APPROXIMATION OF COMMON FIXED POINTS OF A FAMILY OF FINITE NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we deal with an iterative scheme for finding common fixed points of a family of finite nonexpansive mappings in a Banach space. We extend a result of Bauschke in a Hilbert space to a Banach space and a result of Shiōjī and Takahashi for a single nonexpansive mapping to a family of finite mappings.

1. Introduction

Let $C$ be a closed convex subset of a Banach space $E$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. We deal with the iterative process: $x_0 \in C$ and

$$x_{n+1} = \alpha_{n+1} x_0 + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad n = 0, 1, 2, \ldots,$$

where $T_1, T_2, \ldots, T_r$ are nonexpansive mappings of $C$ into itself, $T_{n+r} = T_n$ and $0 < \alpha_{n+1} < 1$. In 1992, Wittmann [11] dealt with the iterative process for $r = 1$ in a Hilbert space and obtained a strong convergence theorem for finding a fixed point of the mapping; see originally Halpern [3]. Shiōjī and Takahashi [7] extended the result of Wittmann to a Banach space. On the other hand, in 1996, Bauschke [1] dealt with the iterative process for finding a common fixed point of finite nonexpansive mappings in a Hilbert space; see also Lions [5].

The objective of this paper is to obtain a strong convergence theorem which unifies the results by Bauschke [1] and Shiōjī and Takahashi [7]. Then, using this result, we consider the problem of image recovery in a Banach space setting.

2. Preliminaries

Throughout this paper, all vector spaces are real. Let $E$ be a Banach space and let $E^*$ be its dual. The value of $f \in E^*$ at $x \in E$ will be denote by $\langle x, f \rangle$. We denote by $I$ the identity mapping on $E$ and by $J$ the duality mapping of $E$ into $2^{E^*}$, i.e., $Jx = \{ f \in E^* \mid \langle x, f \rangle = \|x\|^2 = \|f\|^2, \quad x \in E \}. \quad \text{Let } U = \{ x \in E \mid \|x\| = 1 \}. \quad \text{A Banach space } E \text{ is said to be strictly convex if } \|x + y\|/2 < 1 \text{ for } x, y \in U. \quad \text{In a strictly convex Banach space, we have that if } \|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\| \text{ for } x, y \in E \text{ and } 0 < \lambda < 1, \text{ then } x = y. \quad \text{For every } \epsilon \text{ with } 0 \leq \epsilon \leq 2, \text{ we define the modulus } \delta(\epsilon) \text{ of convexity of } E \text{ by}

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} \mid \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}.$$ 

A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A Banach space $E$ is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = 0.$$ 

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exists for each $x, y \in U$. The norm of $E$ is said to be uniformly Gâteaux differentiable if, for each $y \in U$, the above limit exists uniformly for $x \in U$. It is known that if $E$ is smooth then the duality mapping $J$ is single-valued. Moreover it is known that if the norm of $E$ is uniformly Gâteaux differentiable then the duality mapping is norm to weak-star, uniformly continuous on each bounded subset of $E$.

Let $C$ be a closed convex subset of $E$ and let $F$ be a subset of $C$. A mapping $P$ of $C$ onto $F$ is said to be sunny if $P(\alpha x + t(x - P\alpha x)) = P\alpha x$ for each $x \in C$ and $t \geq 0$ with $P\alpha x + t(x - P\alpha x) \in C$. A subset $F$ of $C$ is said to be a nonexpansive retract of $C$ if there exists a nonexpansive retraction of $C$ onto $F$. We know the following [6] (see also [9]):

**Theorem 2.1.** Let $E$ be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let $C$ be a closed convex subset of $E$ and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. Let $x_0 \in C$ and let $z_n$ be a unique element of $C$ which satisfies $z_n = t x_0 + (1 - t)T z_n$ and $0 < t < 1$. Then $\{z_n\}$ converges strongly to a fixed point of $T$ as $n \to 0$. Moreover $\langle z_n - y, J(y - z) \rangle \geq 0$ for all $z \in F(T)$. Further if $x_0 = \lim_{n \to 0} z_n$ for each $x_0 \in C$, then $P$ is a sunny nonexpansive retraction of $C$ onto $F(T)$.

We use this result in the proof of Theorem 3.2.

Let $\mu$ be a continuous linear functional on $l^\infty$ and let $(a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. We call $\mu$ a Banach limit when $\mu$ satisfies $|\mu| = \mu_n(1) = 1$ and $\mu_n(a_{n+1})$ for all $(a_0, a_1, \ldots) \in l^\infty$. Let $\mu$ be a Banach limit. Then $\liminf_{n \to \infty} a_n \leq \mu(a) \leq \limsup_{n \to \infty} a_n$ for each $(a_0, a_1, \ldots) \in l^\infty$. Specially, if $a_n \to p$, then $\mu(a) = p$; see [8] for more details.

### 3. Strong Convergence Theorem

In this section, we give our main theorem. Before giving it, we prove the following:

**Proposition 3.1.** Let $E$ be a strictly convex Banach space and let $C$ be a closed convex subset of $E$. Let $S_1, S_2, \ldots, S_r$ be nonexpansive mappings of $C$ into itself such that the set of common fixed points of $S_1, S_2, \ldots, S_r$ is nonempty. Let $T_1, T_2, \ldots, T_r$ be mappings of $C$ into itself given by $T_i = (1 - \lambda_i)I + \lambda_i S_i$ where $0 < \lambda_i < 1$ for each $i = 1, 2, \ldots, r$. Then $\bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r F(S_i)$ and

$$\bigcap_{i=1}^r F(T_i) = F(T_rT_{r-1}\cdots T_1) = F(T_1T_r\cdots T_2) = \cdots = F(T_{r-1}\cdots T_1T_r).$$

**Proof.** For simplicity, we give the proof of Proposition for $r = 2$. It is clear that $F(S_1) \cap F(S_2) = F(T_1) \cap F(T_2).$ Let $T_1 = (1 - \lambda_1)I + \lambda_1 S_1$ and $T_2 = (1 - \lambda_2)I + \lambda_2 S_2$. Since $\|z - w\| = \|\lambda_2 S_2[(1 - \lambda_1)z + \lambda_1 S_1z] + (1 - \lambda_2)\|[(1 - \lambda_1)z + \lambda_1 S_1z] - w\|$$
\leq \lambda_2 \|S_2[(1 - \lambda_1)z + \lambda_1 S_1z] - w\| + (1 - \lambda_2)\|[1 - \lambda_1)z + \lambda_1 S_1z - w\|
\leq (1 - \lambda_1)\|z - w\| + \lambda_1 \|S_1z - w\|
\leq \|z - w\|,
we have

$$\|z - w\| = \|\lambda_2 S_2[(1 - \lambda_1)z + \lambda_1 S_1z] + (1 - \lambda_2)\|[(1 - \lambda_1)z + \lambda_1 S_1z] - w\|$$
$$= \|S_2[(1 - \lambda_1)z + \lambda_1 S_1z] - w\|$$
$$= \|(1 - \lambda_1)z + \lambda_1 S_1z - w\|$$
$$= \|S_1z - w\|.$$
By the strict convexity of $E$, we have $S_2[(1 - \lambda_1)z + \lambda_1 S_1 z] - w = (1 - \lambda_1)z + \lambda_1 S_1 z - w$ and $z - w = S_1 z - w$. Therefore we obtain that $z = S_1 z = S_2 z$. This completes the proof.

Now we state our main result.

**Theorem 3.2.** Let $E$ be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $C$ be a closed convex subset of $E$. Let $T_1, T_2, \ldots, T_r$ be nonexpansive mappings of $C$ into itself such that the set $F = \bigcap_{i=1}^r F(T_i)$ of common fixed points of $T_1, T_2, \ldots, T_r$ is nonempty and

$$
\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_2 \cdots T_r) = \cdots = F(T_{r-1} \cdots T_1 T_r).
$$

Let $\{\alpha_n\} \subset (0, 1)$ be a sequence which satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_{n+r} - \alpha_n| < \infty$. Define a sequence $\{x_n\}$ by $x_0 \in C$ and

$$
x_{n+1} = \alpha_{n+1} x_0 + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad n = 0, 1, 2, \ldots,
$$

where $T_{n+r} = T_n$. Then $\{x_n\}$ converges strongly to a point $z$ in $F$. Further, if $P z_0 = \lim_{n \to \infty} x_n$ for each $z_0 \in C$, then $P$ is a sunny nonexpansive retraction of $C$ onto $F$.

**Proof.** We first show that $\lim_{n \to \infty} \|x_{n+r} - x_n\| = 0$. Since $F \neq \emptyset$, $\{x_n\}$ and $\{T_{n+1} x_n\}$ are bounded. Then there exists $L > 0$ such that $\|x_{n+r} - x_n\| \leq L |\alpha_{n+r} - \alpha_n| + (1 - \alpha_{n+r}) \|x_{n+r-1} - x_{n-1}\|$ for each $n = 1, 2, \ldots$. Therefore we have

$$
\|x_{n+r} - x_n\| \leq L \sum_{k=m+1}^n |\alpha_{k+r} - \alpha_k| + \|x_{n+r} - x_m\| \prod_{k=m+1}^n (1 - \alpha_{k+r})
$$

for all $n \geq m$. This yields $\limsup_{n \to \infty} \|x_{n+r} - x_n\| \leq L \sum_{k=m+1}^\infty |\alpha_{k+r} - \alpha_k| \text{ by } \sum_{k=1}^\infty \alpha_k = \infty$. Hence by $\sum_{k=1}^\infty |\alpha_{k+r} - \alpha_k| < \infty$, we obtain $\lim_{n \to \infty} \|x_{n+r} - x_n\| = 0$. Next we prove $\lim_{n \to \infty} \|x_n - T_{n+r} \cdots T_{n+1} x_n\| = 0$. It suffices to show that $\lim_{n \to \infty} \|x_{n+r} - T_{n+r} \cdots T_{n+1} x_n\| = 0$. Since $x_{n+r} - T_{n+r} x_{n+r} = \alpha_{n+r} (x_0 - T_{n+r} x_{n+r-1})$ and $\lim_{n \to \infty} \alpha_n = 0$, we have $x_{n+r} - T_{n+r} x_{n+r-1} \to 0$. From

$$
\|x_{n+r} - T_{n+r} T_{n+r-1} x_{n+r-1}\| \leq \|x_{n+r} - T_{n+r} x_{n+r-1}\|
$$

we also have $x_{n+r} - T_{n+r} T_{n+r-1} x_{n+r-1} \to 0$. Similarly, we obtain the conclusion.

Let $z^n$ be a unique element of $C$ which satisfies $0 < t < 1$ and $z^n = t x_0 + (1 - t) T_{n+r} \cdots T_{n+1} z^n$. From $F(T_{n+r} \cdots T_{n+1} z^n) = F$ and Theorem 2.1, $\{z^n\}$ converges strongly to $P x_0$ of as $t \downarrow 0$, where $P$ is a sunny nonexpansive retraction of $C$ onto $F$. We show $\lim_{n \to \infty} \|x_n - P x_0, J(x_n - P x_0)\| \leq 0$. Let $A = \lim_{n \to \infty} \|x_n - P x_0, J(x_n - P x_0)\|$. Then there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that $A = \lim_{i \to \infty} \|x_i - P x_0, J(x_i - P x_0)\|$. We assume that $n_i \equiv k \pmod{r}$ for some $k \in \{1, 2, \ldots, r\}$. Since

$$
\|x_{n_i} - T_{n+r} \cdots T_{n+1} z_i\|^2 \leq \|x_{n_i} - T_{n_i+r} \cdots T_{n_i+1} x_{n_i}\|^2 + \|T_{n+r} \cdots T_{n+1} x_{n_i} - T_{n+r} \cdots T_{n+1} z_i\|^2
$$

we also have $x_{n+r} - T_{n+r} T_{n+r-1} x_{n+r-1} \to 0$. Similarly, we obtain the conclusion.
and \(\|x_{n_i} - T_{n_i + r} \cdots T_{n_{i+1}} x_{n_i}\| \to 0\), we have
\[
\mu_i \|x_{n_i} - T_{n_i + r} \cdots T_{n_{i+1}} z_i^k\|^2 \leq \mu_i \|x_{n_i} - z_i^k\|^2.
\]
From \((1-t) (x_{n_i} - T_{n_i + r} \cdots T_{n_{i+1}} z_i^k) = (x_{n_i} - z_i^k) - t(x_{n_i} - x_0)\), we have
\[
(1-t)^2 \|x_{n_i} - T_{n_i + r} \cdots T_{n_{i+1}} z_i^k\|^2 \geq \|x_{n_i} - z_i^k\|^2 - 2t \langle x_{n_i} - x_0, J(x_{n_i} - z_i^k) \rangle
= (1-2t) \|x_{n_i} - z_i^k\|^2 + 2t \langle x_0 - z_i^k, J(x_{n_i} - z_i^k) \rangle
\]
for each \(i\). These inequalities yield
\[
\mu_i \langle x_0 - z_i^k, J(x_{n_i} - z_i^k) \rangle \leq \frac{t}{2} \mu_i \|x_{n_i} - z_i^k\|^2.
\]
As \(t\) tends to 0, we obtain
\[
\mu_i \langle x_0 - P x_0, J(x_{n_i} - P x_0) \rangle \leq 0
\]
because \(E\) has a uniformly Gâteaux differentiable norm. Hence we have
\[
A = \lim_{i \to \infty} \langle x_0 - P x_0, J(x_{n_i} - P x_0) \rangle = \mu_i \langle x_0 - P x_0, J(x_{n_i} - P x_0) \rangle \leq 0.
\]

Now we can prove \(\{x_n\}\) converges strongly to \(P x_0\). Let \(\varepsilon > 0\). By \(\limsup_{n \to \infty} \langle x_0 - P x_0, J(x_{n} - P x_0) \rangle \leq 0\), there exists a positive integer \(n_0\) such that
\[
\langle x_0 - P x_0, J(x_{n} - P x_0) \rangle \leq \frac{\varepsilon}{2}
\]
for all \(n \geq n_0\). Since \((1 - \alpha_n) (T_{n} x_{n-1} - P x_0) = (x_n - P x_0) - \alpha_n (x_0 - P x_0)\), we have
\[
(1 - \alpha_n)^2 \|T_{n} x_{n-1} - P x_0\|^2 \geq \|x_n - P x_0\|^2 - 2 \alpha_n \langle x_0 - P x_0, J(x_{n} - P x_0) \rangle
\geq \|x_n - P x_0\|^2 - \alpha_n \varepsilon
\]
for all \(n \geq n_0\). This yields
\[
\|x_n - P x_0\|^2 \leq (1 - \alpha_n) \|T_{n} x_{n-1} - P x_0\|^2 + \alpha_n \varepsilon
\]
for all \(n \geq n_0\). Then we have
\[
\|x_n - P x_0\|^2 \leq \left( \prod_{k=n_0+1}^{n} (1 - \alpha_k) \right) \|x_{n_0} - P x_0\|^2 + \left\{ 1 - \prod_{k=n_0+1}^{n} (1 - \alpha_k) \right\} \varepsilon.
\]
Hence by \(\sum_{k=1}^{\infty} \alpha_k = \infty\), we obtain \(\limsup_{n \to \infty} \|x_n - P x_0\|^2 \leq \varepsilon\). Since \(\varepsilon\) is arbitrary positive real number, \(\{x_n\}\) converges strongly to \(P x_0\). \(\square\)

**Remark.** When \(E\) is a Hilbert space, Theorem 3.2 is the result of Bauschke.

## 4. Applications to the Feasibility Problem

In this section, we deal with strong convergence theorems which are connected with the feasibility problem. Using a nonlinear ergodic theorem, Crambez [2] considered the feasibility problem in a Hilbert space setting. Let \(H\) be a Hilbert space, let \(C_1, C_2, \ldots, C_r\) be closed convex subsets of \(H\) and let \(I\) be the identity operator on \(H\). Then the feasibility problem in a Hilbert space setting may be stated as follows: The original (unknown) image \(z\) is known a priori to belong to the intersection \(C_0\) of \(r\) well-defined sets \(C_1, C_2, \ldots, C_r\) in a Hilbert space; given only the metric projections \(P_{C_i}\) of \(H\) onto \(C_i\) \((i = 1, 2, \ldots, r)\), recover \(z\) by an iterative scheme. Crambez [2] proved the following: Let \(T = c_0 I + \sum_{i=1}^{r} \alpha_i T_i\) with \(T_i = (1 - \lambda_i) I + \lambda_i P_{C_i}\) for all \(i\), \(0 < \lambda_i < 2\), \(\alpha_i \geq 0\) for \(i = 0, 1, 2, \ldots, r\), \(\sum_{i=1}^{r} \alpha_i = 1\) where \(C_0 = \bigcap_{i=1}^{r} C_i\) is nonempty. Then starting from an arbitrary element \(x\) of \(H\), the sequence \(\{T^n x\}\) converges weakly to an element of \(C_0\). Later, Kitahara and Takahashi [4]

Using Theorem and Proposition in Section 3, we obtain two results.

**Corollary 4.1.** Let $E$ be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $C$ be a closed convex subset of $E$. Let $S_1, S_2, \ldots, S_r$ be nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. Define a family of finite nonexpansive mappings $\{T_i, T_1, \ldots, T_r\}$ by $T_i = (1 - \lambda_i)I + \lambda_i S_i$ for $i = 1, 2, \ldots, r$, $0 < \lambda_i < 1$. Let $\{\alpha_n\} \subset (0, 1)$ be a sequence which satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$. Define a sequence $\{x_n\}$ by $x_0 \in C$ and

$$x_{n+1} = \alpha_{n+1}x_0 + (1 - \alpha_{n+1})T_{n+1}x_n \quad \text{for all } n = 0, 1, 2, \ldots,$$

where $T_{n+r} = T_n$. Then $\{x_n\}$ converges strongly to a common fixed point $z$ of $S_1, S_2, \ldots, S_r$.

Further, if $Pz_0 = \lim_{n \to \infty} x_n$ for each $x_0 \in C$, then $P$ is a sunny nonexpansive retraction of $C$ onto $\bigcap_{i=1}^{r} F(S_i)$.

**Proof.** By Proposition 3.1 and Theorem 3.2, we have $\{x_n\}$ converges to a common fixed point of $S_1, \ldots, S_r$. □

**Corollary 4.2.** Let $E$ be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $C$ be a closed convex subset of $E$. Let $C_1, C_2, \ldots, C_r$ be nonexpansive retractions of $C$ such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Define a family of finite nonexpansive mappings $\{T_i, T_1, \ldots, T_r\}$ by $T_i = (1 - \lambda_i)I + \lambda_i P_{C_i}$, where $0 < \lambda_i < 1$ and $P_{C_i}$ is a nonexpansive retraction of $C$ onto $C_i$ for $i = 1, 2, \ldots, r$. Let $\{\alpha_n\} \subset (0, 1)$ be a sequence which satisfies $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$. Define a sequence $\{x_n\}$ by $x_0 \in C$ and

$$x_{n+1} = \alpha_{n+1}x_0 + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n = 0, 1, 2, \ldots,$$

where $T_{n+r} = T_n$. Then $\{x_n\}$ converges strongly to a point $z$ of $\bigcap_{i=1}^{r} C_i$. Further, if $Pz_0 = \lim_{n \to \infty} x_n$ for each $x_0 \in C$, then $P$ is a sunny nonexpansive retraction of $C$ onto $\bigcap_{i=1}^{r} C_i$.

**Proof.** By Corollary 4.1 and $\bigcap_{i=1}^{r} F(P_{C_i}) = \bigcap_{i=1}^{r} C_i$, we obtain the conclusion. □

**References**


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