ROUGHNESS OF IDEALS IN BCK-ALGEBRAS

YOUNG BAE JUN

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Abstract. As a generalization of ideals in BCK-algebras, the notion of rough ideals is discussed.

1. Introduction

In 1982, Pawlak introduced the concept of a rough set (see [5]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [6]). Rough set theory is applied to semigroups and groups (see [2, 3]). In this paper, we apply the rough set theory to BCK-algebras, and we introduce the notion of upper/lower rough subalgebras/ideals which is an extended notion of an ideal in a BCK-algebra.

2. Preliminaries

Recall that a BCK-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the following axioms: for every \(x, y, z \in X\),

- \((x * y) * (z * x) = (z * y) * (x * z) = 0\),
- \(x * (x * y) * y = 0\),
- \(x * x = 0\),
- \(0 * x = 0\),
- \(x * y = 0\) and \(y * x = 0\) imply \(x = y\).

For any BCK-algebra \(X\), the relation \(\leq\) defined by \(x \leq y\) if and only if \(x * y = 0\) is a partial order on \(X\). A nonempty subset \(S\) of a BCK-algebra \(X\) is said to be a subalgebra of \(X\) if \(x * y \in S\) whenever \(x, y \in S\). A nonempty subset \(A\) of a BCK-algebra \(X\) is called an ideal of \(X\), denoted by \(A \triangleleft X\), if it satisfies

- \(0 \in A\),
- \(x * y \in A\) and \(y \in A\) imply \(x \in A\) for all \(x, y \in X\).

Note that every ideal of a BCK-algebra \(X\) is a subalgebra of \(X\).

Let \((V, E)\) be a set and \(E\) an equivalence relation on \(V\) and let \(\mathcal{P}(V)\) denote the power set of \(V\). For all \(x \in V\), let \([x]_E\) denote the equivalence class of \(x\) with respect to \(E\). Define the functions \(E_+: \mathcal{P}(V) \rightarrow \mathcal{P}(V)\) as follows: \(\forall S \in \mathcal{P}(V)\),

\[
E_+(S) = \{x \in V \mid [x]_E \subseteq S\} \quad \text{and} \quad E_-(S) = \{x \in V \mid [x]_E \cap S \neq \emptyset\}.
\]

The pair \((V, E)\) is called an approximation space. Let \(S\) be a subset of \(V\). Then \(S\) is said to be definable if \(E_+(S) = E_-(S)\) and rough otherwise. \(E_+(S)\) is called the lower approximation of \(S\) while \(E_-(S)\) is called the upper approximation.

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3. Roughness of ideals

Throughout this section $X$ is a $BCK$-algebra. Let $A$ be an ideal of $X$. Define a relation $\Theta$ on $X$ by

$$(x, y) \in \Theta \text{ if and only if } x \cdot y \in A \text{ and } y \cdot x \in A.$$  

Then $\Theta$ is an equivalence relation on $X$ related to an ideal $A$ of $X$. Moreover $\Theta$ satisfies

$$(x, y) \in \Theta \text{ and } (u, v) \in \Theta \text{ imply } (x \cdot u, y \cdot v) \in \Theta.$$  

Hence $\Theta$ is a congruence relation on $X$. Let $A_x$ denote the equivalence class of $x$ with respect to the equivalence relation $\Theta$ related to the ideal $A$ of $X$, and $X/A$ denote the collection of all equivalence classes, that is, $X/A = \{ A_x \mid x \in X \}$. Then $A_0 = A$. If $A_x \cdot A_y$ is defined as the class containing $x \cdot y$, that is, $A_x \cdot A_y = A_{x \cdot y}$, then $(X/A, \cdot, A_0)$ is a $BCK$-algebra (see [4]). Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. For any nonempty subset $S$ of $X$, the lower and upper approximation of $S$ by $\Theta$ are denoted by $\Theta(A; S)$ and $\overline{\Theta}(A; S)$ respectively, that is,

$$\Theta(A; S) = \{ x \in X \mid A_x \subseteq S \} \quad \text{and} \quad \overline{\Theta}(A; S) = \{ x \in X \mid A_x \cap S \neq \emptyset \}.$$  

If $A = S$, then $\Theta(A; S)$ and $\overline{\Theta}(A; S)$ are denoted by $\Theta(A)$ and $\overline{\Theta}(A)$, respectively.

**Example 3.1.** (1) Let $X = \{0, 1, 2, 3\}$ be a $BCK$-algebra with the Cayley table as follows (see [4]).

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Let $A = \{0, 1\} \triangleleft X$ and let $\Theta$ be an equivalence relation on $X$ related to $A$. Then $A_0 = A_1 = A$, $A_2 = \{2\}$, and $A_3 = \{3\}$. Hence $\Theta(A; \{0, 2\}) = \{2\} = \Theta(A; \{2\})$, $\Theta(A; \{0\}) = \emptyset$, $\Theta(A; \{0, 3\}) = \{3\}$, $\Theta(A; \{0, 1, 3\}) = \{0, 1, 3\} \triangleleft X$, $\Theta(A; \{0, 2\}) = \{0, 1, 2\} \triangleleft X$, and $\overline{\Theta}(A; \{0, 3\}) = \{0, 1, 3\} \triangleleft X$.

(2) Let $X = \{0, 1, 2, 3, 4\}$ be a $BCK$-algebra with the Cayley table as follows (see [4]).

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Consider $A = \{0, 1, 2\} \triangleleft X$ and let $\Theta$ be an equivalence relation on $X$ related to $A$. Then the equivalence classes are as follows: $A_0 = A_1 = A_2 = A$, $A_3 = \{3\}$, and $A_4 = \{4\}$. Thus $\Theta(A; \{0, 1, 3\}) = \{3\}$, $\Theta(A; \{0, 2, 4\}) = \{4\}$, $\Theta(A; \{0, 1, 2, 3\}) = \{0, 1, 2, 3\} \triangleleft X$, $\Theta(A; \{0, 1, 2, 4\}) = \{0, 1, 2, 4\} \triangleleft X$, $\Theta(A; \{0, 2\}) = \{0, 1, 2\} \triangleleft X$, and $\overline{\Theta}(A; \{0, 3\}) = \{0, 1, 2, 3\} \triangleleft X$.

In Example 3.1, we know that there exists a non-ideal $U$ of $X$ such that $\Theta(A; U) \triangleleft X$; and there exists a non-ideal $V$ of $X$ such that $\overline{\Theta}(A; V) \triangleleft X$, where $\Theta$ is an equivalence relation on $X$ related to $A \triangleleft X$.

**Proposition 3.2.** Let $\Theta$ and $\Psi$ be equivalence relations on $X$ related to ideals $A$ and $B$ of $X$, respectively. If $A \subseteq B$, then $\Theta \subseteq \Psi$.

**Proof.** If $(x, y) \in \Theta$, then $x \cdot y \in A \subseteq B$ and $y \cdot x \in A \subseteq B$. Hence $(x, y) \in \Psi$, and so $\Theta \subseteq \Psi$. \(\square\)
Proposition 3.3. Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. Then

1. $\Theta(A; S) \subseteq S \subseteq \overline{\Theta(A; S)}$, $\forall S \in \mathcal{P}(X)$.
2. $\Theta(A; S \cup T) = \Theta(A; S) \cup \Theta(A; T)$, $\forall S, T \in \mathcal{P}(X)$.
3. $\Theta(A; S \cap T) = \Theta(A; S) \cap \Theta(A; T)$, $\forall S, T \in \mathcal{P}(X)$.
4. $\Theta(A; S) \subseteq \Theta(A; T)$ and $\Theta(A; S) \subseteq \Theta(A; T)$. $\forall S, T \in \mathcal{P}(X)$.
5. $\Theta(A; S \cup T) \supseteq \Theta(A; S) \cup \Theta(A; T)$, $\forall S, T \in \mathcal{P}(X)$.
6. $\Theta(A; S \cap T) \subseteq \Theta(A; S) \cap \Theta(A; T)$, $\forall S, T \in \mathcal{P}(X)$.
7. If $\Psi$ is an equivalence relation on $X$ related to an ideal $B$ of $X$ and if $A \subseteq B$, then $\Theta(A; S) \subseteq \Theta(B; S)$, $\forall S \in \mathcal{P}(X)$.

Proof. (1) is straightforward.

(2) For any subsets $S$ and $T$ of $X$, we have
\[
x \in \Theta(A; S \cup T) \Leftrightarrow A_x \cap (S \cup T) \neq \emptyset
\]
\[
\Leftrightarrow (A_x \cap S) \cup (A_x \cap T) \neq \emptyset
\]
\[
\Leftrightarrow A_x \cap S \neq \emptyset \text{ or } A_x \cap T \neq \emptyset
\]
\[
\Leftrightarrow x \in \Theta(A; S) \text{ or } x \in \Theta(A; T)
\]
\[
\Rightarrow x \in \Theta(A; S \cup T) = \Theta(A; S) \cup \Theta(A; T),
\]
and hence $\Theta(A; S \cup T) = \Theta(A; S) \cup \Theta(A; T)$.

(3) For any subsets $S$ and $T$ of $X$, we have
\[
x \in \Theta(A; S \cap T) \Leftrightarrow A_x \subseteq S \cap T
\]
\[
\Leftrightarrow A_x \subseteq S \text{ and } A_x \subseteq T
\]
\[
\Leftrightarrow x \in \Theta(A; S) \text{ and } x \in \Theta(A; T)
\]
\[
\Rightarrow x \in \Theta(A; S \cap T) = \Theta(A; S) \cap \Theta(A; T).
\]

Hence $\Theta(A; S \cap T) = \Theta(A; S) \cap \Theta(A; T)$.

(4) Let $S, T \in \mathcal{P}(X)$ be such that $S \subseteq T$. Then $S \cap T = S$ and $S \cup T = T$. It follows from (3) and (2) that
\[
\Theta(A; S) = \Theta(A; S \cap T) = \Theta(A; S) \cap \Theta(A; T)
\]
and
\[
\Theta(A; T) = \Theta(A; S \cup T) = \Theta(A; S) \cup \Theta(A; T),
\]
which yield $\Theta(A; S) \subseteq \Theta(A; T)$ and $\Theta(A; S) \subseteq \Theta(A; T)$, respectively.

(5) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, it follows from (4) that
\[
\Theta(A; S) \subseteq \Theta(A; S \cup T) \text{ and } \Theta(A; T) \subseteq \Theta(A; S \cup T).
\]

Thus $\Theta(A; S) \cup \Theta(A; T) \subseteq \Theta(A; S \cup T)$.

(6) Since $S \cap T \subseteq S, T$, it follows from (4) that
\[
\Theta(A; S \cap T) \subseteq \Theta(A; S \cap T) \text{ and } \Theta(A; S \cap T) \subseteq \Theta(A; T).
\]

(7) If $x \in \Theta(A; S)$, then $A_x \cap S \neq \emptyset$, and so there exists $a \in S$ such that $a \in A_x$. Hence $(a, x) \in \Theta$, that is, $a \ast x \in A$ and $x \ast a \in A$. Since $A \subseteq B$, it follows that $a \ast x \in B$ and $x \ast a \in B$ so that $(a, x) \in \Psi$, that is, $a \in B_x$. Therefore $a \in B_x \cap S$, which means $x \in \Psi(B; S)$. This completes the proof.

Proposition 3.4. Let $\Theta$ be an equivalence relation on $X$ related to any ideal $A$ of $X$. Then $\Theta(A; X) = X = \Theta(A; X)$, that is, $X$ is definable.

Proof. It is straightforward.
Proposition 3.5. Let $\Theta$ be an equivalence relation on $X$ related to the trivial ideal $\{0\}$ of $X$. Then $\Theta(0); S) = S = \Theta(0); S$ for every nonempty subset $S$ of $X$, that is, every nonempty subset of $X$ is definable.

Proof. Note that $\{0\}_x = \{x\}$ for all $x \in X$, since if $a \in \{0\}_x$ then $(a, x) \in \Theta$ and hence $a \ast x = 0$ and $x \ast a = 0$. It follows that $a = x$. Hence

$$\Theta(0); S) = \{x \in X \mid \{0\}_x \subseteq S\} = S$$

and

$$\Theta(0); S) = \{x \in X \mid \{0\}_x \cap S \neq \emptyset\} = S.$$

This completes the proof. \hfill \Box

Remark 3.6. Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. If $B$ is an ideal of $X$ such that $A \neq B$, then $\Theta(A; B)$ is not an ideal of $X$ in general. For, consider a $BC$-algebra $X$ in Example 3.1(2) and an equivalence relation $\Theta$ on $X$ related to the ideal $A = \{0, 1, 2\}$. If we take an ideal $B = \{0, 1, 3\}$ of $X$, then $A \neq B$ and $\Theta(A; B) = \{3\}$ which is not an ideal of $X$.

Definition 3.7. Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. A nonempty subset $S$ of $X$ is called an upper (resp. a lower) rough subalgebra/ideal of $X$ if the upper (resp. nonempty lower) approximation of $S$ is a subalgebra/ideal of $X$. If $S$ is both an upper and a lower rough subalgebra/ideal of $X$, we say that $S$ is a rough subalgebra/ideal of $X$.

Theorem 3.8. Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. Then every subalgebra $S$ of $X$ is a rough subalgebra of $X$.

Proof. Let $x, y \in \Theta(A; S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since $S$ is a subalgebra of $X$, it follows that $A_{x+y} = A_x \ast A_y \subseteq S$ so that $x \ast y \in \Theta(A; S)$. Hence $\Theta(A; S)$ is a subalgebra of $X$. Now if $x, y \in \Theta(A; S)$, then $A_x \cap S \neq \emptyset$ and $A_y \cap S \neq \emptyset$, and so there exist $a, b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a, x) \in \Theta$ and $(b, y) \in \Theta$. Since $\Theta$ is a congruence relation on $X$, we have $(a \ast b, x \ast y) \in \Theta$. Hence $a \ast b \in A_{x+y}$. Since $S$ is a subalgebra of $X$, we get $a \ast b \in S$, and therefore $a \ast b \in A_{x+y} \cap S$, that is, $A_{x+y} \cap S \neq \emptyset$. This shows that $x \ast y \in \Theta(A; S)$, and consequently $\Theta(A; S)$ is a subalgebra of $X$. This completes the proof. \hfill \Box

Corollary 3.9. Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. Then $\Theta(A) (\neq \emptyset)$ and $\Theta(A)$ are subalgebras of $X$, that is, $A$ is a rough subalgebra of $X$.

Proof. It is straightforward. \hfill \Box

Theorem 3.10. Let $\Theta$ be an equivalence relation on $X$ related to an ideal $A$ of $X$. If $U$ is an ideal of $X$ containing $A$, then

1. $\Theta(A; U) (\neq \emptyset)$ is an ideal of $X$, that is, $U$ is a lower rough ideal of $X$.
2. $\Theta(A; U)$ is an ideal of $X$, that is, $U$ is an upper rough ideal of $X$.

Proof. Let $U$ be an ideal of $X$ containing $A$. Let $x \in A_0$. Then $x \in A \subseteq U$, and so $A_0 \subseteq U$. Hence $0 \in \Theta(A; U)$. Let $x, y \in X$ be such that $y \in \Theta(A; U)$ and $x \ast y \in \Theta(A; U).$ Then $A_y \subseteq U$ and $A_x \ast A_y = A_{x+y} \subseteq U$. Let $a \in A_x$ and $b \in A_y$. Then $(a, x) \in \Theta$ and $(b, y) \in \Theta$, which implies $(a \ast b, x \ast y) \in \Theta$. Hence $a \ast b \in A_{x+y} \subseteq U$. Since $b \in A_y \subseteq U$ and $U$ is an ideal, it follows that $a \in U$, so that $A_x \subseteq U$. Thus $x \in \Theta(A; U)$. This shows that $\Theta(A; U)$ is an ideal of $X$, that is, $U$ is a lower rough ideal of $X$. Now, obviously $0 \in \Theta(A; U)$. Let $x, y \in X$ be such that $y \in \Theta(A; U)$ and $x \ast y \in \Theta(A; U)$. Then $A_y \cap U \neq \emptyset$ and $A_{x+y} \cap U \neq \emptyset$, and so there exist $a, b \in U$ such that $a \in A_y$ and $b \in A_{x+y}$. Hence $(a, y) \in \Theta$ and $(b, x \ast y) \in \Theta$, which implies $(a \ast b, x \ast y) \in \Theta$. Hence $a \ast b \in A_{x+y} \subseteq U$. This shows that $\Theta(A; U)$ is an ideal of $X$, that is, $U$ is a lower rough ideal of $X$. Now, obviously $0 \in \Theta(A; U)$. Let $x, y \in X$ be such that $y \in \Theta(A; U)$ and $x \ast y \in \Theta(A; U)$. Then $A_y \cap U \neq \emptyset$ and $A_{x+y} \cap U \neq \emptyset$, and so there exist $a, b \in U$ such that $a \in A_y$ and $b \in A_{x+y}$. Hence $(a, y) \in \Theta$ and $(b, x \ast y) \in \Theta$, which implies $(a \ast b, x \ast y) \in \Theta$. Hence $a \ast b \in A_{x+y} \subseteq U$. This shows that $\Theta(A; U)$ is an ideal of $X$, that is, $U$ is a lower rough ideal of $X$. Now, obviously $0 \in \Theta(A; U)$. Let $x, y \in X$ be such that $y \in \Theta(A; U)$ and $x \ast y \in \Theta(A; U)$. Then $A_y \cap U \neq \emptyset$ and $A_{x+y} \cap U \neq \emptyset$, and so there exist $a, b \in U$ such that $a \in A_y$ and $b \in A_{x+y}$. Hence $(a, y) \in \Theta$ and $(b, x \ast y) \in \Theta$, which implies $(a \ast b, x \ast y) \in \Theta$. Hence $a \ast b \in A_{x+y} \subseteq U$. This shows that $\Theta(A; U)$ is an ideal of $X
which implies \( y \ast a \in A \subseteq U \) and \((x \ast y) \ast b \in A \subseteq U \). Since \( a, b \in U \) and \( U \) is an ideal, we get \( y \in U \) and \((x \ast y) \in U \); hence \( x \in U \). Note that \( x \in A_x \), thus \( x \in A_x \cap U \), that is, \( A_x \cap U \neq \emptyset \). Therefore \( x \in \overline{\Theta}(A; U) \), and consequently \( U \) is an upper rough ideal of \( X \). □

**Corollary 3.11.** Let \( \Theta \) be an equivalence relation on \( X \) related to an ideal \( A \) of \( X \). Then \( \Theta(A) \neq \emptyset \) and \( \overline{\Theta}(A) \) are ideals of \( X \), that is, \( A \) is a rough ideal of \( X \).

Theorem 3.10 shows that the notion of an upper (resp. a lower) rough ideal is an extended notion of an ideal in a \( BCK \)-algebra. The following example provides that the converse of Theorem 3.10 may not be true.

**Example 3.12.** (1) Let \( X = \{0,1,2,3,4\} \) be a \( BCK \)-algebra with the Cayley table as follows (see [4]).

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Consider \( A = \{0,2\} \leq X \) and a subset \( U = \{0,2,3\} \) of \( X \) which is not an ideal of \( X \). Let \( \Theta \) be an equivalence relation on \( X \) related to \( A \). Then \( A_0 = A_2 = A, A_1 = \{1\}, A_3 = \{3\}, \) and \( A_4 = \{4\} \). Hence \( \overline{\Theta}(A; U) = \{0,2\} \leq X \).

(2) Let \( X = \{0,1,2,3,4\} \) be a \( BCK \)-algebra with the Cayley table as follows (see [4]).

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Consider \( B = \{0,2\} \leq X \) and let \( \Psi \) be an equivalence relation on \( X \) related to \( B \). Then all equivalence classes are \( B_0 = B_2 = \{0,2\}, B_1 = \{1\}, B_3 = \{3\} \) and \( B_4 = \{4\} \). Note that \( V = \{0,1,4\} \) is not an ideal of \( X \), but \( \overline{\Psi}(B; V) = \{0,1,2,4\} \leq X \).

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**References**


Department of Mathematics Education, Gyeongsang National University, Chinju (Jinju) 660-701, Korea. E-mail: ybjun@mongae.gsmu.ac.kr