_{n+1} \leq \left| \frac{1}{m+1} \sum_{j=0}^{m} T_j \left( \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right) \right|
\]

\[
+ \left| \frac{1}{m+1} \sum_{j=0}^{m} T_j \left( \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right|
\]

\[
\leq \left( \frac{1}{m+1} \sum_{j=0}^{m} k_j + 1 \right) \left| x_{n+1} - \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right|
\]

\[
+ \frac{1}{m+1} \sum_{j=0}^{m} T_j \left( \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right|
\]

\[
\leq (K_0 + 1) \cdot \alpha_n \left( \| x \| + \sup_{j,n} \| T_j y_n \| \right)
\]

\[
+ \frac{1}{m+1} \sum_{j=0}^{m} T_j \left( \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right|
\]

It follows from Lemma 3.2 that

\[
\lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m+1} \sum_{j=0}^{m} T_j \left( \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right) - \frac{1}{n+1} \sum_{l=0}^{n} T_l y_n \right| = 0.
\]

Hence by \( \lim_{n \to \infty} \alpha_n = 0 \), we have

\[
\lim_{m \to \infty} \lim_{n \to \infty} \left| \frac{1}{m+1} \sum_{j=0}^{m} T_j x_{n+1} - x_{n+1} \right| = \lim_{m \to \infty} \lim_{n \to \infty} \left| \frac{1}{m+1} \sum_{j=0}^{m} T_j x_n - x_n \right| = 0.
\]

Then, we may also assume that

\[
\lim_{n \to \infty} \left| \frac{1}{m+1} \sum_{j=0}^{m} T_j x_n - x_n \right| \leq d_m^2
\]

for each \( m \in \mathbb{N} \). Set \( R = \sup \{ \| T_j z_m \| : j, m \in \mathbb{N} \} \cup \{ \| T_j x_n \| : j, n \in \mathbb{N} \} \). From

\[
(1 - d_m) \left( \frac{1}{m+1} \sum_{j=0}^{m} T_j z_m - x_n \right) = (z_m - x_n) - d_m (x - x_n),
\]
we obtain

\[
(1-d_m)^2 \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right\|^2 \geq \|z_m - x_n\|^2 - 2d_m \langle x - x_n, J(z_m - x_n) \rangle
\]

\[= \|z_m - x_n\|^2 - 2d_m \langle x - z_m + z_m - x_n, J(z_m - x_n) \rangle \]

\[= (1 - 2d_m) \|z_m - x_n\|^2 + 2d_m \langle x - z_m, J(x_n - z_m) \rangle \]

for each \(m, n \in \mathbb{N}\). Then, it follows from (18) that

\[
\langle x - z_m, J(x_n - z_m) \rangle \leq \frac{1}{2d_m} \left(\left(1 - d_m\right)^2 \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right\|^2 - \left(1 - 2d_m\right) \|z_m - x_n\|^2 \right)
\]

\[\leq \frac{1 - 2d_m}{2d_m} \left(\left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right\|^2 - \|z_m - x_n\|^2 \right) + \frac{d_m}{2} \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right\|^2
\]

\[\leq \frac{1 - 2d_m}{2d_m} \left(\left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right\|^2 - \|z_m - x_n\|^2 \right) + 2R^2 d_m
\]

\[\leq \frac{1}{2d_m} \left(\left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j z_m - x_n \right\|^2 - \|z_m - x_n\|^2 \right) + 2R^2 d_m
\]

\[\leq \frac{1}{2d_m} \left(\left\| \frac{1}{m+1} \sum_{j=0}^{m} k_j \right\|^2 - 1 \right) \|z_m - x_n\|^2 + 2R^2 d_m
\]

\[\leq \frac{1}{2d_m} \left(\left\| \frac{1}{m+1} \sum_{j=0}^{m} k_j \right\|^2 - 1 \right) \|z_m - x_n\|^2 + 6R \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n - x_n \right\| + 2R^2 d_m
\]

\[\leq 4R^2 d_m + 3R \left\| \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n - x_n \right\| + 2R^2 d_m
\]

for each \(m, n \in \mathbb{N}\). Hence by (19), we have

\[
\lim_{n \to \infty} \langle x - z_m, J(x_n - z_m) \rangle \leq (4R^2 + 3R)d_m
\]

for each \(m \in \mathbb{N}\). Since \(\{z_m\}\) converges strongly to \(P x\) and the norm of \(E\) is uniformly Gâteaux differentiable, we have

\[
\lim_{n \to \infty} \langle x - P x, J(x_n - P x) \rangle \leq 0.
\]

Let \(\varepsilon > 0\). Then, there exists \(n_0 \in \mathbb{N}\) such that \(\langle x - P x, J(x_n - P x) \rangle < \frac{\varepsilon}{2}\) for each \(n \geq n_0\).

From

\[
(1 - \alpha_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} T^j y_n - P x \right) = (x_{n+1} - P x) - \alpha_n (x - P x),
\]
we also obtain

$$\|x_{n+1} - P_x\|^2 \leq 2\alpha_n \langle x - P_x, J(x_{n+1} - P_x) \rangle + (1 - \alpha_n)^2 \left( \frac{1}{n+1} \sum_{j=0}^{n} T_j y_n - P_x \right)^2$$

for each $n \in \mathbb{N}$. So, we get

$$\|x_{n+1} - P_x\|^2 \leq \alpha_n \varepsilon + (1 - \alpha_n)^2 \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right)^2 \|y_n - P_x\|^2$$

(21)

$$\leq \alpha_n \varepsilon + (1 - \alpha_n)^2 \left( \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right)^2 \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right)^2 \right) \|x_n - P_x\|^2$$

$$= \alpha_n \varepsilon + (1 - \alpha_n)^2 M_n^2 \|x_n - P_x\|^2$$

for each $n \geq n_0$. Set $p_n = \|x_n - P_x\|^2$, $L_n = M_n^2$ and $c_n = ((1 - \alpha_n)L_n - 1)\varepsilon$. Then, for each $n \in \mathbb{N}$, we have

$$c_n = ((1 - \alpha_n)L_n - 1)\varepsilon \leq (1 - \alpha_n)(L_n - 1)$$

$$= (1 - \alpha_n)(M_n + 1)(M_n - 1) \leq (K_0^2 + 1)(1 - \alpha_n)(M_n - 1).$$

Hence by (15), $\sum_{i=0}^{\infty} c_i < \infty$. Let $n \in \mathbb{N}$ with $n \geq n_0$. Then, for each $m \in \mathbb{N}$, we have

$$p_{n+m} \leq \alpha_n \cdot \varepsilon + (1 - \alpha_n)^2 L_{n+m-1} p_{n+m-1}$$

$$\leq \{ \alpha_n + (1 - \alpha_n)^2 L_{n+m-1} \varepsilon \} + (1 - \alpha_n)^2 L_{n+m-1}(1 - \alpha_n) p_{n+m-2}$$

$$\vdots$$

$$\leq \left( \alpha_n + \sum_{j=n+1}^{n+m-1} \prod_{i=j}^{n-1} \left( 1 - \alpha_i \right) \right) \varepsilon + \prod_{i=n}^{n+m-1} \left( 1 - \alpha_i \right) \cdot p_n$$

$$\leq \prod_{i=n+1}^{n+m-1} \left( 1 + c_i \right) \left( \alpha_n + \sum_{j=n+1}^{n+m-1} \prod_{i=j}^{n-1} \left( 1 - \alpha_i \right) \right) \varepsilon$$

$$+ \prod_{i=n+1}^{n+m-1} \left( 1 + c_i \right) \cdot \prod_{i=n}^{n+m-1} \left( 1 - \alpha_i \right) \cdot p_n$$

$$\leq \prod_{i=n+1}^{n+m-1} \left( 1 + c_i \right) \left( \frac{1}{1 - \alpha_i} \right) \varepsilon + \prod_{i=n}^{n+m-1} \left( 1 + c_i \right) \cdot \prod_{i=n}^{n+m-1} \left( 1 - \alpha_i \right) \cdot p_n$$

$$\leq \varepsilon \cdot \exp \left( \sum_{i=n+1}^{n+m-1} c_i \right) + \exp \left( \sum_{i=n+1}^{n+m-1} c_i \right) \cdot \exp \left( - \sum_{i=n}^{n+m-1} \alpha_i \right) \cdot p_n$$

$$\leq \exp \left( \sum_{i=0}^{\infty} c_i \right) \left( \varepsilon + \exp \left( - \sum_{i=n}^{n+m-1} \alpha_i \right) \cdot p_n \right).$$
By \( \sum_{i=0}^{\infty} \alpha_i = \infty \), we get
\[
\lim_{m \to \infty} p_m = \lim_{n \to \infty} p_{n+m} \leq \epsilon \cdot \exp \left( \sum_{i=0}^{\infty} c_i \right).
\]
Since \( \exp \left( \sum_{i=0}^{\infty} c_i \right) < \infty \) and \( \epsilon > 0 \) is arbitrary, \( \{x_n\} \) converges strongly to \( Px \in F(T) \). \( \square \)

**Remark 4.2.** \( \sum_{n=0}^{\infty} (1 - \alpha_n)(M_n - 1) < \infty \) yields \( \sum_{n=0}^{\infty} c_n < \infty \). So, by the proofs of Lemma 3.1 and Theorem 4.1, we see the following: Let \( E, C, T, x \) and \( \{k_n\} \) be as in Theorem 4.1. Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \), and let \( \{\beta_n\} \) be a sequence of real numbers such that \( 0 \leq \beta_n \leq 1 \). Assume
\[
\sum_{n=0}^{\infty} \left( (1 - \alpha_n)M_n^2 - 1 \right) < \infty,
\]
where \( M_n = \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \left( \beta_n + (1 - \beta_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \right) \right) \). Let \( \{x_n\} \) be the sequence defined by (16). Then, \( \{x_n\} \) converges strongly to a fixed point of \( T \) if and only if \( \{x_n\} \) is bounded.

Since \( \sum_{n=0}^{\infty} (1 - \alpha_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j - 1 \right) < \infty \) yields \( \sum_{n=0}^{\infty} (1 - \alpha_n)(M_n - 1) < \infty \), we get the following.

**Corollary 4.3.** Let \( E, C, T, x \) and \( \{k_n\} \) be as in Theorem 4.1. Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \), and let \( \{\beta_n\} \) be any sequence of real numbers such that \( 0 \leq \beta_n \leq 1 \). Assume
\[
\sum_{n=0}^{\infty} (1 - \alpha_n) \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j - 1 \right) < \infty.
\]
Let \( \{x_n\} \) be the sequence defined by (16). Then, \( \{x_n\} \) converges strongly to \( Px \), where \( P \) is the sunny nonexpansive retraction from \( C \) onto \( F(T) \).

In the case when \( T \) is nonexpansive, by \( \sum_{n=0}^{\infty} (1 - \alpha_n)(M_n - 1) = 0 \), we can directly obtain the following.

**Theorem 4.4.** Let \( E \) be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let \( C \) be a nonempty closed convex subset of \( E \). Let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \), and let \( \{\beta_n\} \) be a sequence of real numbers such that \( 0 \leq \beta_n \leq 1 \). Let \( x \in C \) and let \( \{x_n\} \) be the sequence defined by (16). Then, \( \{x_n\} \) converges strongly to \( Px \), where \( P \) is the sunny nonexpansive retraction from \( C \) onto \( F(T) \).

**References**


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