ON CLASSES OF OPERATORS GENERALIZING
CLASS A AND PARANORMALITY

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Received July 1, 2002

Dedicated to Professor Hisaharu Umegaki on his 77th birthday.

Abstract. Recently, we introduced class A defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. As generalizations of class A and paranormality, Fujii-Nakamoto introduced class $F(p, r, q)$ and $(p, r, q)$-paranormality respectively. These classes are related to p-hyponormality and log-hyponormality. The author showed more precise inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality than the results by Fujii-Nakamoto, and he also showed the results on powers of class $F(p, r, q)$ operators. But some of the results on class $F(p, r, q)$ require the assumption of invertibility.

In this paper, we shall remove the assumption of invertibility from the results on invertible class $F(p, r, q)$ operators. Moreover we shall show that the families of class $F(p, r, \frac{1}{p} \frac{1}{r})$ and $(p, r, \frac{1}{p} \frac{1}{r})$-paranormality are proper on $p$.

1. Introduction

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that $p$-hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$ and invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator $T$, and also an operator $T$ is said to be $p$-quasi-hyponormal for $p > 0$ if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$.

We remark that we treat only invertible log-hyponormal operators in this paper (see also [19]). It is easily obtained that every $p$-hyponormal operator is $q$-hyponormal for $p > q > 0$ by Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$," and every invertible $p$-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotonically increasing function. We remark that log-hyponormality is sometimes regarded as $0$-hyponormality since $\frac{X^2}{p^2} \rightarrow \log X$ as $p \rightarrow +0$ for $X > 0$. An operator $T$ is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. Ando [2] showed that every $p$-hyponormal operator for $p > 0$ and (invertible) log-hyponormal operator is paranormal.

Recently, in [10], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms. And also Fujii-Jung-S. H. Lee-M. Y. Lee-Nakamoto [6] introduced class $A(p, r)$...
and Yamazaki-Yanagida [21] introduced absolute-\((p, r)\)-paranormality as follows: An operator \(T\) belongs to class \(A(p, r)\) for \(p > 0\) and \(r > 0\) if \(|\langle T^*|T|T^*|T^*\rangle^{\frac{r}{p+r}} \geq |T^*|^{r/p} \), and also an operator \(T\) is absolute-\((p, r)\)-paranormal if \(\left\| |T|^{p} |T^*|^{r} x \right\| \geq \left\| |T^*|^{r} x \right\|^{p+r} \) for every unit vector \(x \in H\). We remark that class \(A(1,1)\) equals class \(A\) and also absolute-(1,1)-paranormality equals paranormality. These classes are generalizations of class \(A(k)\) and absolute-\(k\)-paranormality introduced as two families of classes based on class \(A\) and paranormal in [10], and also absolute-\((p, r)\)-paranormality is a generalization of \(p\)-paranormality in [3]. We should remark that the families of class \(A(p, r)\) determined by operator inequalities and absolute-\((p, r)\)-paranormality determined by norm inequalities constitute two increasing lines on \(p > 0\) and \(r > 0\) whose origin is (invertible) log-hypnormality.

Moreover, as a continuation of the discussion in [6], Fujii-Nakamoto [7] introduced the following classes of operators.

**Definition ([7]).** For each \(p > 0\), \(r > 0\) and \(q > 0\),

(i) An operator \(T\) belongs to class \(F(p, r, q)\) if

\[
|\langle T^*|T|T^*|T^*\rangle^{\frac{r}{p+r}} \geq |T^*|^{r/p} \tag{1.1}
\]

(ii) An operator \(T\) is \((p, r, q)\)-paranormal if

\[
\left\| |T|^{p} |T^*|^{r} x \right\| \geq \left\| |T^*|^{r} x \right\|^{p+r} \tag{1.2}
\]

for every unit vector \(x \in H\), where \(T = U|T|\) is the polar decomposition of \(T\). In particular, if \(r > 0\) and \(q \geq 1\), then (1.2) is equivalent to

\[
\left\| |T^*|^{r} x \right\| \geq \left\| |T^*|^{r} x \right\|^{r/p} \tag{1.3}
\]

for every unit vector \(x \in H\) ([13]).

We remark that class \(F(p, r, \frac{p+r}{r})\) equals class \(A(p, r)\) and also \((p, r, \frac{p+r}{r})\)-paranormality equals absolute-\((p, r)\)-paranormality. In [13], we obtained the parallel result to that of class \(A(p, r)\) and absolute-\((p, r)\)-paranormality that invertible class \(F(p, r, q)\) and \((p, r, q)\)-paranormality constitute two increasing lines on \(p \geq \delta > 0\) and \(r \geq \gamma_0 > 0\) whose origin is \(\delta\)-quasihypnormality. And also we showed the result on powers of invertible class \(F(p, r, q)\) operators. Thus many researchers have been discussed parallel families of classes of operators which are generalizations of class \(A\) and paranormality.

In this paper, we shall remove the assumption of invertibility from the results on invertible class \(F(p, r, q)\) operators in [13]. Moreover we shall show that the families of class \(F(p, r, \frac{p+r}{r})\) and \((p, r, \frac{p+r}{r})\)-paranormality are proper on \(p\).

2. Preliminaries

Fujii-Nakamoto [7] observed that class \(F(p, r, q)\) derives from the following Theorem 2. A shown in [8] and \((p, r, q)\)-paranormality corresponds to class \(F(p, r, q)\).

We remark that alternative proofs of Theorem 2 A were given in [4] and [16] and also an elementary one page proof in [9]. Tanahashi [17] showed that the domain drawn for \(p, q\) and \(r\) in the Figure 1 is the best possible one for Theorem 2 A.
**Theorem 2.A (Furuta inequality [8]).**

If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

(i) \( (B^* A^p B^*)^\frac{1}{p} \geq (B^* B^p B^*)^\frac{1}{p} \)

and

(ii) \( (A^* A^p A^*)^\frac{1}{p} \geq (A^* B^p A^*)^\frac{1}{p} \)

hold for \( p \geq 0 \) and \( q \geq 1 \) with \( (1 + r)q \geq p + r \).

Fujii-Nakamoto [7] and the author [13] obtained the results on inclusion relations among the families of class \( F(p, r, q) \) and \((p, r, q)\)-paranormality.

**Theorem 2.B ([7]).**

(i) For a fixed \( k > 0 \), \( T \) is \( k \)-hyponormal if and only if \( T \) belongs to class \( F(2k, 2kr, q) \) for all \( p > 0 \), \( r > 0 \) and \( q \geq 1 \) with \((1 + 2r)q \geq 2(p + r)\), i.e., \( T \) belongs to class \( F(p, r, q) \) for all \( p > 0 \), \( r > 0 \) and \( q \geq 1 \) with \((k + r)q \geq p + r \).

(ii) If \( T \) belongs to class \( F(p_0, r_0, q_0) \) for \( p_0 > 0 \), \( r_0 \geq 0 \) and \( q_0 \geq 1 \), then \( T \) belongs to class \( F(p_0, 2kr, q) \) for any \( q \geq q_0 \).

(iii) If \( T \) is \((p_0, r_0, q)\)-paranormal for \( p_0 > 0 \), \( r_0 \geq 0 \) and \( q_0 > 0 \), then \( T \) is \((p_0, r_0, q)\)-paranormal for any \( q \geq q_0 \).

(iv) If \( T \) belongs to class \( F(p, r, q) \) for \( p > 0 \), \( r \geq 0 \) and \( q \geq 1 \), then \( T \) is \((p, r, q)\)-paranormal.

**Theorem 2.C ([13]).**

(i) For each \( p > 0 \) and \( r > 0 \),

(i-1) \( T \) is \( p \)-quasihyponormal if and only if \( T \) belongs to class \( F(p, r, 1) \) if and only if \( T \) is \((p, r, 1)\)-paranormal.

(i-2) \( T \) is \( p \)-quasihyponormal if and only if \( T \) is \((p, 0, 1)\)-paranormal.

(ii) Let \( T \) be a class \( F(p_0, r_0, \frac{p_0 + r_0}{p + r_0}) \) operator for \( p_0 > 0 \), \( r_0 \geq 0 \) and \( \delta > -r_0 \).

(ii-1) If \( T \) is invertible and \( 0 \leq \delta \leq p_0 \), then \( T \) belongs to class \( F(p, r, \frac{p_0 + r}{p + r}) \) for any \( p \geq p_0 \) and \( r \geq r_0 \).

(ii-2) If \( -r_0 < \delta \leq p_0 \), then \( T \) belongs to class \( F(p_0, r, \frac{p_0 + r}{p + r}) \) for any \( r \geq r_0 \).

(iii) Let \( T \) be a \((p_0, r_0, \frac{p_0 + r_0}{p + r_0})\)-paranormal operator for \( p_0 > 0 \), \( r_0 \geq 0 \) and \( \delta > -r_0 \).

(iii-1) If \( 0 \leq \delta \leq p_0 \), then \( T \) is \((p, r, \frac{p_0 + r}{p + r})\)-paranormal for any \( p \geq p_0 \) and \( r \geq r_0 \).

(iii-2) If \( -r_0 < \delta \leq p_0 \), then \( T \) is \((p_0, r, \frac{p_0 + r}{p + r})\)-paranormal for any \( r \geq r_0 \).

(iii-3) If \( 0 \leq \delta \), then \( T \) is \((p, r, \frac{p_0 + r}{p + r})\)-paranormal for any \( p \geq p_0 \).

We remark that only (ii-1) of Theorem 2.C requires invertibility of \( T \), and also we obtained in [14] that every class \( A(p_0, r_0) \) operator for \( p_0 > 0 \) and \( r_0 > 0 \) belongs to class \( A(p, r) \) for any \( p \geq p_0 \) and \( r \geq r_0 \) (without assumption of invertibility).

The following Figure 2 represents the inclusion relations among the families of class \( F(p, r, q) \) and \((p, r, q)\)-paranormality.
On the other hand, we obtained the results on powers of $p$-hyponormal, class $A(p, r)$ and invertible class $F(p, r, q)$ operators.

**Theorem 2.D.**

(i) Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then $T^n$ is $\frac{p}{n}$-hyponormal for all positive integer $n$ ([1]).

(ii) Let $T$ be a class $A(p, r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then $T^n$ belongs to class $A(\frac{p}{n}, \frac{r}{n})$ for all positive integer $n$ ([14]).

(iii) Let $T$ be an invertible class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 < r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then $T^n$ belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer $n$ ([13]).

We remark that (iii) interpolates (i) and (ii) if $T$ is invertible in Theorem 2.D. In fact, (iii) yields (i) by putting $q = 1$ and $r = 0$, and also (iii) yields (ii) by putting $q = \frac{p + r}{r}$.

Moreover we have another result on powers of class $A$ operators by combining [22, Theorem 1] and [14, Theorem 3].

**Theorem 1.** If $T$ is a class $A$ operator, then

\[ |T|^1 \leq |T|^2 \leq \cdots \leq |T|^n |^\frac{p}{n} \quad \text{and} \quad |T|^n |^\frac{p}{n} \geq |T|^n |^\frac{p}{n} \geq \cdots \geq |T|^n |^\frac{p}{n} \]

hold for all positive integer $n$.

We remark that (ii) of Theorem 2.D and Theorem 1 in case of invertible operators were shown in [20] and [12], respectively.
3. Main Results

In this section, we shall show the results which remove the assumption of invertibility from (ii-1) of Theorem 2.C and (iii) of Theorem 2.D.

**Theorem 2.** Let $T$ be a class $F(p_0, r_0, \frac{p_0 + r_0}{p_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $0 \leq \delta \leq p_0$. Then $T$ belongs to class $F(p, r, \frac{p + r}{\delta + r})$ for any $p \geq p_0$ and $r \geq r_0$.

**Theorem 3.** Let $T$ be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then $T^n$ belongs to class $F(\frac{p}{p + r}, \frac{r}{p + r}, q)$ for all positive integer $n$.

We need the following two results in order to prove Theorem 2.

**Theorem 3.A ([14]).** Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$,
\begin{enumerate}
  \item[(i)] If $(B^{\frac{\gamma}{p}} A^p B^{\frac{\gamma}{r}})^{\frac{1}{\gamma}} \geq B^r$, then $A^p \geq (A\frac{p}{r} B^r A \frac{p}{r})^{\frac{p}{p + r}}$.
  \item[(ii)] If $A^p \geq (A\frac{p}{r} B^r A \frac{p}{r})^{\frac{p}{p + r}}$ and $N(A) \subset N(B)$, then $(B^{\frac{\gamma}{p}} A^p B^{\frac{\gamma}{r}})^{\frac{1}{\gamma}} \geq B^r$.
\end{enumerate}

**Theorem 3.B ([22]).** If $A^\alpha \geq (A^\frac{\alpha}{p} B^\frac{\alpha}{q} A^\frac{\alpha}{q})^{\frac{\alpha}{\alpha + \beta}}$ holds for positive operators $A$ and $B$ and fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then
\begin{equation}
A^\alpha \geq (A^\frac{\alpha}{p} B^\frac{\alpha}{q} A^\frac{\alpha}{q})^{\frac{\alpha}{\alpha + \beta}}
\end{equation}
holds for any $\alpha \geq \alpha_0$. Moreover, for each fixed $\gamma \geq -\beta_0$,
\begin{equation}
g_{\beta_0, \gamma}(\alpha) = (B^{\frac{\gamma}{p}} A^\alpha B^{\frac{\gamma}{q}})^{\frac{\alpha + \beta}{\alpha + \beta + \gamma}}
\end{equation}
is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence $(B^{\frac{\gamma}{p}} A^\alpha B^{\frac{\gamma}{q}})^{\frac{\alpha + \beta + \gamma}{\alpha + \beta + \gamma + \gamma}} \geq B^{\frac{\gamma}{p}} A^{\alpha_1} B^{\frac{\gamma}{q}}$ holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

**Proof of Theorem 2.** In case $r_0 = 0$, it is already shown in (i) of Theorem 2.B since class $F(p_0, 0, \frac{p_0}{\delta})$ for $0 < \delta \leq p_0$ equals $\delta$-hyponormality. So we may assume $r_0 > 0$. Suppose that $T$ belongs to class $F(p_0, r_0, \frac{p_0 + r_0}{p_0})$ for $p_0 > 0$, $r_0 > 0$ and $0 \leq \delta \leq p_0$, i.e.,
\begin{equation}
(|T^*|^p |T|^p |T^*|^p)^{\frac{p_0}{p + r}} \geq |T^*|^p (\delta + r_0).
\end{equation}
Applying L"{o}wner-Heinz theorem to (3.1), we have
\begin{equation}
(|T^*|^p |T|^p |T^*|^p)^{\frac{p_0}{p + r}} \geq |T^*|^p,
\end{equation}
and also we have
\begin{equation}
|T|^p \geq (|T|^p |T^*|^p |T|^p)^{\frac{p_0}{p + r}}
\end{equation}
by (i) of Theorem 3.A. By applying Theorem 3.B to (3.2), we obtain that
\begin{equation}
g_{\gamma_0, \delta}(p) = (|T|^p |T|^p |T|^p)^{\frac{\delta + \gamma_0}{p + r}}
\end{equation}
is an increasing function for $p \geq \max\{p_0, \delta\} = p_0$. 

Therefore we have
\[
(|T^*|^q |T|^{2p} |T^*|^r)^{\frac{2+r}{2p}} = g_{r_0, \delta}(p)
\]
\[
\geq g_{r_0, \delta}(p_0) \quad \text{by (3.3)}
\]
\[
= (|T^*|^q |T|^{2p_0} |T^*|^r)^{\frac{2+r_0}{2p_0}}
\]
\[
\geq |T^*|^3(|T|^{2(p_0 + r_0)})^{\frac{2+r_0}{2p_0}} \quad \text{by (3.1)}
\]
for any \( p \geq p_0 \), i.e., \( T \) belongs to class \( \mathcal{F}(p, r_0, \frac{2+r_0}{2p_0}) \) for any \( p \geq p_0 \). Hence \( T \) belongs to class \( \mathcal{F}(p, r, \frac{2+r}{2p}) \) for any \( p \geq p_0 \) and \( r \geq r_0 \) by (ii-2) of Theorem 2C. \( \Box \)

To prove Theorem 3, we prepare the following result which is a slight modification of [22, Lemma 5].

**Lemma 4.** Let \( A, B \) and \( C \) be positive operators, \( p > 0 \), \( 0 < r \leq 1 \) and \( q \geq 1 \) with \( rq \leq p + r \leq (1 + r)q \). If \( (B^* A^p B)^{\frac{1}{q}} \geq B^{\frac{2+r}{q}} \) and \( B \geq C \), then \( (C^* A^p C)^{\frac{1}{q}} \geq C^{\frac{2+r}{q}} \).

**Proof.** The hypothesis \( B \geq C \) ensures \( B^r \geq C^r \) for \( r \in (0,1] \) by Löwner-Heinz theorem. By Douglas' theorem [3], there exists an operator \( X \) such that

\[
B^{\frac{1}{q}} X = X^* B^{\frac{1}{q}} = C^{\frac{1}{q}}
\]

and \( \|X\| \leq 1 \). Then we have

\[
(C^* A^p C)^{\frac{1}{q}} = (X^* B^{\frac{1}{q}} A^p B X)^{\frac{1}{q}}
\]
\[
\geq X^* (B^{\frac{1}{q}} A^p B)^{\frac{1}{q}} X \quad \text{by Hansen's inequality [11]}
\]
\[
\geq X^* B^{\frac{2+r}{q}} X \quad \text{by the hypothesis}
\]
\[
= C^* B^{\frac{2+r}{q}} C \quad \text{by (3.4)} \quad \text{since} \quad \frac{2+r}{q} - r \in [0,1]
\]
\[
\geq C^{\frac{2+r}{q}} \quad \text{by Löwner-Heinz theorem.}
\]

Hence the proof is complete. \( \Box \)

**Proof of Theorem 3.** Let \( T \) be a class \( \mathcal{F}(p, r, q) \) operator for \( 0 < p \leq 1 \), \( 0 \leq r \leq 1 \) and \( q \geq 1 \) with \( rq \leq p + r \), i.e.,

\[
(|T^*|^q |T|^{2p} |T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2+r}{q}}.
\]

Class \( \mathcal{F}(p, r, q) \) operator \( T \) for \( 0 < p \leq 1 \), \( 0 \leq r \leq 1 \) and \( q \geq 1 \) with \( rq \leq p + r \) belongs to class \( \mathcal{F}(1,1,2) \), i.e., class A by (ii) of Theorem 2.B and Theorem 2, and also

\[
|T^n|^q \geq |T|^2
\]

and

\[
|T^*|^2 \geq |T^n|^q
\]

hold for all positive integer \( n \) by Theorem 1. By applying Lemma 4 to (1.1) and (3.6), we have

\[
(|T^n|^q |T|^{2p} |T^n|^r)^{\frac{1}{q}} \geq |T^n|^{\frac{2+r}{q}}
\]
for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ since $p + r \leq (1 + r)q$ always holds. Hence we obtain
\[
\left( |T^n|^\frac{p}{r} |T^n|^\frac{q}{r} |T^n'|^\frac{p}{r} \right)^{\frac{1}{2}} \geq \left( |T^n|^\frac{p}{r} |T^n|^\frac{q}{r} |T^n'|^\frac{p}{r} \right)^{\frac{1}{2}} \text{ by (3.5) and L"owner-Heinz theorem}
\]
\[
\geq |T^n|^\frac{p}{r} \left( \frac{rq}{p + r} \right)^{\frac{1}{2}} \text{ by (3.7)}
\]
for all positive integer $n$, that is, $T^n$ belongs to class $F\left( \frac{p}{r}, \frac{q}{r}, q \right)$ for all positive integer $n$. □

4. Properness of Class $F(p, r, \frac{p+r}{r+\delta})$ and $(p, r, \frac{p+r}{r+\delta})$-Paranormality

In this section, we shall show the results on inclusion relation among the families of $p$-quasihyponormality, class $F(p, r, q)$ and $(p, r, q)$-paranormality.

**Theorem 5.** For each $p_0 > 0$, there exists a $p_0$-quasihyponormal operator $T$ such that $T$ is not $(p, r, \frac{p+r}{r+\delta})$-paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.

**Theorem 6.** For each $p_0 > 0$, $m > 0$ and $-m < \delta \leq p_0$.

(i) There exists a $p_0$-quasihyponormal operator $T$ such that $T$ is not $p$-quasihyponormal for any $p > 0$ such that $0 < p < p_0$.

(ii) There exists a class $F(p_0, r_0, \frac{p_0+r_0}{r_0+\delta})$ operator $T$ such that $T$ does not belong to class $F(p, r_0, \frac{p+r_0}{r+\delta})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.

(iii) There exists a $(p_0, r_0, \frac{p_0+r_0}{r_0+\delta})$-paranormal operator $T$ such that $T$ is not $(p, r_0, \frac{p+r_0}{r_0+\delta})$-paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.

In Theorem 6, (i) has been obtained in [18], and also (ii) and (iii) asserts that the families of class $F(p, r, \frac{p+r}{r+\delta})$ and $(p, r, \frac{p+r}{r+\delta})$-paranormality are proper on $p$. Moreover we remark that these properness on $p$ has no connection with $r$, and also we have the following corollary by putting $r = r_0$ in Theorem 6.

**Corollary 7.** For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$.

(i) There exists a class $F(p_0, r_0, \frac{p_0+r_0}{r_0+\delta})$ operator $T$ such that $T$ does not belong to class $F(p, r_0, \frac{p+r_0}{r_0+\delta})$ for any $p > 0$ such that $\delta \leq p < p_0$.

(ii) There exists a $(p_0, r_0, \frac{p_0+r_0}{r_0+\delta})$-paranormal operator $T$ such that $T$ is not $(p, r_0, \frac{p+r_0}{r_0+\delta})$-paranormal for any $p > 0$ such that $\delta \leq p < p_0$.

Here we shall show two propositions as a preparation of the proof of Theorem 5. We remark that these propositions are similar arguments to [2], [10], [13] and so on.

Firstly we shall give a characterization of $(p, r, q)$-paranormal operators.

**Proposition 8.** For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, an operator $T$ is $(p, r, \frac{p+r}{r+\delta})$-paranormal if and only if
\[
(\delta + r)|T^n|^\frac{p}{r} |T^n|^\frac{q}{r} |T^n'|^\frac{p}{r} - (p + r)\lambda^{p-\delta} |T^n|^\frac{p}{r} |T^n'|^\frac{p}{r} + (p - \delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0.
\]
Proof. Suppose that $T$ is $(p, r, \frac{p-r}{p+r})$-paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$, i.e.,
\begin{equation}
\| |T|^p |T^*|^r \| \frac{p-r}{p+r} \| \| |T^*|^{|r+\delta} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H.
\end{equation}
(1.3) holds iff
\begin{equation}
\| |T|^p |T^*|^r \| \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H
\end{equation}
iff
\begin{equation}
( |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H
\end{equation}
By arithmetic-geometric mean inequality,
\begin{equation}
( |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H
\end{equation}
\begin{align}
&= \left\{ \left( \frac{1}{\lambda} \right)^{p-\delta} ( |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H
\end{align}
\begin{equation}
\leq \frac{\delta + r}{p + r} \lambda^{p-r} ( |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H
\end{equation}
for all $x \in H$ and all $\lambda > 0$, so (4.1) ensures the following (4.3) by (4.2).
\begin{equation}
\frac{\delta + r}{p + r} \lambda^{p-r} ( |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H
\end{equation}
Conversely, (4.1) follows from (4.3) by putting $\lambda = \left( \frac{\| |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) \frac{p-r}{p+r} \| \| \| |T^*|^{|r+\delta} \| \| \| x \in H \right)^{1/p}$. (In case
\begin{equation}
( |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) = 0, let \lambda \to +0.) Hence (4.3) holds if and only if
\begin{equation}
( \delta + r) |T^*|^{|r+\delta} |T^*|^{|r+\delta} \| (x, x) - (p + r) \lambda^{p-\delta} |T^*|^{|r+\delta} \| (x, x) \geq 0 \quad \text{for all } \lambda > 0,
\end{equation}
so that the proof is complete.
\[ \square \]
Secondly we shall give the following Proposition 9. But we omit to describe these calculation because it is obtained by easy calculation.

Proposition 9. Let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong H$. For given positive operators $A, B$ on $H$, define the operator $T_{A, B}$ on $K$ as follows:
\begin{equation}
T_{A, B} = \begin{pmatrix}
& & & & \vdots & & & \\
& & & 0 & B^\frac{1}{2} & 0 & & \\
& & B^\frac{1}{2} & 0 & \square & 0 & & \\
& 0 & A^\frac{1}{2} & 0 & \square & 0 & & \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{pmatrix},
\end{equation}
where $\square$ shows the place of the $(0, 0)$ matrix element.
(i) For each $p > 0$, $T_{A, B}$ is $p$-quasihyponormal if and only if
\[ B^\frac{1}{2} A^p B^\frac{1}{2} \geq B^{p+1}. \]
(ii) For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, $T_{A,B}$ is $(p, r, \frac{e^{-p}}{p + r})$-paranormal if and only if
\[(\delta + r)B^{\ast}A^p B^{\ast} - (p + r)\lambda^{p-r}B^{\delta+r} + (p - \delta)\lambda^{p+r}I \geq 0\quad \text{for all } \lambda > 0.

Proof of Theorem 5. Let
\[A = U\Lambda U^* \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\]
where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} (2 - e^{-p})e^{-p} & 0 \\ 0 & e^{-r} \end{pmatrix}$,
and also let $K = \bigoplus_{n=\infty} H_n$, where $H_n \cong \mathbb{R}^2$. For positive matrices $A, B$ on $\mathbb{R}^2$ given in (4.5), define the operator $T_{A,B}$ on $K$ as (4.4) in Proposition 9. By (i) of Proposition 9, $T_{A,B}$ is $p$-quasi-hyponormal for $p > 0$ if and only if
\[B^{\ast}A^p B^{\ast} - B^{p+1} = \frac{1}{2}(2 - e^{-p})e^{-p} + e^{-2p} - 1 \geq 0\]
if and only if
\[f(p) = \frac{1}{2}(2 - e^{-p})e^{-p} + e^{-2p} - 1 \geq 0.

On the other hand, let $X_p(\lambda)$ as
\[X_p(\lambda) = (\delta + r)B^{\ast}A^p B^{\ast} - (p + r)\lambda^{p-r}B^{\delta+r} + (p - \delta)\lambda^{p+r}I\]
= \(\frac{1}{2}(\delta + r)(2 - e^{-p})e^{-p} + e^{-2p} - (p + r)\lambda^{p-r} + (p - \delta)\lambda^{p+r} - 0\)
\[(p - \delta)\lambda^{p+r} \geq 0\]
By (ii) of Proposition 9, $T_{A,B}$ is $(p, r, \frac{e^{-p}}{p + r})$-paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$ if and only if $X_p(\lambda) \geq 0$ for all $\lambda > 0$ if and only if
\[g_p(\lambda) = \frac{1}{2}(\delta + r)(2 - e^{-p})e^{-p} + e^{-2p} - (p + r)\lambda^{p-r} + (p - \delta)\lambda^{p+r} - 0\]
\[(p - \delta)\lambda^{p+r} \geq 0\]
so that (4.6) holds if and only if $f(p) \geq 0$.

$f(p)$ is a convex function for $p > 0$ since
\[f''(p) = -\frac{1}{2}(2 - e^{-p})e^{-p}[\log(2 - e^{-p})e^{-p}]^2 + 4e^{-2p} > 0\quad \text{for all } p > 0,
\]
and also $f(p) = 0$ if $p = p_0$. So we have $f(p_0) = 0$ but $f(p) < 0$ for $0 < p < p_0$. Therefore $g_p(1) < 0$, that is $X_p(1) \not\geq 0$ for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.

Hence $T_{A,B}$ is $p_0$-quasi-hyponormal but non-$(p, r, \frac{e^{-p}}{p + r})$-paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$, so the proof is complete. \(\square\)

Proof of Theorem 6. Let $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$.

Proof of (i). By (i-1) of Theorem 2.C, $T$ is $p$-quasi-hyponormal if and only if $T$ is $(p, r, 1)$-paranormal for some $p > 0$ and $r > 0$. Therefore there exists a $p_0$-quasi-hyponormal operator $T$ such that $T$ is not $p$-quasi-hyponormal for any $0 < p < p_0$ by putting $\delta = p$ in Theorem 5.
Proof of (ii). By (i-1) of Theorem 2.2 and (ii) of Theorem 2.2, every $p_0$-quasihyponormal operator belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$. And also, by (iv) of Theorem 2.2, $T$ does not belong to class $F(p, r, \frac{p+r}{\delta+r})$ if $T$ is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for each $p > 0$, $r > 0$ and $-r < \delta \leq p$. Therefore there exists a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator $T$ such that $T$ does not belong to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$ by Theorem 5.

Proof of (iii). By (i-1) of Theorem 2.2 and (iii) of Theorem 2.2, every $p_0$-quasihyponormal operator is $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal. Therefore there exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator $T$ such that $T$ is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$ by Theorem 5.

Remark. In [10], we introduced two families of classes of operators based on class $A$ and paranormality as follows: An operator $T$ belongs to class $A(k)$ for $k > 0$ if $(\|T^k T\|^{\frac{k+1}{k}} \geq \|T\|)$, and also an operator $T$ is absolute-$k$-paranormal for $k > 0$ if $\|T^k T \| \geq \|T\|^{k+1}$ for every unit vector $x \in H$. In [5], Fujii-Izumino-Nakamoto introduced $p$-paranormality for $p > 0$ defined by $\|T^p U T^p x\| \geq \|T^p x\|^2$ for every unit vector $x \in H$, where $T = U |T|$ is the polar decomposition of $T$. It was pointed out in [20] that class $A(k)$ equals class $A(k, 1)$, and also it was shown in [21] that absolute-$k$-paranormality equals absolute-($k, 1$)-paranormality and $p$-paranormality equals absolute-($p, p$)-paranormality. We shall also get the results on inclusion relation among the families of these classes.

Corollary 10.

(i) For each $k_0 > 0$, there exists a class $A(k_0)$ operator $T$ such that $T$ does not belong to class $A(k)$ for any $0 < k < k_0$.

(ii) For each $k_0 > 0$, there exists an absolute-$k_0$-paranormal operator $T$ such that $T$ is not absolute-$k$-paranormal for any $0 < k < k_0$.

(iii) For each $p_0 > 0$, there exists a $p_0$-paranormal operator $T$ such that $T$ is not $p$-paranormal for any $0 < p < p_0$.

Proof of Corollary 10.

Proof of (i) and (ii). By putting $p_0 = k_0$, $r = 1$, $\delta = 0$ and $p = k$ in Corollary 7, we have (i) and (ii) since class $A(k)$ equals class $F(k, 1, k + 1)$ and absolute-$k$-paranormality equals $(k, 1, k + 1)$-paranormality.

Proof of (iii). By putting $p_0 = r_0$, $\delta = 0$ and $p = r$ in (iii) of Theorem 6, we have (iii) since $p$-paranormality equals $(p, p, 2)$-paranormality.

References

[8] T. Furuta, \( A \geq B \geq 0 \) assures \( (B^p A^q B^r)^{1/q} \geq B^{[p+2r]/q} \) \( \geq 0 \) \( p \geq 0, q \geq 1 \) with \( (1+2r)q \geq p+2r \), Proc. Amer. Math. Soc., 101 (1987), 85–88.


[14] M. Ito and T. Yamazaki, Relations between two inequalities \( (B^p A^q B^r)^{1/q} \geq B^p \) and \( A^p \geq (A^p B^q A^r)^{1/p} \) and their applications, to appear in Integral Equations Operator Theory.


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