Fuzzy congruence on $BCI$-algebras

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Abstract. In this paper we define fuzzy congruences on $BCI$-algebras and their quotient algebras, and prove some fundamental results:

1. There is a one to one correspondence between the set $FC(X)$ of all fuzzy closed ideals of $X$ and the set $FCon_R(X)$ of all fuzzy regular congruences on $X$.

2. Let $X,Y$ be $BCI$-algebras and $f : X \rightarrow Y$ be a $BCI$-homomorphism. If $\bar{A}$ is a fuzzy ideal of $Y$, then the quotient algebra $X/f^{-1}(\bar{A})$ and $f(X)/\bar{A}$ are $BCI$-algebras and $X/f^{-1}(\bar{A}) \cong f(X)/\bar{A}$

1 Introduction While there are many papers about fuzzy $BCK/BCI$-algebras and fuzzy ideals of those, we find few papers about fuzzy congruences. In the usual theory of crisp $BCK/BCI$-algebras, there exists a close relationship between ideals and congruences. It is a natural question to extend the relationship to the case of fuzzy $BCK/BCI$-algebras. In this paper we define fuzzy congruences on $BCI$-algebras and quotient fuzzy $BCI$-algebras by those and investigate their properties.

2 Preliminaries By a $BCI$-algebra we mean an algebraic structure $(X, s, 0)$ of type $(2,0)$ satisfying the following conditions: For all $x, y, z \in X$,

1. $(x * y) * (x * z) = (x * z) * (x * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = y * x = 0$ implies $x = y$

We define a relation $\leq$ on $X$ by $x \leq y$ if and only if $x * y = 0$. It is clear from definition that $\leq$ is a partial order on $X$. If a $BCI$-algebra $X$ satisfies the extra condition $0 * x = 0$ for all $x \in X$, then it is called a $BCK$-algebra. In any $BCI$-algebra $X$, we have:

$(P1)$ $x * 0 = x$
$(P2)$ $x * y \leq x$
$(P3)$ $(x * y) * z = (x * z) * y$
$(P4)$ $(x * z) * (y * z) \leq x * y$
$(P5)$ $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

A non-empty subset $A$ of a $BCI$-algebra $X$ is said to be an ideal of $X$ if

$(I1)$ $0 \in A$
$(I2)$ $x * y \in A$ and $y \in A$ imply $x \in A$.

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Moreover an ideal \( A \) is called \textit{closed} if \( x \in A \) implies \( 0 \preceq x \in A \).

We denote by \( C(X) \) the set of all closed ideals of \( X \).

A binary relation \( \theta \) on \( X \) is called a \textit{congruence} on \( X \) if

\[
(C1) \quad \theta \text{ is an equivalence relation on } X
\]

\[
(C2) \quad (x, y) \in \theta \text{ implies } (x \circ z, y \circ z) \in \theta \text{ and } (z \circ x, z \circ y) \in \theta \text{ for all } x, y, z \in X
\]

Also a relation \( \theta \) is called \textit{regular} if

\[
(R) \quad (x \circ y, 0) \in \theta \text{ and } (y \circ x, 0) \in \theta \text{ imply } (x, y) \in \theta
\]

By \( \text{Con}_R(X) \) we mean the set of all regular congruences on \( X \). We have the following result ([1, 2]):

**Proposition 1.** Let \( X \) be a BCI-algebra. Then \( C(X) \) and \( \text{Con}_R(X) \) are lattices with respect to set inclusion and they are isomorphic as lattices, that is, \( C(X) \cong \text{Con}_R(X) \).

Let \( X \) be a BCI-algebra. By a fuzzy set of \( X \) we mean a mapping from \( X \) to \([0, 1]\). A fuzzy set \( \tilde{A} \) of \( X \) (i.e. \( \tilde{A}: X \to [0, 1] \)) is called a \textit{fuzzy ideal} if, for all \( x, y, z \in X \),

\[
(i) \quad \tilde{A}(0) \preceq \tilde{A}(x)
\]

\[
(ii) \quad \tilde{A}(x) \preceq \tilde{A}(x \circ y) \wedge \tilde{A}(y)(= \min \{ \tilde{A}(x \circ y), \tilde{A}(y) \})
\]

A fuzzy ideal \( \tilde{A} \) of \( X \) is called \textit{closed} if \( \tilde{A}(0 \circ x) \preceq \tilde{A}(x) \) for every \( x \in X \). It is easy to show the next result. So we omit the proof.

**Lemma 1.** Let \( \tilde{A} \) be a fuzzy ideal of \( X \). Then

(1) If \( x \preceq y \) then \( \tilde{A}(x) \preceq \tilde{A}(y) \)

(2) \( \tilde{A}(x \circ y) \preceq \tilde{A}(x) \wedge \tilde{A}(y) \wedge \tilde{A}(y \circ z) \)

We define a fuzzy congruence on a BCI-algebra \( X \). A binary function \( \tilde{\theta} \) from \( X \times X \) to \([0, 1]\) is called a \textit{fuzzy congruence} on \( X \) if it satisfies the conditions: For all \( x, y, z \in X \),

1. \( \tilde{\theta}(0, 0) = \tilde{\theta}(x, x) \)
2. \( \tilde{\theta}(x, y) = \tilde{\theta}(y, x) \)
3. \( \tilde{\theta}(x, z) \preceq \tilde{\theta}(x, y) \wedge \tilde{\theta}(y, z) \)
4. \( \tilde{\theta}(x \circ u, y \circ u), \tilde{\theta}(u \circ x, u \circ y) \preceq \tilde{\theta}(x, y) \)

**Lemma 2.** If \( \tilde{\theta} \) satisfies the conditions (2), (3), and (4) above, then (1) \( \tilde{\theta}(0, 0) = \tilde{\theta}(x, x) \) if and only if (1) \( \tilde{\theta}(0, 0) \preceq \tilde{\theta}(x, y) \), for all \( x, y \in X \).

**Proof.** Suppose that \( \tilde{\theta}(0, 0) = \tilde{\theta}(x, x) \). Since \( \tilde{\theta} \) satisfies the conditions (2) and (3), we have \( \tilde{\theta}(0, 0) = \tilde{\theta}(x, x) \preceq \tilde{\theta}(x, y) \wedge \tilde{\theta}(y, x) = \tilde{\theta}(x, y) \).

Conversely, it is sufficient to prove \( \tilde{\theta}(0, 0) \leq \tilde{\theta}(x, x) \). From (4), we have \( \tilde{\theta}(0, 0) \leq \tilde{\theta}(x \circ 0, y \circ 0) = \tilde{\theta}(x, x) \).

**Theorem 1.** If \( \tilde{A} \) is a fuzzy ideal of \( X \), then the fuzzy relation \( \tilde{\theta}_A(x, y) \) defined by \( \tilde{\theta}_A(x, y) = \tilde{A}(x \circ y) \wedge \tilde{A}(y \circ x) \) is a fuzzy congruence.
Proof. We only show that \( \tilde{\sigma}_A \) satisfies the conditions (3) and (4). For the case of (3), we have
\[
\tilde{\sigma}_A(x, z) = \tilde{\alpha}(x * z) \land \tilde{\alpha}(z * x) \geq \tilde{\alpha}(x * y) \land \tilde{\alpha}(y * z) \land \tilde{\alpha}(z * y) \land \tilde{\alpha}(y * x) \\
= (\tilde{\alpha}(x * y) \land \tilde{\alpha}(y * x)) \land (\tilde{\alpha}(y * z) \land \tilde{\alpha}(z * y)) \\
= \tilde{\sigma}_A(x, y) \land \tilde{\sigma}_A(y, z)
\]
For the case of (4), it follows from lemma 1 that
\[
\tilde{\sigma}(x * u, y * u) = \tilde{\alpha}((x * u) * (y * u)) \land \tilde{\alpha}((y * u) * (x * u)) \\
\geq \tilde{\alpha}(x * y) \land \tilde{\alpha}(y * x) \\
= \tilde{\sigma}_A(x, y)
\]
It is similar the case of \( \tilde{\sigma}_A(u * x, u * y) \geq \tilde{\sigma}_A(x, y) \).

Conversely,

**Theorem 2.** If \( \tilde{\sigma} \) is a fuzzy congruence, then the function \( \tilde{\alpha}_\theta \) from X to \([0,1]\) defined by \( \tilde{\alpha}_\theta(x) = \tilde{\sigma}(x, 0) \) is a fuzzy ideal of X.

**Proof.** By lemma 2, \( \tilde{\alpha}_\theta(0) = \tilde{\sigma}(0, 0) \geq \tilde{\sigma}(x, 0) = \tilde{\alpha}_\theta(x) \) and \( \tilde{\alpha}_\theta(x) = \tilde{\sigma}(x, 0) \geq \tilde{\sigma}(x, x * y) \land \tilde{\sigma}(x * y, 0) \geq \tilde{\alpha}(x, 0) \land \tilde{\alpha}(y, 0) = \tilde{\alpha}_\theta(y) \land \tilde{\alpha}_\theta(x * y)

Hence \( \tilde{\alpha}_\theta \) is the fuzzy ideal of X.

In general, for every fuzzy ideal \( \tilde{\alpha} \) of X, we have \( \tilde{\alpha}_\theta\tilde{\alpha}(x) = \tilde{\sigma}_\theta(x, 0) = \tilde{\alpha}(x * 0) \land \tilde{\alpha}(0 * x) = \tilde{\alpha}(x) \land \tilde{\alpha}(0 * x) \leq \tilde{\alpha}(x) \)

In particular if X is a BCK-algebra then we have \( \tilde{\alpha}_\theta\tilde{\alpha} = \tilde{\alpha} \) for every fuzzy BCK-ideal \( \tilde{\alpha} \) of X.

**Lemma 3.** If \( \tilde{\alpha} \) is a fuzzy closed ideal, then we have \( \tilde{\sigma}_A(x * y, 0) \land \tilde{\sigma}_A(y * x, 0) = \tilde{\sigma}_A(x, y), \)
that is, \( \tilde{\sigma}_A \) is a fuzzy regular congruence.

**Proof.** Since \( \tilde{\alpha} \) is closed, it follows that \( \tilde{\sigma}_A(x * y, 0) = \tilde{\alpha}(x * y) \land \tilde{\alpha}(0 * (x * y)) = \tilde{\alpha}(x * y) \) and similarly \( \tilde{\sigma}_A(y * x, 0) = \tilde{\alpha}(y * x) \). Hence \( \tilde{\sigma}_A(x * y, 0) \land \tilde{\sigma}_A(y * x, 0) = \tilde{\alpha}(x * y) \land \tilde{\alpha}(y * x) = \tilde{\sigma}_A(x, y) \).

This means that if \( \tilde{\alpha} \in FC(X) \) then \( \tilde{\sigma}_A \in FC_{on}(X) \).

Conversely we have

**Lemma 4.** If \( \tilde{\sigma} \) is a fuzzy regular congruence, then \( \tilde{\alpha}_\theta \) is a fuzzy closed ideal.

**Proof.** It follows from definition that
\[
\tilde{\alpha}_\theta(0 * x) = \tilde{\sigma}(0 * x, 0) \\
= \tilde{\sigma}(0 * x, x * x) \\
\geq \tilde{\sigma}(0, x) = \tilde{\sigma}(x, 0) = \tilde{\alpha}_\theta(x)
\]
Thus \( \tilde{\alpha}_\theta \) is closed.

From the above we can conclude that

1. For any fuzzy closed ideal \( \tilde{\alpha} \) of X, \( \tilde{\alpha} = \tilde{\alpha}_\theta \tilde{\alpha} \).
2. For any fuzzy regular congruence \( \tilde{\sigma} \) of X, \( \tilde{\sigma} = \tilde{\sigma}_A \).
Because, for the case of (1), we have $	ilde{A}(x) = \hat{A}(x, 0) = \hat{A}(x) \wedge \hat{A}(0, x) = \tilde{A}(x)$ and for the case of (2), since $\tilde{A}$ is regular, $\hat{A}(x, y) = \hat{A}(0, x) \wedge \hat{A}(0, y) = \hat{A}(x, y)$.

Thus we get one of main theorems of the paper.

**Theorem 3.** Let $X$ be a $BCI$-algebra. Then we have $FC(X) \cong FC_{on}(X)$

**Proof.** We define a map $\xi$ from $FC(X)$ to $FC_{on}(X)$ by $\xi(A) = \tilde{A}$ for any fuzzy closed ideal $\tilde{A}$ of $X$. It is clear from the above that $\xi$ is an isomorphism. We note that $FC(X)$ and $FC_{on}(X)$ are lattices with set inclusion orders, respectively.

We can also show the next theorem, which is so-called the transfer principle ([3]).

**Theorem 4.** If $\tilde{A}$ is a fuzzy relation on $X$, then $\tilde{A}$ is a fuzzy congruence if and only if for all $\alpha \in [0, 1]$ if $U(\tilde{A} : \alpha) \neq \emptyset$ then $U(\tilde{A} : \alpha)$ is a congruence on $X$, where $U(\tilde{A} : \alpha) = \{(x, y) \in X \times X | \tilde{A}(x, y) \geq \alpha \}$

**Proof.** ($\Rightarrow$) Suppose that $\tilde{A}$ is a fuzzy congruence relation on $X$. Take any $\alpha \in [0, 1]$ such that $U(\tilde{A} : \alpha)$ is not empty. It is sufficient to show that $U(\tilde{A} : \alpha)$ is a congruence on $X$. Since $U(\tilde{A} : \alpha)$ is not empty, there is an element $(u, v) \in X \times X$ such that $(u, v) \in U(\tilde{A} : \alpha)$. This means that $\alpha \leq \tilde{A}(u, v)$. Since $\tilde{A}$ is the congruence, we have $\alpha \leq \tilde{A}(u, v) \leq \tilde{A}(0, 0) = \tilde{A}(x, y)$.

That is, $(x, y) \in U(\tilde{A} : \alpha)$.

Suppose that $(x, y), (y, z) \in U(\tilde{A} : \alpha)$. Since $\alpha \leq \tilde{A}(x, y), \tilde{A}(y, z)$, we have $\alpha \leq \tilde{A}(x, y) \wedge \tilde{A}(y, z) = \tilde{A}(x, z)$.

At last we assume that $(x, y) \in U(\tilde{A} : \alpha)$. Since $\alpha \leq \tilde{A}(x, y) \leq \tilde{A}(x, y) \wedge \tilde{A}(y, z)$, we have $(x * u, y * u, y * y) \in U(\tilde{A} : \alpha)$.

Hence from the above we can conclude that $U(\tilde{A} : \alpha)$ is the congruence on $X$ if it is not empty.

($\Leftarrow$) Conversely, suppose that for all $\alpha \in [0, 1]$ if $U(\tilde{A} : \alpha) \neq \emptyset$ then $U(\tilde{A} : \alpha)$ is a congruence on $X$. We only show that $\tilde{A}(x, z) \geq \tilde{A}(x, y) \wedge \tilde{A}(y, z)$. Take any $\alpha \in [0, 1]$ such that $U(\tilde{A} : \alpha)$ is not empty. Since the relation $U(\tilde{A} : \alpha)$ is transitive, if $(x, y), (y, z) \in U(\tilde{A} : \alpha)$ then $(x, z) \in U(\tilde{A} : \alpha)$. This means that if $\tilde{A}(x, y), \tilde{A}(y, z) \geq \alpha$ then $\tilde{A}(x, z) \geq \alpha$ for any $\alpha$.

Hence we have $\tilde{A}(x, z) \geq \tilde{A}(x, y) \wedge \tilde{A}(y, z)$.

The other cases can be proved similarly.

Now we will define a quotient algebra by a fuzzy ideal. Let $X$ be a $BCI$-algebra and $\tilde{A}$ be a fuzzy ideal of $X$. For any element $x, y \in X$, we define $x \sim_{\tilde{A}} y$ by

$\tilde{A}(x, y) = \tilde{A}(y, x) = \tilde{A}(0)$,

that is, $\theta_{\tilde{A}}(x, y) = \tilde{A}(x)$. Then it is clear that

**Lemma 5.** $\sim_{\tilde{A}}$ is a congruence relation on $X$.

We define $X/\tilde{A} = \{x/\tilde{A} | x \in X\}$ and $x/\tilde{A} = \{y \in X | x \sim_{\tilde{A}} y\}$. We note that these sets are not fuzzy sets but crisp ones. By a fuzzy congruent $BCI$-algebra induced by a fuzzy ideal $\tilde{A}$, we mean a map $\xi$ from $X/\tilde{A}$ to $[0, 1]$ which is defined by $\xi(x/\tilde{A}) = \tilde{A}(x)$. It is obvious that the map $\xi$ is well-defined. Now we consider the property of a crisp set $X/\tilde{A}$.

For any element $x/\tilde{A}, y/\tilde{A} \in X/\tilde{A}$, we define $x/\tilde{A} * y/\tilde{A} = (x * y)/\tilde{A}$. It is easy to show

**Theorem 5.** For any $BCI$-algebra $X$ and fuzzy ideal $\tilde{A}$ of $X$, $X/\tilde{A}$ is a $BCI$-algebra.

**Proof.** We only show that $X/\tilde{A}$ satisfies the condition (4) : $x/\tilde{A} * y/\tilde{A} = y/\tilde{A} * x/\tilde{A} = 0/\tilde{A}$ implies $x/\tilde{A} = y/\tilde{A}$. Suppose that $x/\tilde{A} * y/\tilde{A} = y/\tilde{A} * x/\tilde{A} = 0/\tilde{A}$. Since $x * y \sim_{\tilde{A}} y * x \sim_{\tilde{A}} 0$, it follows from definition that $\tilde{A}(x * y) = \tilde{A}(y * x) = \tilde{A}(0)$ and hence $x \sim_{\tilde{A}} y$. This means that $x/\tilde{A} = y/\tilde{A}$. □
We have some applications. A BCK-algebra $X$ is called commutative when it satisfies $x \ast (x \ast y) = y \ast (y \ast x)$ for all $x, y \in X$. It is well-known that the condition is equivalent to the following: $x \ast y = 0$ implies $x \ast (y \ast (y \ast x)) = 0$. For a fuzzy ideal $\tilde{A}$ of a BCK-algebra $X$ is called fuzzy commutative if it satisfies the condition $\tilde{A}(x \ast (y \ast (y \ast x))) \geq \tilde{A}(x \ast y)$ for all $x, y \in X$. In this case we have the following.

**Theorem 6.** Let $\tilde{A}$ be a fuzzy ideal of a BCK-algebra $X$. Then we have $\tilde{A}$ : fuzzy commutative ideal $\iff X/\tilde{A}$ : commutative BCK-algebra.

**Proof.** $(\Rightarrow)$ It is sufficient to prove that $x/\tilde{A} \ast y/\tilde{A} = 0/\tilde{A}$ implies $x/\tilde{A} \ast (y/\tilde{A}) (x/\tilde{A} \ast y) = 0/\tilde{A}$. That is, $x \ast y \sim 0$ implies $x \ast (y \ast (y \ast x)) \sim 0$. Suppose that $x \ast y \sim 0$. It follows from definition that $\tilde{A}(x \ast y) = \tilde{A}(0)$. Since $\tilde{A}$ is commutative, we have $\tilde{A}(0) = \tilde{A}(x \ast y) \leq \tilde{A}(x \ast (y \ast (y \ast x)))$ and hence $\tilde{A}(x \ast (y \ast (y \ast x))) = \tilde{A}(0)$. On the other hand, since $X$ is the BCK-algebra, it follows that $\tilde{A}(0) \ast (x \ast y) = \tilde{A}(0)$. Hence we get that $x \ast (y \ast (y \ast x)) \sim 0$.

$(\Leftarrow)$ Suppose that $X/\tilde{A}$ is a commutative BCK-algebra. Since $\tilde{A}$ is a fuzzy ideal, we have $\tilde{A}(x \ast (y \ast (y \ast x))) \geq \tilde{A}(x \ast (y \ast (y \ast x))) \ast (y \ast (y \ast x)) = \tilde{A}(x \ast (y \ast (y \ast x))) \wedge \tilde{A}(x \ast y) = \tilde{A}(x \ast (y \ast (y \ast x))) \wedge \tilde{A}(x \ast y)$. That $X/\tilde{A}$ is the commutative BCK-algebra implies $x \ast (y \ast (y \ast x)) / \tilde{A} = y / \tilde{A} \ast (y \ast (y \ast x)) / \tilde{A}$, hence $x \ast y \sim y \ast (y \ast x)$. This means that $\tilde{A}(x \ast y) \ast y \ast (y \ast x) = \tilde{A}(0)$. From the above we get $\tilde{A}(x \ast (y \ast (y \ast x))) \geq \tilde{A}(0) \wedge \tilde{A}(x \ast y) = \tilde{A}(x \ast y)$. Thus $\tilde{A}$ is the fuzzy commutative ideal. $\square$

For the other cases, we can show the similar result. For example, we can show the following for the positive implicative BCK-algebra. A BCK-algebra $X$ is called positive implicative if $(x \ast y) \ast y = 0$ implies $x \ast y = 0$ for all $x, y \in X$. For a fuzzy ideal $\tilde{A}$ of $X$, $\tilde{A}$ is said to be fuzzy positive implicative if $\tilde{A}(x \ast y) \geq \tilde{A}(x \ast y \ast y)$ for all $x, y \in X$. In this case, we can show the next. The proof is clear, so we omit it.

**Theorem 7.** For any BCK-algebra $X$ and a fuzzy ideal $\tilde{A}$ of $X$, $X/\tilde{A}$ is a positive implicative BCK-algebra if and only if $\tilde{A}$ is a fuzzy positive implicative ideal of $X$.

These results are extenstions of the following results respectively: For any BCK-algebra $X$ and ideal $A$ of $X$,

1. $X/A$ : commutative BCK-algebra $\iff A$ : commutative ideal
2. $X/A$ : positive implicative BCK-algebra $\iff A$ : positive implicative ideal

Let $X, Y$ be BCI-algebras and $f$ be a BCI-homomorphism, that is, a map satisfying $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$. If $\tilde{B}$ is a fuzzy ideal of $Y$, then the map $f^{-1}(\tilde{B})$ defined by $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$ for all $x \in X$ is a fuzzy ideal of $X$ ([4]).

In this case we can show the following result which is an extension of homomorphism theorem.

**Theorem 8.** Let $X, Y$ be BCI-algebras, $f$ a BCI-homomorphism, and $\tilde{B}$ a fuzzy ideal of $Y$. Then there is a bijective BCI-homomorphism from $X/ f^{-1}(\tilde{B})$ onto $f(X)/ \tilde{B}$, that is, $X/ f^{-1}(\tilde{B}) \cong f(X)/ \tilde{B}$.

**Proof.** We define a map $h$ from $X/ f^{-1}(\tilde{B})$ to $f(X)/ \tilde{B}$ by $h(x/ f^{-1}(\tilde{B})) = f(x)/ \tilde{B}$ for all $x \in X$. The map $h$ is well-defined. Because, if $x/ f^{-1}(\tilde{B}) = y/ f^{-1}(\tilde{B})$, since $x \sim_{f^{-1}(\tilde{B})} y$, then we have $f^{-1}(\tilde{B})(x \ast y) = f^{-1}(\tilde{B})(y \ast x) = f^{-1}(\tilde{B})(0)$ and hence $\tilde{B}(f(x) \ast f(y)) = \tilde{B}(f(x) \ast f(y)) = \tilde{B}(f(0)) = \tilde{B}(0)$ by definition of $f^{-1}(\tilde{B})$. This means that $f(x) \sim_{\tilde{B}} f(y)$, that is, $f(x)/ \tilde{B} = f(y)/ \tilde{B}$. Hence $h$ is well-defined.
For injectiveness of \( h \), we suppose that \( h(x/f^{-1}(B)) = h(y/f^{-1}(B)) \), that is, \( f(x)/B = f(y)/\bar{B} \). Since \( f(x) \sim_B f(y) \), we have \( \bar{B}(f(x) * f(y)) = \bar{B}(f(y) * f(x)) = \bar{B}(0') \). It follows from definition that \( f^{-1}(B)(x * y) = f^{-1}(B)(y * x) = f^{-1}(B)(0) \) and hence that \( x/f^{-1}(B) = y/f^{-1}(B) \).

It is easy to show that \( h \) is a surjective BCI-homomorphism.

Thus we can conclude that \( X/f^{-1}(\bar{B}) \cong f(X)/\bar{B} \). □

In particular, if \( f \) is surjective then we have \( X/f^{-1}(B) \cong Y/B \).

From the above we can prove that two quotient algebras \( X/f^{-1}(B) \) and \( f(X)/B \) are isomorphic as fuzzy quotient algebras, that is,

**Theorem 9.** For two fuzzy quotient algebras \( \xi \) and \( \eta \) which are defined by
\[
\xi : X/f^{-1}(B) \rightarrow [0,1], \quad \xi(x/f^{-1}(B)) = f^{-1}(B)(x),
\]
\[
\eta : f(X)/B \rightarrow [0,1], \quad \eta(f(x)/B) = B(f(x)), \text{ respectively},
\]
there exists a bijective map \( h \) from \( X/f^{-1}(B) \) to \( f(X)/B \) such that \( \eta \circ h = \xi \).

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