ON THE EXISTENCE OF \((\gamma_p)\) \(k\)-SET CONTRACTIVE RETRACTIONS IN 
\(L_p[0, 1]\) SPACES, \(1 \leq p < \infty\)

ALESSANDRO TROMBETTA* AND GIULIO TROMBETTA**

Received August 3, 2001; revised December 20, 2001

Abstract. We prove that for any \(\varepsilon > 0\) there exists a retraction of the closed unit ball in the space \(L_p[0, 1]\), \(1 \leq p < \infty\), onto the unit sphere being a \((\gamma_p)\) \((2 + \varepsilon)\)-set contractive retraction.

1 Introduction. Let \(X\) be an infinite-dimensional Banach space with the closed unit ball \(B\) and the unit sphere \(S\). A continuous mapping \(R: B \to S\) with \(Rx = x\) for any \(x \in S\) is a retraction of the ball onto the sphere. Since the works of Nowak [3] and Benyamini and Sternfeld [2] it is known that, for any infinite-dimensional Banach space \(X\), there exists a \(k\)-lipschitzian retraction \(R: B \to S\) (i.e. a retraction satisfying the Lipschitz condition \(\|Rx - Ry\| \leq k \|x - y\|\), for all \(x, y \in B\)). Let \(\psi\) be a measure of noncompactness defined on \(X\) (see Section 2). A mapping \(T: D(T) \subset X \to X\) is said to be a \((\psi)\) \(k\)-set contraction if there exists a constant \(k \geq 0\) such that
\[
\psi(TA) \leq k\psi(A), \quad \text{for all bounded sets } A \subset D(T).
\]

We set
\[
k_0(X) := \inf\{k \geq 1 : \text{there is a } k\text{-lipschitzian retraction } R: B \to S\},
\]
\[
k_\psi(X) := \inf\{k \geq 1 : \text{there is a } (\psi) k\text{-set contractive retraction } R: B \to S\}.
\]

In [3] it is proved that \(k_0(X) \geq 3\). Recall that the Hausdorff measure of noncompactness \(\gamma\) on a Banach space \(X\) is defined by
\[
\gamma(A) := \inf\{r > 0 : A \text{ can be covered by a finite number of balls centered in } X\},
\]
for all bounded sets \(A \subset X\). If \(R\) is a \(k\)-lipschitzian retraction it is also \((\gamma)\) \(k\)-set contractive. So that \(k_\gamma(X) \leq k_0(X)\) for any infinite-dimensional Banach space \(X\). See the book of Teledano, Benavides and Acedo [7] and the references therein for more details concerning measures of noncompactness and \((\psi)\) \(k\)-set contractions. In [9], the author proved that \(k_\gamma(C[0, 1]) = 1\) and that, for any infinite-dimensional Banach space \(X\), there is no retraction \(R: B \to S\) being both, \(k\)-lipschitzian for some constant \(k\) and \((\gamma)\) \(1\)-set contractive.

Further he posed the problem to estimate \(k_\gamma(X)\) for particular classical Banach spaces and to establish for which spaces it is \(k_\gamma(X) < k_0(X)\). For \(1 \leq p < \infty\), let \(\gamma_p\) be the Hausdorff measure of noncompactness on \(L_p[0, 1]\). In the present note we prove that \(k_{\gamma_p}(L_p[0, 1]) \leq 2, 1 \leq p < \infty\). Moreover, we observe that, for any infinite-dimensional Banach space \(X\) and for any measure of noncompactness \(\psi\) defined on \(X\), there is no \((\gamma)\) \(1\)-set contractive retraction \(R: B \to S\) being \(k\)-lipschitzian for some constant \(k\).

2000 Mathematics Subject Classification. 46E30, 47H09.
Key words and phrases. retraction, \(k\)-set contraction, measure of noncompactness.
2 Notations and definitions. Let $X$ be a Banach space and $B$ the family of all bounded subsets of $X$. A mapping $\psi : B \to [0, +\infty]$ is called a measure of noncompactness on $X$ if it satisfies the following properties:

1) $\psi(A) = 0$ if and only if $A$ is precompact;
2) $\psi(\overline{\partial}A) = \psi(A)$, where $\overline{\partial}A$ denotes the closed convex hull of $A$;
3) $\psi(A \cup B) = \max\{\psi(A), \psi(B)\}$;
4) $\psi(A + B) \leq \psi(A) + \psi(B)$;
5) $\psi(\lambda A) = |\lambda| \psi(A)$, $\lambda \in \mathbb{R}$.

Let $L_p := L_p[0, 1]$, $1 \leq p < \infty$, be the classical Lebesgue spaces with the usual norm denoted by $\| \cdot \|_p$. In the following we will assume $1 \leq p < \infty$ and we will always denote by $S_p$ and $B_p$ the unit sphere and the unit closed ball of $L_p$, respectively. Moreover, every function $f \in L_p$ will be extended outside $[0, 1]$ by 0. Then for $f \in L_p$ and $h > 0$ consider the Steklov function

$$f_h(t) = \frac{1}{2h} \int_{[t-h, t+h]} f(s) ds,$$

for each $t \in [0, 1]$. For any bounded set $A \subset L_p[0, 1]$, we set

$$\omega_p(A) := \limsup_{\delta \to 0} \max_{f \in A, \|f\|_p \leq \delta} \|f - f_h\|_p.$$

It can be shown that $\omega_p$ is a measure of noncompactness on $L_p$. Moreover, as a straightforward consequence of the Kolmogorov compactness criterion in the spaces $L_p$ (see, e. g., [4]) we get the following

**Theorem 1** Let $A$ be a bounded subset of $L_p$. Then

$$\frac{1}{2} \omega_p(A) \leq \gamma_p(A) \leq \omega_p(A).$$

**Remark 2** In [8], Vîth notes that the precise formula for the Hausdorff measure of noncompactness in $L_p$

$$\gamma_p(A) = \frac{1}{2} \omega_p(A),$$

given in [1] is false; see also [16].

3 Results. Define a mapping $Q_p : B_p \to B_p$ by

$$(Q_p f)(t) = \begin{cases} \left( \frac{2}{1 + \|f\|_p} \right)^{\frac{1}{p}} f \left( \frac{2}{1 + \|f\|_p} \right), & \text{for } t \in \left[ 0, \frac{1 + \|f\|_p}{2} \right], \\
0, & \text{for } t \in \left( \frac{1 + \|f\|_p}{2}, 1 \right]. \end{cases}$$

It is easy to see that $\|Q_p f\|_p = \|f\|_p$ for all $f \in B_p$ and $Q_p f = f$ for all $f \in S_p$.

**Proposition 3** The mapping $Q_p$ is continuous.
\textbf{Proof.} Let \( \{ f_n \} \) be a sequence of elements of \( B_p \) such that \( f_n \to f(n \to \infty) \) with respect to the norm \( \| \cdot \|_p \). Set

\[ A_n := \left[ 0, \frac{1 + \| f_n \|_p}{2} \right] \cap \left[ 0, \frac{1 + \| f \|_p}{2} \right], \]

\[ B_n := \left[ 0, \frac{1 + \| f_n \|_p}{2} \right] \triangle \left[ 0, \frac{1 + \| f \|_p}{2} \right], \]

for all \( n \in \mathbb{N} \), where \( \triangle \) denotes the symmetric difference. Let \( \varepsilon > 0 \). Since the family \( \{ f, f_1, f_2, \ldots \} \) has uniformly continuous norms and \( \| f_n - f \|_p \to 0(n \to \infty) \), we can find an \( n_1 \in \mathbb{N} \) such that \( \| f_n - f \|_p \leq \frac{\varepsilon}{2} \) and

\[ \| Q_p f_n - Q_p f \|_p \leq \| (Q_p f_n - Q_p f) \chi_{A_n} \|_p + \| (Q_p f_n - Q_p f) \chi_{B_n} \|_p \]

\[ \leq \| (Q_p f_n - Q_p f) \chi_{A_n} \|_p + \frac{\varepsilon}{3}, \text{ for all } n \geq n_1. \]

Suppose \( \| f_n \|_p \leq \| f \|_p \). Then, by the change of variables

\[ s := \frac{2}{1 + \| f_n \|_p} t \left( t \in \left[ 0, \frac{1 + \| f_n \|_p}{2} \right] \right), \]

it follows that

\[ \| (Q_p f_n - Q_p f) \chi_{A_n} \|_p \]

\[ = \left[ \int_{0}^{\frac{1 + \| f_n \|_p}{2}} \left( \frac{2}{1 + \| f_n \|_p} \right)^\frac{p}{2} f_n \left( \frac{2}{1 + \| f \|_p} - \left( \frac{2}{1 + \| f \|_p} \right)^\frac{p}{2} \right)^{p - 1} dt \right]^{\frac{1}{p}} \]

\[ = \left[ \int_{0}^{1} f_n(s) - \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \right)^\frac{p}{2} f \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \right)^{p - 1} ds \right]^{\frac{1}{p}} \]

\[ \leq \| f_n - f \|_p + \left[ \int_{0}^{1} f(s) - \left( \frac{1 + \| f \|_p}{1 + \| f \|_p} \right)^\frac{p}{2} f \left( \frac{1 + \| f \|_p}{1 + \| f \|_p} \right)^{p - 1} ds \right]^{\frac{1}{p}}. \]

Now, suppose \( \| f_n \|_p > \| f \|_p \). Then, by the change of variables

\[ s := \frac{2}{1 + \| f \|_p} t \left( t \in \left[ 0, \frac{1 + \| f \|_p}{2} \right] \right), \]

it follows that

\[ \| (Q_p f_n - Q_p f) \chi_{A_n} \|_p \]

\[ \leq \| f_n - f \|_p + \left[ \int_{0}^{1} f(s) - \left( \frac{1 + \| f \|_p}{1 + \| f \|_p} \right)^\frac{p}{2} f \left( \frac{1 + \| f \|_p}{1 + \| f \|_p} \right)^{p - 1} ds \right]^{\frac{1}{p}}. \]
\[
\left[ \int_{[0, \frac{1+t\lambda}{1+t\lambda}]} \left( \left( \frac{2}{1 + \|f_n\|_p} \right)^p f_n \left( \frac{2}{1 + \|f\|_p} \right)^p - \left( \frac{2}{1 + \|f\|_p} \right)^p f \left( \frac{2}{1 + \|f\|_p} \right)^p \right) \right]^{\frac{1}{p'}}
\]
Choose a continuous function \( g : [0, 1] \to \mathbb{R} \) such that \( \| g - f \|_p \leq \frac{\varepsilon}{5} \). We put

\[
g_n(t) := \begin{cases} \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \right) \frac{p}{p-1} g \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} t \right), \quad & \text{if } \| f_n \|_p \leq \| f \|_p, \\ \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \right) \frac{p}{p-1} g \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} t \right), \quad & \text{if } \| f_n \|_p > \| f \|_p, \end{cases} \quad (t \in [0, 1]),
\]

for any \( n \in \mathbb{N} \). Suppose \( \| f_n \|_p \leq \| f \|_p \). By the change of variables \( s := \frac{1 + \| f_n \|_p}{1 + \| f \|_p} t (t \in [0, 1]) \), we obtain that

\[
\| g_n - h_n \|_p \leq \frac{\varepsilon}{5}.
\]

If \( \| f_n \|_p > \| f \|_p \), by the change of variables \( s := \frac{1 + \| f_n \|_p}{1 + \| f \|_p} t (t \in [0, 1]) \), it follows again \( \| g_n - h_n \|_p \leq \| g - f \|_p \leq \frac{\varepsilon}{5} \). Since \( g \) is continuous, \( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \to 1 (n \to \infty) \) and \( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \to 1 (n \to \infty) \) we have that

\[
\| g_n(t) - g(t) \| \to 0 (n \to \infty),
\]

for each \( t \in [0, 1] \). Then \( g_n(t) \to g(t) (n \to \infty) \), for each \( t \in [0, 1] \). On the other hand

\[
\| g - g_n \|_p = \begin{cases} \left[ \int_{[0, 1]} \left| g (t) - \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \right)^{\frac{p}{p-1}} g \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} t \right) \right|^p dt \right]^\frac{1}{p}, \quad & \text{if } \| f_n \|_p \leq \| f \|_p, \\ \left[ \int_{[0, 1]} \left| g (t) - \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} \right)^{\frac{p}{p-1}} g \left( \frac{1 + \| f_n \|_p}{1 + \| f \|_p} t \right) \right|^p dt \right]^\frac{1}{p}, \quad & \text{if } \| f_n \|_p > \| f \|_p. \end{cases}
\]

Moreover, we have that

\[
\| g_n \|_p = \begin{cases} \left[ \int_{[0, 1]} \left| g (s) \right|^p ds \right]^\frac{1}{p}, \quad & \text{if } \| f_n \|_p \leq \| f \|_p, \\ \left[ \int_{[0, 1]} \left| g (s) \right|^p ds \right]^\frac{1}{p}, \quad & \text{if } \| f_n \|_p > \| f \|_p. \end{cases}
\]

Then

\[
\lim_n \| g_n \|_p = \| g \|_p.
\]

So that \( \| g_n - g \|_p = 0 \). Let \( n_2 \in \mathbb{N} \) such that \( \| g_n - g \|_p \leq \frac{\varepsilon}{5} \), for any \( n \geq n_2 \). Set \( n := \{ n_1, n_2 \} \), we have

\[
\| Q_p f_n - Q_p f \|_p \leq \| (Q_p f_n - Q_p f) \chi_{A_n} \|_p + \frac{\varepsilon}{5}.
\]
\[\leq \|f_n - f\|_p + \|f - h_n\|_p + \frac{\varepsilon}{5}\]

\[\leq \|f_n - f\|_p + \|f - g\|_p + \|g - g_n\|_p + \|g_n - h_n\|_p + \frac{\varepsilon}{5} \leq \varepsilon\]

for all \(n \geq \nu.\]

**Proposition 4** The mapping \(Q_p\) is an \((\gamma_p)\) 2-set contraction.

**Proof.** Let \(f \in B_p\) and \(0 < h \leq \frac{1}{t}\). Set \(\alpha := \frac{1 + \|f\|_p}{2h}\). In this proof we consider the Steklov function \(f_h\) of \(f\) defined by

\[f_h(t) = \frac{\alpha}{2h} \int_{[t, t + \frac{h}{\alpha}]} f(s) \, ds,
\]

for \(t \in [0, 3/2]\) and equal to 0 elsewhere. Moreover, we still denote by \(\|\cdot\|_p\) the usual norm on \(L_p[0, 3/2]\). We start to prove that

\[\|f - f_h\|_p = \|Q_p f - (Q_p f)_h\|_p\]

In fact

\[\|f - f_h\|_p^p = \int_{[0, \frac{1}{\alpha}]} \left|f(\tau) - f_h(\tau)\right|^p \, d\tau
\]

\[+ \int_{[\frac{1}{\alpha}, 1]} \left|f(\tau) - f_h(\tau)\right|^p \, d\tau + \int_{[1, 1 + \frac{1}{\alpha}]} \left|f(\tau) - f_h(\tau)\right|^p \, d\tau
\]

\[+ \int_{[1 - \frac{1}{\alpha}, 1]} \left|f(\tau) - f_h(\tau)\right|^p \, d\tau + \int_{[1 + \frac{1}{\alpha}, 1 + \frac{1}{\alpha}]} \left|f(\tau) - f_h(\tau)\right|^p \, d\tau
\]

\[+ \int_{[1 - \frac{1}{\alpha}, 1]} \left|f(\tau) - \frac{\alpha}{2h} \int_{[0, \tau + \frac{1}{\alpha}]} f(s) \, ds \right|^p \, d\tau
\]

\[+ \int_{[1 - \frac{1}{\alpha}, 1]} \left|f(\tau) - \frac{\alpha}{2h} \int_{[0, \tau + \frac{1}{\alpha}]} f(s) \, ds \right|^p \, d\tau + \int_{[1, 1 + \frac{1}{\alpha}]} \left|\frac{\alpha}{2h} \int_{[0, \tau + \frac{1}{\alpha}]} f(s) \, ds \right|^p \, d\tau.\]

By the change of variables \(\tau = \frac{\alpha}{\alpha} t\) and \(s = \frac{\alpha}{\alpha} s\), we obtain that

\[\|f - f_h\|_p^p = \int_{[0, 1]} \left|\frac{1}{\alpha^p} f(t) - \frac{1}{2h} \int_{[0, t + h]} \frac{1}{\alpha^p} f(t) \, dt\right|^p \, dt
\]

\[+ \int_{[h, a - h]} \left|\frac{1}{\alpha^p} f(t) - \frac{1}{2h} \int_{[t - h, t + h]} \frac{1}{\alpha^p} f(t) \, dt\right|^p \, dt\]
\[
\begin{align*}
&\left[1\right. + \int_{[a-h,a]} \left| \frac{1}{\alpha f} f(t) \right| - \frac{1}{2h} \int_{[a-h,a]} \left| \frac{x}{\alpha f} \right| dt \right)^p \left[1\right. \\
&\left. + \int_{[a,a+h]} \left| \frac{1}{\alpha f} f(t) \right| - \frac{1}{2h} \int_{[a-h,a]} \left| \frac{x}{\alpha f} \right| dt \right]^p dt.
\end{align*}
\]

Hence
\[
\| f - f_h \|_p = \int_{[0,h]} \| Q_p f(t) - (Q_p f)_h(t) \|_p dt + \int_{[h,a-h]} \| Q_p f(t) - (Q_p f)_h(t) \|_p dt
\]
\[
+ \int_{[a-h,a]} \| Q_p f(t) - (Q_p f)_h(t) \|_p dt + \int_{[a,a+h]} \| Q_p f(t) - (Q_p f)_h(t) \|_p dt
\]
\[
= \| Q_p f - (Q_p f)_h \|_p.
\]

Then, for any set \( A \subset B_p \), we have that
\[
\omega'_p(Q_p A) = \lim_{\delta \to 0} \sup_{f \in A, 0 < \varepsilon \leq \delta} \| Q_p f - (Q_p f)_h \|_p \leq \lim_{\delta \to 0} \sup_{f \in A, 0 < \varepsilon \leq \delta} \| f - f_h \|_p = \omega'_p(A),
\]

where \( \omega'_p \) is defined on \( L_p[0,3/2] \) as \( \omega_p \). Let \( \gamma'_p \) be the Hausdorff measure of noncompactness on \( L_p[0,3/2] \). Then, since \( \gamma_p(C) = \gamma'_p(C) \) for all sets \( C \subset B_p \) and an analogous to Theorem 1 holds in \( L_p[0,3/2] \), we have that
\[
\gamma_p(Q_p A) \leq \omega'_p(Q_p A) \leq \omega'_p(A) \leq 2\gamma_p(A),
\]

for all sets \( A \subset B_p \). 

For any \( u > 0 \), we define the mapping \( P_{p,u} : B_p \to L_p \) putting
\[
(P_{p,u} f)(t) := \max \left\{ 0, \frac{u}{2} \left( 2t - \| f \|_p - 1 \right) \right\}, (t \in [0,1]).
\]

Observe that, for any \( u > 0 \) and for all \( f \in B_p \), we have \( (P_{p,u} f)(t) = 0 \) for any \( t \in \left[ 0, \frac{1 + \| f \|_p}{2} \right] \). Moreover, it easy to see that, for any \( u > 0 \), the mapping \( P_{p,u} \) is continuous and compact. We set, for any \( u > 0 \),
\[
F_{p,u}(\lambda) := \lambda^p + \frac{1}{2(p+1)} \left( \frac{u}{2} \right)^p \left( 1 - \lambda \right)^{p+1}, (\lambda \in [0,1]).
\]

Then, it is simple to verify that \( F_{p,u} \) attains its minimum for a unique \( \lambda_u \in [0,1] \). Moreover, \( 0 < F_{p,u}(\lambda_u) < 1 \) and
\[
\lim_{u \to \infty} \lambda_u = 1.
\]

For any \( u > 0 \), consider the mapping \( T_{p,u} : B_p \to L_p \) defined by
\[
T_{p,u} f = Q_p f + P_{p,u} f.
\]
Clearly, the mapping $T_{p, u}$ is an $(\gamma_p)$ 2-set contraction, and $T_{p, u}f = f$ for any $f \in B_p$. Further, for any $u > 0$ and for all $f \in B_p$, we have that

$$
\|T_{p, u}f\|_p^p = \int_{[0, 1/\sqrt{2}]} |Q_p f(t)|^p \, dt + \int_{[1/\sqrt{2}, 1]} |Q_p f(t) + P_{p, u} f(t)|^p \, dt
$$

$$
= \|f\|_p^p + \int_{[1/\sqrt{2}, 1]} \frac{u}{2} \left(2t - \|f\|_p - 1\right)^p \, dt
$$

$$
= \|f\|_p^p + \frac{1}{2(p + 1)} \left(\frac{u}{2}\right)^p \left(1 - \|f\|_p\right)^{p+1} = F_{p, u}(\|f\|_p^p).
$$

So that, $\|T_{p, u}f\|_p^p \geq F_{p, u}(\lambda_u)$ for any $f \in B_p$. Now, we define

$$
R_{p, u}f = \frac{1}{\|T_{p, u}f\|_p^p} T_{p, u}f.
$$

Then, for any set $A \subseteq B_p$, we have that

$$
\gamma_p(R_{p, u}A) \leq \left(\frac{1}{F_{p, u}(\lambda_u)}\right)^\frac{p}{p+1} 2\gamma_p(A).
$$

Therefore $R_{p, u} : B_p \to S_p$ is a $(\gamma_p)$ 2-set contractive retraction. Since $\lim_{u \to \infty} F_{p, u}(\lambda_u) = 1$, for any $\varepsilon > 0$ there exists $u > 0$ such that the mapping $R_{p, u} : B_p \to S_p$ is a $(\gamma_p)$ $(2 + \varepsilon)$-set contractive retraction. Thus, the following result holds.

**Theorem 5** $k_{\gamma_p}(L_p) \leq 2$.

In the context above described the following question remains open.

**Problem 6** Let $X$ be an infinite-dimensional Banach space and let $\psi$ be a measure of noncompactness on $X$. Does there exist a $(\psi)$ 1-set contractive retraction $R : B \to S$?

However, we have that

**Theorem 7** Let $X$ be an infinite-dimensional Banach space and let $\psi$ be a measure of noncompactness on $X$. If $R : B \to S$ is a $(\psi)$ 1-set contractive retraction, then it is not $k$-Lipschitzian for any constant $k$.

The proof of the above theorem is carried out analogously to the proof of Theorem II in [9].

**References**


*Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Università di Napoli "Federico II", Via Cintia - 80126, Napoli, Italy

email: aletromb@unical.it

**Dipartimento di Matematica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

email: trombetta@unical.it*