APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS

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ABSTRACT. Our purpose is to show two strong convergence theorems for nonexpansive nonself-mappings in a Hilbert space; these are generalizations of Wittmann’s result[7], and are proved without any boundary conditions. For this purpose, a boundary condition, called nowhere normal-outward condition, is investigated and characterized.

1 Introduction Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $T$ be a nonexpansive nonself-mapping from $C$ into $H$ such that the set $F(T)$ of all fixed points of $T$ is nonempty. In 1992, Marino and Trombetta[2] defined two contraction mappings $S_t$ and $U_t$ as follows: For a given $u \in C$ and each $t \in (0,1),$

(1.1) \[ S_t x = t P T x + (1 - t) u \quad \text{for all} \quad x \in C \]

and

(1.2) \[ U_t x = P (t T x + (1 - t) u) \quad \text{for all} \quad x \in C, \]

where $P$ is the metric projection from $H$ onto $C$. Then by the Banach contraction principle, there exists a unique element $x_t \in F(S_t)$ (resp. $y_t \in F(U_t)$), i.e.

(1.3) \[ x_t = t P T x_t + (1 - t) u \]

and

(1.4) \[ y_t = P (t T y_t + (1 - t) u). \]

Recently, Xu and Yin[8] proved that if $T$ is a nonexpansive nonself-mapping from $C$ into $H$ satisfying the weak inwardness condition, then \{x_t\} (resp. \{y_t\}) defined by (1.3) (resp. (1.4)) converges strongly as $t \to 1$ to an element of $F(T)$ which is nearest to $u$ in $F(T)$. This result was extended to a Banach space by Takahashi and Kim[6]. On the other hand, Wittmann[7] proved the following strong convergence theorem; see also [4]:

**Theorem (Wittmann 1992).**

Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $S$ be a nonexpansive mapping from $C$ into itself. Let \{a_n\} be a sequence of real numbers such that $0 \leq a_n \leq 1$, $\lim_{n \to \infty} a_n = 0$, $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$, and $\sum_{n=1}^{\infty} a_n = \infty$. Define a sequence \{x_n\} as follows: $x_1 = x \in C$ and

(1.5) \[ x_{n+1} = a_n x + (1 - a_n) S x_n \quad \text{for} \quad n \geq 1. \]
If $F(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to $Px \in F(S)$, where $P$ is the metric projection from $C$ onto $F(S)$.

In this paper, we extend the above Wittmann’s result to nonexpansive nonself-mappings without any boundary conditions. For this purpose, we consider about a boundary condition in Section 2, which is called nowhere normal-outward condition. Also we show two propositions between the boundary condition and $F(T)$ when $T$ is a nonexpansive nonself-mapping; the propositions play important roles in this paper. Finally, we introduce two iteration schemes for $T$ by using the metric projection from $H$ onto $C$, and show two strong convergence theorems, which are generalizations of the Wittmann’s result in Section 3.

2 Preliminaries Throughout this paper, we denote the set of all positive integers by $\mathbb{N}$. Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and with inner product $\langle \cdot , \cdot \rangle$, let $C$ be a closed convex subset of $H$, and let $T$ be a nonself-mapping from $C$ into $H$. We denote the set of all fixed points of $T$ by $F(T)$. Then $T$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$  

For all $x \in H$, there exists a unique element $Px$ of $C$ satisfying

$$\|x - Px\| = \min_{y \in C} \|x - y\| \text{ for all } x \in H.$$  

This mapping $P$ is said to be the metric projection from $H$ onto $C$. We know that $P$ is nonexpansive and for all $x \in H$, $z = Px$ if and only if $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$. It is known that $H$ satisfies Opial’s condition [3]; see also [5]: if $\{x_n\}$ converges weakly to $x$, then

$$\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\|$$  

for all $y \neq x$.

Next, we introduce several boundary conditions upon the nonself-mapping,

(i) Rothe’s condition: $T(\partial C) \subset C$, where $\partial C$ is the boundary set of $C$;

(ii) inwardness condition [1]: $Tx \in I_a(x)$ for all $x \in C$, where

$$I_a(x) = \{y \in H \mid y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\};$$

(iii) weak inwardness condition [1]: $Tx \in \text{cl} I_a(x)$ for all $x \in C$, where $\text{cl}$ denotes the norm-closure; and

(iv) nowhere normal-outward condition [1]: $Tx \in S_x^C$ for all $x \in C$, where $P$ is the metric projection from $H$ onto $C$, and

$$S_x = \{y \in H \mid y \neq x, Py = x\}.$$  

It is easily seen that there hold implications: (i)$\Rightarrow$(ii)$\Rightarrow$(iii). It also holds that (iii)$\Rightarrow$(iv); see [1], p.354. To prove our results, we need the following propositions:

**Proposition 2.1** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, let $P$ be the metric projection from $H$ onto $C$, and let $T$ be a nonself-mapping from $C$ into $H$ satisfying the nowhere normal-outward condition. Then $F(T) = F(PT)$. Moreover, if $C$ is bounded and $T$ is nonexpansive, then $T$ has a fixed point.
**Proof.** At first we show $F(T) = F(PT)$. It is sufficient to prove that $F(PT)$ is a subset of $F(T)$. Let $x \in F(PT)$, that is $PTx = x$. Since $Tx \in S_x^T$, we obtain $Tx = x$. Next, suppose that $C$ is bounded and $T$ is nonexpansive. Then $PT$ is a nonexpansive mapping from $C$ into itself. Therefore $F(T) = F(PT) \neq \emptyset$, see [5].

**Proposition 2.2** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, let $T$ be a nonexpansive nonself-mapping from $C$ into $H$. If $F(T) \neq \emptyset$, then $T$ satisfies nowhere normal-outward condition.

**Proof.** If there exists $x_0 \in C$ such that $Tx_0 \in S_{x_0}$, then $Tx_0 \neq x_0$ and $PTx_0 = x_0$, where $P$ is the metric projection from $H$ onto $C$. Let $z \in F(T)$, we have

$$
\|Tx_0 - z\|^2 = \|Tx_0 - x_0\|^2 + 2\langle Tx_0 - PTx_0, PTx_0 - z \rangle + \|PTx_0 - z\|^2
$$

This contradicts that $T$ is nonexpansive. Therefore, $Tx \in S_x^T$ for all $x \in C$.

**Remark 2.1** By using Proposition 2.1 and Proposition 2.2, we can consider generalizations of fixed point theorems from self-mappings to nonself-mappings. When $T$ is a nonexpansive nonself-mapping, applying the fixed point theorems to self-mapping $PT$, we have some results with respect to nonself-mapping $T$. For example, we can show the following, which is a generalization result of Xu and Yin’s result, see [8], and also note that it is proved without any boundary conditions:

Let $H$ be a real Hilbert space, let $C$ be a nonempty closed convex subset of $H$, let $T$ be the metric projection from $H$ onto $C$, and let $T$ be a nonexpansive nonself-mapping from $C$ into $H$. Let $\{x_n\}$ and $\{x_n\}$ be the nets defined by (1.3) and (1.4), respectively. If $T$ satisfies nowhere normal-outward condition, then the following three conditions are equivalent:

- $F(T) \neq \emptyset$,
- $\{x_n\}$ remains bounded as $t \to 1$,
- $\{y_n\}$ remains bounded as $t \to 1$.

Also, if $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{y_n\}$ converge strongly as $t \to 1$ to some fixed points of $T$.

In the next section, we can apply the idea to Theorem 3.1. However, we can not apply it to Theorem 3.2 simply; it is more complicated.

**3 Main Results** In this section, we prove two strong convergence theorems for nonexpansive nonself-mappings, which are generalizations of Wittmann’s result[7], and also, which are not required any boundary conditions.

**Theorem 3.1** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, let $P_1$ be the metric projection from $H$ onto $C$, and let $T$ be a nonexpansive nonself-mapping from $C$ into $H$. Let $\{c_n\}$ be a sequence of real numbers such that $0 \leq c_n \leq 1$, $\lim_{n \to \infty} c_n = 0$, $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$, and $\sum_{n=1}^{\infty} c_n = \infty$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and

$$
x_{n+1} = c_n x + (1 - c_n)P_1 Tx_n \quad \text{for } n \geq 1.
$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to $P_2x \in F(T)$, where $P_2$ is the metric projection from $C$ onto $F(T)$.
This theorem is proved easily by using Proposition 2.1 and Proposition 2.2, as shown in Remark 2.1.

**Proof.** Since \( P_1 T \) is a nonexpansive mapping from \( C \) into itself, applying Wittmann’s result, we obtain that \( \{x_n\} \) converges strongly as \( n \to \infty \) to a fixed point \( z \) of \( P_1 T \) nearest to \( x \). Using Proposition 2.1 and Proposition 2.2, we obtain \( F(P_1 T) = F(T) \). Hence \( \{x_n\} \) converges strongly as \( n \to \infty \) to a fixed point \( z \) of \( T \) nearest to \( x \). \( \square \)

**Theorem 3.2** Let \( H \) be a Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \), let \( P_1 \) be the metric projection from \( H \) onto \( C \), and let \( T \) be a nonexpansive nonself-mapping from \( C \) into \( H \). Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \), and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Define a sequence \( \{y_n\} \) as follows: \( y_1 = y \in C \) and

\[
y_{n+1} = P_1(\alpha_n y + (1 - \alpha_n)Ty_n) \quad \text{for} \quad n \geq 1.
\]

If \( F(T) \neq \emptyset \), then \( \{y_n\} \) converges strongly to \( P_2 y \in F(T) \), where \( P_2 \) is the metric projection from \( C \) onto \( F(T) \).

**Proof.** Let \( z \in F(T) \). Then we have

\[
\|y_2 - z\| = \|P_1(\alpha_1 y + (1 - \alpha_1)Ty_1) - P_1z\|
\leq \|\alpha_1 y + (1 - \alpha_1)Ty_1 - z\|
\leq \alpha_1 \|y - z\| + (1 - \alpha_1)\|y_1 - z\|
= \|y - z\|.
\]

If \( \|y_n - z\| \leq \|y - z\| \) for some \( n \in \mathbb{N} \), then we can show that \( \|y_{n+1} - z\| \leq \|y - z\| \) similarly. Therefore, by induction, we obtain \( \|y_n - z\| \leq \|y - z\| \) for all \( n \in \mathbb{N} \) and hence \( \{y_n\} \) and \( \{Ty_n\} \) are bounded. Set \( K = \sup\{\|Ty_n\| : n \in \mathbb{N}\} \). Then

\[
\|y_{n+1} - y_n\| = \|P_1(\alpha_n y + (1 - \alpha_n)Ty_n) - P_1(\alpha_{n-1} y + (1 - \alpha_{n-1})Ty_{n-1})\|
\leq \|\alpha_n y + (1 - \alpha_n)Ty_n - \{\alpha_{n-1} y + (1 - \alpha_{n-1})Ty_{n-1}\}\|
= \|\alpha_{n-1} y + (1 - \alpha_{n-1})Ty_{n-1} - Ty_{n-1}\|
\leq |\alpha_{n-1} - \alpha_n|\|y\| + (1 - \alpha_n)\|y_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|Ty_{n-1}\|
\leq |\alpha_{n-1} - \alpha_n|\|y\| + (1 - \alpha_n)\|y_{n-1}\|
\]

for each \( n \in \mathbb{N} \). By induction, we have

\[
\|y_{n+m+1} - y_{n+m}\| \leq \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\|y\| + K + m \prod_{k=m}^{n+m-1} (1 - \alpha_{k+1})\|y_{m+1} - y_m\|
\]

for all \( m, n \in \mathbb{N} \). By \( \sum_{n=1}^{\infty} \alpha_n = \infty \), we have \( \prod_{n=1}^{\infty} (1 - \alpha_n) = 0 \); see [4]. Hence we obtain

\[
\limsup_{n \to \infty} \|y_{n+1} - y_n\| \leq \limsup_{n \to \infty} \|y_{n+m+1} - y_{n+m}\| \leq \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\|y\| + K
\]

for all \( m \in \mathbb{N} \). By \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \), we get \( \lim_{n \to \infty} \|y_{n+1} - y_n\| = 0 \). Also, from

\[
\|y_n - P_1 Ty_n\| = \|P_1(\alpha_{n+1} y + (1 - \alpha_{n+1})Ty_{n+1}) - P_1 Ty_n\|
\leq \|\alpha_{n+1} y + (1 - \alpha_{n+1})Ty_{n+1} - Ty_n\|
\leq \alpha_{n+1} \|y - Ty_n\| + (1 - \alpha_{n+1})\|y_{n+1} - y_n\|
\]

for all \( n \in \mathbb{N} \).
we obtain
\[(3.5) \quad \lim_{n \to \infty} \|y_n - P_1 T y_n\| = 0.\]

Next we prove
\[(3.6) \quad \limsup_{n \to \infty} \langle y_n - P_2 y, y - P_2 y \rangle \leq 0.\]

Let \(\{y_{n_k}\}\) be a subsequence of \(\{y_n\}\) which satisfies
\[\lim_{k \to \infty} \langle y_{n_k} - P_2 y, y - P_2 y \rangle = \limsup_{n \to \infty} \langle y_n - P_2 y, y - P_2 y \rangle,
\]
and which converges weakly as \(k \to \infty\) to \(y_0 \in C\). By (3.5) and Opial’s condition, we obtain \(y_0 \in F(P_1 T)\). Applying Proposition 2.1 and Proposition 2.2, we conclude \(y_0 \in F(T)\). Then we have
\[\limsup_{n \to \infty} \langle y_n - P_2 y, y - P_2 y \rangle = \lim_{k \to \infty} \langle y_{n_k} - P_2 y, y - P_2 y \rangle = \langle y_0 - P_2 y, y - P_2 y \rangle \leq 0.
\]

By (3.6), for any \(\varepsilon > 0\), there exists \(m \in \mathbb{N}\) such that
\[\langle y_n - P_2 y, y - P_2 y \rangle \leq \varepsilon\]
for all \(n \geq m\). On the other hand, from
\[P_1(\alpha_n y + (1 - \alpha_n) T y_n) - P_1(\alpha_n y + (1 - \alpha_n) P_2 y) = y_{n+1} - P_2 y + \alpha_n (P_2 y - y),\]
we have
\[\|P_1(\alpha_n y + (1 - \alpha_n) T y_n) - P_1(\alpha_n y + (1 - \alpha_n) P_2 y)\|^2 \geq \|y_{n+1} - P_2 y\|^2 + 2\alpha_n \langle y_{n+1} - P_2 y, y - P_2 y \rangle.
\]
This implies
\[\|y_{n+1} - P_2 y\|^2 \leq (1 - \alpha_n)\|T y_{n} - P_2 y\|^2 + 2\alpha_n \langle y_{n+1} - P_2 y, y - P_2 y \rangle
\]
for all \(n \geq m\). By (3.7), we have
\[\|y_{n+1} - P_2 y\|^2 \leq 2\alpha_n \|y_{n+1} - P_2 y, y - P_2 y\|^2 + (1 - \alpha_n)^2\|T y_{n} - P_2 y\|^2 \leq 2\alpha_n \varepsilon + (1 - \alpha_n)^2\|y_{n} - P_2 y\|^2
\]
for all \(n \geq m\). This implies
\[\|y_{n+1} - P_2 y\|^2 \leq 2\varepsilon \{1 - (1 - \alpha_n)\}
\]
\[+ 2\varepsilon \left\{1 - \alpha_n\right\}(1 - (1 - \alpha_{n-1})\|y_{n-1} - P_2 y\|^2)
\]
\[= 2\varepsilon \left\{1 - (1 - \alpha_n)\right\}(1 - (1 - \alpha_{n-1})\|y_{n-1} - P_2 y\|^2
\]
for all \(n \geq m\). By induction, we obtain
\[\|y_{n+1} - P_2 y\|^2 \leq 2\varepsilon \left\{1 - \prod_{k=m}^{n} (1 - \alpha_k)\right\} + \prod_{k=m}^{n} (1 - \alpha_k)\|y_m - P_2 y\|^2.
\]
Therefore, from \(\sum_{n=1}^{\infty} \alpha_n = \infty\), we obtain
\[\limsup_{n \to \infty} \|y_{n+1} - P_2 y\|^2 \leq 2\varepsilon.
\]
Since \(\varepsilon\) is arbitrary, we can conclude that \(\{y_n\}\) converges strongly to \(P_2 y\). \(\square\)
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References


