A WEIGHTED VERSION OF OZEKI’S INEQUALITY

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Received July 5, 2001; revised November 30, 2001

ABSTRACT. As an extension of Ozeki’s inequality we give an inequality which estimates the difference

\[ \sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 - \left( \sum_{i=1}^{n} p_i a_i b_i \right)^2 \]

derived from the weighted Cauchy-Schwarz inequality for n-tuples \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \) and \( p = (p_1, \ldots, p_n) \) of positive numbers under certain conditions. We discuss the upper bound of the difference not only in the general case but also in the special cases that \( a \) and \( b \) are monotonic in the opposite sense and in the same sense.

1 Introduction As a complement of Cauchy-Schwartz inequality, the following inequality was given in [4] (cf. [7, p. 121]) which was originally presented by Ozeki [8]: If \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) are n-tuples of positive numbers satisfying

\[ m_1 \leq a_k \leq M_1, \quad m_2 \leq b_k \leq M_2 \quad (k = 1, 2, \ldots, n), \quad 0 < m_1 < M_1 \quad \text{and} \quad 0 < m_2 < M_2, \]

then

(2) \[ \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{3} (M_1 M_2 - m_1 m_2)^2. \]

Put \( T(a, b) \) the left-hand side of the above inequality, then \( T(a, b) \) is considered as a function on the product \( [m_1, M_1]^n \times [m_2, M_2]^n \) of n-dimensional cubes \( [m_1, M_1]^n \) and \( [m_2, M_2]^n \). Then it is Ozeki’s idea to make use of the following two facts in order to prove the inequality (2) (and the technique was also useful for further results in [3, 5]):

(i) \( T(a, b) \) is a separately convex function with respect to \( a \) and \( b \), so that its maximum is attained at an extreme point, namely, vertex of \( 2n \)-dimensional rectangle \( [m_1, M_1]^n \times [m_2, M_2]^n \).

(ii) Denote by \( \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) \) and \( \bar{\tau} = (\bar{\tau}_1, \ldots, \bar{\tau}_n) \) the rearrangements of a nonnegative n-tuple \( c = (c_1, \ldots, c_n) \) in nonincreasing order and in nondecreasing order, respectively. Then for \( a \) and \( b \)

(3) \[ T(\bar{a}, \bar{b}) = T(\bar{\tau}, \bar{b}) \geq T(a, b). \]

As a result, from (3) the inequality (2) was obtained by considering \( T(a, b) \) for \( a \) and \( b \) such that they are monotonic in the opposite sense.

2000 Mathematics Subject Classification. 47A63.

Key words and phrases. Cauchy-Schwartz inequality, Ozeki’s inequality, Čebyšev’s inequality, Grüss’ inequality, rearrangement of numbers.
Now let $D(a, b) = n \sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k,$ which is $n^2$ times of the covariance between $a$ and $b.$ As an estimation of $D(a, b),$ Biermaak, Pidek and Ryll-Nardzewski [1] (cf. [7, p. 299]) presented the following result:

$$|D(a, b)| \leq \left( \frac{n}{2} \right) \left( \frac{n}{2} \right) (M_1 - m_1)(M_2 - m_2) \quad \text{for} \ (a, b) \text{ satisfying (1)}.$$ 

In particular, taking $D(a, b)$ for $a$ and $b$ such that they are monotonic in the same sense, (say, $a = \sigma$ and $b = \delta$), we obtain an inequality, which is nothing but a complement of the well-known Čebyšev’s inequality, a kind of Grüss type inequalities. It is a problem to estimate $T(a, b)$ with the restriction that $a$ and $b$ are monotonic in the same sense, likely to the above consideration and several works [6], [9], [10], etc., related to Grüss’ inequality. 

Now to consider the problem more generally, define by

$$T(a, b; p) = \sum_{k=1}^{n} \sum_{k=1}^{n} p_k a_k^2 - \sum_{k=1}^{n} p_k a_k b_k^2$$

the difference derived from the weighted Cauchy-Schwartz inequality with a positive $n$-weight $(n$-tuple $p = (p_1, \ldots, p_n),$ $\sum_{k=1}^{n} p_k = 1.$ Then unlike $T(a, b)$ the equality-inequality $T(\sigma, \delta; p) = T(\sigma, \sigma; p)$ corresponding to (3) are false in general. (For example, if $a = (1, 1, 1),$ $b = (2, 1, 2)$ and $p = (\frac{1}{13}, \frac{1}{13}, \frac{1}{13})$ then $T(\sigma, \delta; p) = \frac{16}{13},$ $T(\sigma, \sigma; p) = \frac{10}{13}$ and $T(a, b; p) = \frac{16}{13}.$) This means that rearrangements of $a$ and $b$ to be monotonic in the opposite sense are not effective to obtain the maximum of $T_p(a, b) = T(a, b; p).$ However, the calculation of the maximum for such $a$ and $b$ yields, in a sense, an extension of (2).

In this paper, using Ozeki’s technique on convex functions, we give upper bounds of (4) not only in the general case for $a$ and $b,$ but also in the special cases that $a$ and $b$ are monotonic in the opposite sense and in the same sense.

2 Preliminaries We prepare some useful facts for our discussion. Let $I_n = \{1, \ldots, n\}$ and define an index set $\Delta$ in $I_n^2 = I_n \times I_n$ by

$$\Delta = \{(i, j) \in I_n^2; i < j\}.$$

Now we state a weighted version of Lagrange’s formula (cf. [7, p. 84]), which we can prove easily.

**Lemma 2.1**

$$T(a, b; p) = \sum_{(i, j) \in \Delta} p_i p_j (a_i b_j - a_j b_i).$$

From this lemma we can see the following:

**Lemma 2.2** $T_p(a, b) = T(a, b; p)$ is a separably convex function on $[m_1, M_1]^n \times [m_2, M_2]^n$ with respect to $a$ and $b,$ that is,

$$T_p(\lambda a + (1 - \lambda) a', b) \leq \lambda T_p(a, b) + (1 - \lambda) T_p(a', b), \quad \lambda \in [0, 1]$$

and

$$T_p(a, \mu b + (1 - \mu) b') \leq \mu T_p(a, b) + (1 - \mu) T_p(a, b'), \quad \mu \in [0, 1].$$
Consequently, we see that $T_p(a, b)$ attains its maximum at a point $(a, b)$ of $[m_1, M_1]^n \times [m_2, M_2]^n$, with both $a$ and $b$ being vertices of $[m_1, M_1]^n$ and $[m_2, M_2]^n$, respectively. (Note that a point $v = (v_1, ..., v_n) \in [m, M]^n$ is a vertex if (and only if) each $v_k$ is equal to $m$ or $M$.)

For two real numbers $m, M$, $m < M$, let

$$K = \{(x_1, ..., x_n) \in [m, M]^n; x_1 \leq \cdots \leq x_n\}$$

and

$$L = \{(x_1, ..., x_n) \in [m, M]^n; x_1 \geq \cdots \geq x_n\}.$$ 

Then $K$ and $L$ are convex subsets in $[m, M]^n$. The following fact related to their extreme points is easily seen, say, by the induction method.

**Lemma 2.3** Every extreme point of $K$ (or $L$) is a vertex of $[m, M]^n$.

Now assume that $A, B, C > 0$, and put

$$\hat{A} = B + C - A, \quad \hat{B} = C + A - B, \quad \hat{C} = A + B - C \quad \text{and}$$

$$D = A\hat{A} + B\hat{B} + C\hat{C} = 2AB + 2BC + 2CA - A^2 - B^2 - C^2.$$ 

Then it is not difficult to see that

(i) at least two of $\hat{A}, \hat{B}$ and $\hat{C}$ are positive, and

(ii) if all of $\hat{A}, \hat{B}$ and $\hat{C}$ are positive then $D > 0$.

The following general fact (cf. [4]) is very useful for our discussion.

**Lemma 2.4** With the same notations as above, consider the function

$$u = f(x, y, z) = Ax + Bxz + Cyz$$

under the condition

$x, y, z \geq 0$, $x + y + z = k > 0$ ($k$ is a constant).

(i) If $\hat{A}, \hat{B}, \hat{C} > 0$, then $D > 0$ and

$$u = -C \left\{ \left( y - \frac{B\hat{B}}{D} k \right) + \frac{\hat{A}}{2C} \left( x - \frac{C\hat{C}}{D} k \right) \right\}^2 - \frac{D}{4C} \left( x - \frac{C\hat{C}}{D} k \right)^2 + \frac{ABC}{D} k^2,$$

so that

$$u \leq u_{\text{max}} (= \text{the maximum of } u) = \frac{ABC}{D} k^2,$$

and $u_{\text{max}}$ is attained at a point

$$(x, y, z) = \left( \frac{C\hat{C}}{D} k, \frac{B\hat{B}}{D} k, \frac{A\hat{A}}{D} k \right).$$

(ii) If one of $\hat{A}, \hat{B}, \hat{C}$ is nonpositive, say, $\hat{B} \leq 0$, (hence $\hat{A}, \hat{C} > 0$), then

$$u = -\hat{B}xz + Ax(k - x) + Cz(k - z)$$
and
\[ u \leq u_{max} = \frac{B}{4} k^2. \]

The value \( u_{max} \) is attained at
\[ (x, y, z) = (k/2, 0, k/2). \]

Proof. (i) Putting \( z = k - x - y \), we have, from (8),
\[ u = -Cy^2 - \left( Ax - Ck \right) y - Bx^2 + Bkx. \]

Taking the 4C times of the both sides, we have
\[
4Cu = -4C^2 y^2 - 4C \left( Ax - Ck \right) y - 4BCx^2 + 4BCkx
\]
\[ = - \left( 2Cy + Ax - Ck \right)^2 - D \left( x - \frac{C\hat{C}}{D} k \right)^2 + \frac{4ABC^2}{D} k^2. \]

Hence we have
\[
u = -C \left( y + \frac{Ax - Ck}{2C} \right)^2 - \frac{D}{4C} \left( x - \frac{C\hat{C}}{D} k \right)^2 + \frac{ABC}{D} k^2
\]
\[ = -C \left( \left( y - \frac{B\hat{B}}{D} k \right) + \frac{A}{2C} \left( x - \frac{C\hat{C}}{D} k \right) \right)^2 - \frac{D}{4C} \left( x - \frac{C\hat{C}}{D} k \right)^2 + \frac{ABC}{D} k^2. \]

Now, if \( x = \frac{C\hat{C}}{D} k, y = \frac{B\hat{B}}{D} k \), (so that \( z = k - x - y = \frac{A\hat{A}}{D} k \)), then \( u = u_{max} = \frac{ABC}{D} k^2 \).

(ii) Putting \( y = k - x - z \), we have, from (8),
\[
u = -Bxz + Ax(k - x) + Cz(k - z). \]
Since \( xz \leq \left( \frac{x + z}{2} \right)^2 \leq \frac{k^2}{4}, x(k - x) \leq \frac{k^2}{4} \) and \( z(k - z) \leq \frac{k^2}{4} \), we have
\[ u \leq -B \cdot \frac{1}{4} k^2 + A \cdot \frac{1}{4} k^2 + C \cdot \frac{1}{4} k^2 = \frac{1}{4} Bk^2. \]

Hence \( u_{max} = \frac{1}{4} Bk^2 \), which is attained at \( (x, y, z) = (k/2, 0, k/2). \)

\[ \square \]

3 Weighted Ozeki’s inequality In this section we give an upper bound of \( T(a, b, p) \) without any assumption of monotony on positive \( n \)-tuples \( a \) and \( b \). Let us define, for a positive \( n \)-weight \( p = (p_1, \ldots, p_n) \) with \( \sum_{k=1}^n p_k = 1 \),
\[
P(X) = \sum_{k \in X} p_k \quad \text{for } X \subset I_n.
\]
say, as in [11]. Then we have:
Lemma 3.1 Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples such that $a_k = 1$ or $\alpha$ and $b_k = 1$ or $\beta$ $(k = 1, \ldots, n)$, and let $p = (p_1, \ldots, p_n)$ be a positive weight with $\sum_{k=1}^n p_k = 1$.

Put

$$J_a = \{k \in I_n; a_k = 1\} \quad \text{and} \quad J_b = \{k \in I_n; b_k = 1\}.$$ 

Then

$$T(a, b; p) = \sum_{J_a \cap J_b} p(J_a \cap J_b) P(J_a \cap J_b) (1 - \beta)^2 + P(J_a \cap J_b) P(J_a \cap J_b) (1 - \alpha)^2 \quad \text{or} \quad \beta = \frac{1}{2}$$

$$+ \sum_{J_a \cap J_b} P(J_a \cap J_b) (1 - \alpha)^2 + P(J_a \cap J_b) P(J_a \cap J_b) (1 - \alpha)^2 \quad \text{or} \quad \beta = \frac{1}{2}.$$ 

Proof. First note that $I_n$ is divided into the four subsets

$$J_1 = J_a \cap J_b, \quad J_2 = J_a \cap J_b^c, \quad J_3 = J_a^c \cap J_b, \quad \text{and} \quad J_4 = J_a^c \cap J_b^c,$$

so that $\Delta = \{(i, j) \in I_n^2; i < j\}$ is divided into the ten subsets

$$\Delta_{k, l} = J_k \times J_l, \quad 1 \leq k \leq l \leq 4.$$ 

Let $\sum_{\Delta_{k, l}} = \sum_{(i, j) \in \Delta_{k, l}} p_i p_j (a_i b_j - a_j b_i)^2$. Then we see that $T(a, b; p)$ is the total of sums $\sum_{\Delta_{k, l}}, 1 \leq k \leq l \leq 4$ by Lemma 2.1. We can easily see that $\sum_{\Delta_{k, l}} = 0$. It is also easy to compute $\sum_{\Delta_{k, l}}$, for $k < l$: say, for $k = 1, l = 2$ we have

$$\sum_{\Delta_{1, 2}} = \sum_{(i, j) \in J_1 \times J_2} p_i p_j (a_i b_j - a_j b_i)^2 = P(J_1) P(J_2) (1 - \beta)^2.$$ 

Consequently, we have

$$T(a, b; p) = \sum_{\Delta_{1, 2}} + \sum_{\Delta_{1, 3}} + \sum_{\Delta_{1, 4}} + \sum_{\Delta_{2, 3}} + \sum_{\Delta_{2, 4}}$$

$$= P(J_1) P(J_2) (1 - \beta)^2 + P(J_1) P(J_2) (1 - \alpha)^2 + P(J_1) P(J_2) (\alpha - \beta)^2$$

$$+ P(J_2) P(J_3) (1 - \alpha)^2 + P(J_2) P(J_3) (1 - \beta)^2 + P(J_2) P(J_4) \alpha^2 (1 - \beta)^2.$$

Now we have the following extension of Ozeki’s inequality (cf. [4, Theorem 2.1]).

Theorem 3.2 Let $a$ and $b$ be positive n-tuples satisfying (1) and let $p$ be a positive weight with $\sum_{k=1}^n p_k = 1$. Assume that $\alpha = M_1/M_1 \geq M_2/M_2 = \beta$. Then

$$T(a, b; p) \leq M_1^2 M_2^2 \max_{X \in L_n} \left\{ \frac{(1 - \alpha \beta)^2}{4} (1 - P(X))^2 + (1 - \beta)^2 P(X) (1 - P(X)) \right\}.$$ 

Proof. We may assume that $M_1 = M_2 = 1$ (and then write $\alpha = m_1$, $\beta = m_2$) for convenience. In order to obtain the maximum or the best upper bound of $T_p(a, b) = T(a, b; p)$, we have to calculate, by convexity of $T(a, b; p)$, its value for $a$ and $b$ such that $a_i = 1$ or $\alpha$, $b_i = 1$ or $\beta$ $(i = 1, \ldots, n)$. Hence we may apply the preceding lemma. Put

$$A = \beta^2 (1 - \alpha)^2, \quad B = (1 - \alpha \beta)^2, \quad C = \alpha^2 (1 - \beta)^2.$$
\[ E = (1 - \beta)^2, \quad F = (\alpha - \beta)^2, \quad G = (1 - \alpha)^2, \]
and furthermore put
\[ x = P(J_a \cap J_b^c), \quad y = P(J_b^c \cap J_a^c), \quad z = P(J_b^c \cap J_b^c) \quad \text{and} \quad w = P(J_a \cap J_b). \]
Then we have
\[ x + y + z + w = 1 \quad (x, y, z, w \geq 0) \]
and from (12)
\[ u := T(a, b, p) = Axy + Bxz + Cyz + Exw + Fyw + Gzw. \]
First note that for positive numbers \( A, B, C \) we have
\[ \hat{B} = C + A - B = \alpha^2(1 - \beta)^2 + \beta^2(1 - \alpha)^2 - (1 - \alpha\beta)^2 \]
\[ = -(1 - \alpha)(1 - \beta)(1 + \alpha + \beta - \alpha\beta) < 0, \]
because \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). Hence since \( x + y + z = 1 - w \), we have, by Lemma 2.4 (ii),
\[ Ax + Bxz + Cyz \leq \frac{B}{4}(1 - w)^2. \]
Next from the assumption \( \alpha \geq \beta \), we see \( E \geq F, G \), so that
\[ Exw + Fyw + Gzw \leq Ew(x + y + z) = Ew(1 - w). \]
Hence we have
\[ (14) \quad T(a, b, p) \leq \frac{B}{4}(1 - w)^2 + Ew(1 - w), \]
from which we obtain the desired inequality (13). \( \square \)

Now we obtain the following result [4, Theorem 4.1] from the preceding theorem.

**Theorem 3.3** With the same notations and the same assumptions as in Theorem 3.2,
\[ T(a, b, p) \leq \frac{1}{3}M_1^2M_2^2(1 - \alpha\beta)^2 = \frac{1}{3}(M_1M_2 - m_1m_2)^2. \]

**Proof.** As before we may assume \( M_1 = M_2 = 1 \). Write \( g(w) \) the right-hand side of (14). Then it suffices to show that
\[ g(w) \leq \frac{1}{3}B \quad (0 \leq w \leq 1). \]
Since \( E \leq B \leq 4E \) and
\[ g(w) = -\frac{4E - B}{4}w^2 + \frac{2E - B}{2}w + \frac{B}{4}, \]
we have, by an elementary computation,
\[ \max_{0 \leq w \leq 1} g(w) = \begin{cases} \frac{E^2}{4E - B} & \text{if } (E \leq B) \leq 2E, \\ \frac{B}{4} & \text{if } 2E \leq B \leq 4E. \end{cases} \]
Furthermore, it is not difficult to see that
\[ \frac{E^2}{4E - B} \leq \frac{1}{3}B \quad (\text{if } E \leq B \leq 2E). \]
Hence we have the desired inequality. \( \square \)
The difference $T(a, b; p)$ for oppositely ordered $a$ and $b$ in this section we give an upper bound of $T_p(a, b) = T(a, b; p)$ for $a$ and $b$ ordered oppositely. We confine ourselves to the case that $a$ is ordered nonincreasingly and $b$ is ordered nondecreasingly. Recall that from Lemmas 2.2 and 2.3 the function $T_p(a, b)$ is separately convex with respect to $a$ and $b$, and attains its maximum at a point $(a, b)$ such that

\[(15) \quad a = (M_1, \ldots, M_1, m_1, \ldots, m_1) \quad \text{and} \quad b = (m_2, \ldots, m_2, M_2, \ldots, M_2), \]
\[s, t \in P_0^* = I_n \cup \{0\}.\]

Now we have

**Lemma 4.1** Let $a^{(s)}$ and $b^{(t)}$ be $n$-tuples of real numbers such that

\[(16) \quad a^{(s)} = (1, \ldots, 1, \alpha_1, \ldots, \alpha_s) \quad \text{and} \quad b^{(t)} = (\beta_1, \ldots, 1, \beta_{t-n}, \ldots, 1), \]
\[s, t \in P_0^* = I_n \cup \{0\},\]

and let $p = (p_1, \ldots, p_n)$ be a positive $n$-weight with $\sum_{i=1}^{n} p_i = 1$. Write $P_k = \sum_{i=1}^{k} p_i$, for $k \in I_n$ ($P_0 = 0$). Then

\[(17) \quad T(a^{(s)}, b^{(t)}; p) = \begin{cases} 
  P_t(P_s - P_t)(1 - \beta)^2 + P_t(P_t - P_s)(1 - \alpha)^2 \\
  + (P_s - P_t)(1 - P_s)(1 - \alpha)^2 \\
  \quad \text{if } 0 \leq t \leq s \leq n, \\
  P_s(P_t - P_s)(1 - \alpha)^2 + P_s(P_s - P_t)(1 - \beta)^2 \\
  + (P_t - P_s)(1 - P_t)(1 - \beta)^2 \\
  \quad \text{if } 0 \leq s \leq t \leq n.
\end{cases}\]

**Proof.**

Case I: $0 \leq t \leq s \leq n$. Rewriting $a = a^{(s)}$ and $b = b^{(t)}$ more precisely, we have

\[a = (1, \ldots, 1, \alpha_1, \ldots, \alpha_s), \quad b = (\beta_1, \ldots, 1, \beta_{t-n}, \ldots, 1).\]

Then with the same notations as in Section 3 we have

\[J_a = \{1, \ldots, s\} \quad \text{and} \quad J_b = \{t + 1, \ldots, n\},\]

and $\Delta = \{(i, j) \in \mathbb{I}^2; i < j\}$ is divided into the three subsets

\[J_a \cap J_b^c (= J_2), \quad J_a \cap J_b \cup J_b^c (= J_1) \quad \text{and} \quad J_a^c \cap J_b (= J_3).\]

Hence similarly as in Lemma 3.1 of Section 3, $T(a, b; p)$ is the sum of $\sum_{S_2} \sum_{S_3}$ and $\sum_{S_2} \sum_{S_3}$. Note that $P(J_2) = P_t$, $P(J_1) = P_s - P_t$ and $P(J_3) = 1 - P_s$. Hence we have

\[T(a, b; p) = P(J_1)P(J_2)(1 - \beta)^2 + P(J_1)P(J_3)(1 - \alpha)^2 + P(J_2)P(J_3)(1 - \alpha\beta)^2 = P_t(P_s - P_t)(1 - \beta)^2 + P_t(P_s - P_t)(1 - \alpha)^2 + P_s(P_s - P_t)(1 - \beta)^2 + (P_s - P_t)(1 - P_t)(1 - \alpha)^2.\]

Case II: $0 \leq s \leq t \leq n$. By the similar argument as in Case I, we have

\[T(a^{(s)}, b^{(t)}; p) = \beta^2(1 - \alpha)^2 P_t(P_s - P_t) + (1 - \alpha \beta)^2 P_s(P_s - P_t) + \alpha^2(1 - \beta)^2(P_s - P_t)(1 - P_t).\]
Summarizing Cases I and II, we obtain (17). \hfill \Box

Now we show the following result stronger than Theorem 3.2 with the restriction that \( a \) and \( b \) are oppositely ordered.

**Theorem 4.2** Let \( a \) and \( b \) be positive \( n \)-tuples satisfying

\[ M_1 \geq a_1 \geq \cdots \geq a_n \geq m_1 \quad \text{and} \quad m_2 \leq b_1 \leq \cdots \leq b_n \leq M_2, \]

and let \( p = (p_1, \ldots, p_n) \) be an \( n \)-weight with \( \sum_{k=1}^{n} p_k = 1 \). Put \( \alpha = m_1/M_1, \beta = m_2/M_2, \)

\[ A = (1 - \beta)^2, \quad B = (1 - \alpha \beta)^2, \quad C = (1 - \alpha)^2, \]

\[ A_1 = \beta^2(1 - \alpha)^2, \quad B_1 = B, \quad C_1 = \alpha^2(1 - \beta)^2, \]

and define \( \tilde{A}, \tilde{B}, \tilde{C} \) and \( D \) similarly as (7). (Furthermore, correspondingly define \( \tilde{A}_1, \tilde{B}_1 \) and \( \tilde{C}_1 \).) Then

\[ D = \left\{ 4 - (1 + \alpha)(1 + \beta) \right\} (1 + \alpha)(1 + \beta)(1 - \alpha^2)(1 - \beta^2) \]

and

\[ \frac{ABC}{D} = \frac{(1 - \alpha \beta)^2}{4(1 + \alpha)(1 + \beta)(1 + \alpha)(1 + \beta^2)}, \]

and the following results hold.

(i) If \((1 + \alpha)(1 + \beta) < 2\), then

\[ T(a, b; p) \leq M_1^2 M_2^2 \max \left\{ \frac{ABC}{D} - C\mu^2 - \frac{D}{4C} \lambda^2, \quad B \left( 1 - \frac{1}{4} - \nu^2 \right) \right\}. \]

(ii) If \((1 + \alpha)(1 + \beta) \geq 2\), then

\[ T(a, b; p) \leq M_1^2 M_2^2 B \left( 1 - \frac{1}{4} - \nu^2 \right). \]

Here, \( \lambda, \mu \) and \( \nu \) are defined as follows:

\[ \left\{ \begin{array}{l}
\lambda = \min_{1 \leq t \leq n-1} \left| P_t - \frac{\tilde{A} \tilde{C}}{D} \right|, \\
\mu = \min_{1 \leq s \leq s \leq n-1} \left| (P_s - P_t) - \frac{\tilde{A} \tilde{B}}{D} + \frac{\tilde{B}}{C} \left( P_t - \frac{\tilde{C}}{D} \right) \right| \\
\nu = \min_{1 \leq t \leq n-1} \left| \frac{B}{P_t} \right|
\end{array} \right. \]

and

\[ \frac{ABC}{D} = \frac{1}{4} \quad \text{and} \quad \frac{ABC}{D} = \frac{1}{4} - \nu^2. \]

**Proof.** We may assume that \( M_1 = M_2 = 1 \), and write \( m_1 = \alpha \) and \( m_2 = \beta \) as in Theorem 3.2. Then by convexity of \( T(a, b; p) = T_p(a, b) \) and Lemma 2.3 we may compute the maximum of \( T_p(a, b) \) for \((a, b) = (a^{(s)}, b^{(s)})\), \( s, t \in \mathcal{I}_n \), where \( a^{(s)} \) and \( b^{(s)} \) are positive \( n \)-tuples defined as (16). First we consider

Case I: \( 0 \leq t \leq s \leq n \). Put

\[ x = P_t, \quad y = P_s - P_t \quad \text{and} \quad z = 1 - P_t. \]

Then from (17) of Lemma 4.1

\[ T(a^{(s)}, b^{(s)}; p) = Axy + Bxz + Cyz. \]
Now consider the two subcases I-(1) and I-(2) as follows.

I-(1): Assume \((1 + \alpha)(1 + \beta) < 2\). Then
\[
\hat{B} = C + A - B = (1 - \alpha)^2 + (1 - \beta)^2 - (1 - \alpha \beta)^2 = 2 - (1 + \alpha)(1 + \beta) > 0.
\]
(Note that \((1 + \alpha)(1 + \beta) < 2\) is equivalent to \(\hat{B} > 0\).) For \(\hat{A}\) and \(\hat{C}\), since \(B = (1 - \alpha \beta)^2 > (1 - \beta)^2 = A\), we have \(A = B + C - A > 0\), and similarly \(C > 0\). By Lemma 2.4 (cf. (10)) we can write
\[
u = -C \left\{ \left( y - \frac{BB}{D} \right) + \frac{\hat{A}}{2C} \left( x - \frac{CC}{D} \right) \right\}^2 - \frac{D}{4C} \left( x - \frac{CC}{D} \right)^2 + \frac{ABC}{D}.
\]
Hence from the above definition of \(\lambda\) and \(\mu\), we have
\[
u \leq -\lambda \mu^2 - \frac{D}{4C} \lambda^2 + \frac{ABC}{D}.
\]
Here, it is an elementary computation to show that \(D\) and \(ABC/D\) are expressed as (18) in \(\alpha\) and \(\beta\).

I-(2): Assume \((1 + \alpha)(1 + \beta) \geq 2\). Then \(\hat{B} \leq 0\), so that \(\hat{A}, \hat{C} > 0\). By Lemma 2.4 (cf. (11)) we can write
\[
u = -\hat{B} xz + Ax(1 - x) + Cz(1 - z),
\]
and since
\[
xz = x(1 - x - y) \leq x(1 - x) = \frac{1}{4} - \left( \frac{1}{2} - x \right)^2 \leq \frac{1}{4} - \nu^2,
\]
\[
z(1 - z) \leq \frac{1}{4} - \nu^2 \quad \text{ (cf. } \nu \text{ is defined in (21))},
\]
we then have
\[
u \leq (-\hat{B} + A + C) \left( \frac{1}{4} - \nu^2 \right) = B \left( \frac{1}{4} - \nu^2 \right).
\]
Case II: \(0 \leq s \leq t \leq n\). Put
\[
x = P_s, \quad y = P_t - P_s \quad \text{and} \quad z = 1 - P_t.
\]
Then similarly as Case I, from Lemma 4.1
\[
u = T(a^{(s)}, b^{(t)}; p) = A_1 xy + B_1 xz + C_1 yz,
\]
and furthermore
\[
\hat{A}_1 = B_1 + C_1 - A_1 = (1 - \alpha \beta)^2 + \alpha^2(1 - \beta)^2 - \beta^2(1 - \alpha)^2
\]
\[
= (1 - \beta) \{ (1 + \alpha^2)(1 - \beta) + 2 \beta(1 - \alpha) \} > 0,
\]
\[
\hat{B}_1 = C_1 + A_1 - B_1 = -(1 - \alpha)(1 - \beta)(1 + \alpha + \beta - \alpha \beta) \leq 0,
\]
\[
\hat{C}_1 = A_1 + B_1 - C_1 = (1 - \alpha) \{ (1 + \beta^2)(1 - \alpha) + 2 \alpha(1 - \beta) \} > 0.
\]
Hence by Lemma 2.4 (ii)
\[
u \leq B_1 \left( \frac{1}{4} - \nu^2 \right) = B \left( \frac{1}{4} - \nu^2 \right).
\]
so that

\[ T(a, b, p) \leq M_1^2 M_2^2 \left( \frac{1}{4} - \nu^2 \right). \]

We notice that the constant \( \nu \) is independent from \( A, B, \ldots \), so that it is identical in Cases I and II. Summarizing the two cases, we obtain the desired facts (i) and (ii). \( \Box \)

Considering the special cases \( \lambda = \mu = 0 \) and \( \nu = 0 \) in the preceding theorem, we have:

**Theorem 4.3** With the same notations and the same assumptions as in Theorem 4.2, the following results hold.

(i) If \((1 + \alpha)(1 + \beta) < 2\), then

\[ T(a, b, p) \leq \frac{M_1^2 M_2^2 ABC}{D} = \frac{M_1^2 M_2^2 (1 - \alpha \beta)^2}{4 - (1 + \alpha)(1 + \beta)} \]

If there are integers \( s_0, t_0 \) (\( s_0 > t_0 \)) such that

\[ P_{s_0} = \frac{C^C}{D} \quad \text{and} \quad P_{s_0} - P_{t_0} = \frac{B^B}{D}, \]

then

\[ T_{\text{max}} (= \text{the maximum of } T_p(a, b) = T(a, b, p)) = \frac{M_1^2 M_2^2 ABC}{D}, \]

which is attained at \((a, b)\) such that

\[ a = (M_1, \ldots, M_1, m_1, \ldots, m_1) \quad \text{and} \quad b = (m_2, \ldots, m_2, M_2, \ldots, M_2). \]

(ii) If \((1 + \alpha)(1 + \beta) \geq 2\) then

\[ T(a, b, p) \leq \frac{M_1^2 M_2^2 B}{4} = \frac{M_1^2 M_2^2 (1 - \alpha \beta)^2}{4}. \]

If there is an integer \( t = t_0 \) such that \( P_{t_0} = \frac{1}{2} \), then

\[ T_{\text{max}} = \frac{M_1^2 M_2^2 B}{4}, \]

which is attained at \((a, b)\) such that

\[ a = (M_1, \ldots, M_1, m_1, \ldots, m_1) \quad \text{and} \quad b = (m_2, \ldots, m_2, M_2, \ldots, M_2). \]

**Proof.** By Theorem 4.2 it suffices to see that

\[ \frac{ABC}{D} \geq \frac{B}{4}, \]

which is easily obtained, say, from (18). \( \Box \)
5 The difference \( T(a, b, p) \) for similarly ordered \( a \) and \( b \) We here give an upper bound of \( T_p(a, b) = T(a, b, p) \) under the condition that \( a \) and \( b \) are similarly ordered. We may confine ourselves for the case that both \( a \) and \( b \) are nondecreasingly ordered.

Theorem 5.1 Let \( a \) and \( b \) be positive \( n \)-tuples satisfying

\[
m_1 \leq a_1 \leq \cdots \leq a_n \leq M_1 \quad \text{and} \quad m_2 \leq b_1 \leq \cdots \leq b_n \leq M_2,
\]

and let \( p = (p_1, \ldots, p_n) \) be an \( n \)-weight with \( \sum_{k=1}^n p_k = 1 \). Put, for \( \alpha = m_1/M_1, \beta = m_2/M_2, \)

\[
A = \alpha^2(1 - \beta)^2, \quad B = (\alpha - \beta)^2, \quad C = (1 - \alpha)^2,
\]

\[
A_1 = \beta^2(1 - \alpha)^2, \quad B_1 = B, \quad C_1 = (1 - \beta)^2,
\]

and define \( \hat{A}, \hat{B}, \hat{C} \) and \( D \), similarly as (7). (Furthermore, correspondingly define \( \hat{A}_1, \hat{B}_1 \) and \( \hat{C}_1 \).) Then

\[
D = (1 + \alpha)(1 + \beta)(1 - \alpha)^2(1 - \beta)^2 \{(3 - \beta)\alpha - (1 + \beta)\}
\]

and

\[
\frac{ABC}{D} = \frac{\alpha^2(\alpha - \beta)^2}{(1 + \alpha)(1 + \beta)\{(3 - \beta)\alpha - (1 + \beta)\}}.
\]

Further assume that

\[
\beta \leq \alpha,
\]

and write

\[
\alpha = \frac{-1 + \sqrt{2 - \beta^2}}{1 - \beta} \quad \text{and} \quad \alpha = \frac{1 + \beta^2}{1 + 2\beta - \beta^2}.
\]

Then

\[
\beta \leq \underline{\alpha} \leq \overline{\alpha} < 1
\]

and the following results hold. (\( \lambda, \mu \) and \( \nu \) are defined similarly as (21) in Theorem 4.2).

(i) If \( \beta \leq \alpha \leq \underline{\alpha} \) then

\[
T(a, b, p) \leq M_1^2 M_2^2 C_1 \left( \frac{1}{4} - \nu^2 \right).
\]

(ii) If \( \underline{\alpha} < \alpha < \overline{\alpha} \), then \( D > 0 \) and

\[
T(a, b, p) \leq M_1^2 M_2^2 \max \left\{ \frac{ABC}{D} - C\mu^2 - \frac{D}{4C}, C_1 \left( \frac{1}{4} - \nu^2 \right) \right\}.
\]

(iii) If \( \overline{\alpha} \leq \alpha \leq 1 \), then

\[
T(a, b, p) \leq M_1^2 M_2^2 C_1 \left( \frac{1}{4} - \nu^2 \right).
\]

Proof. By Lemma 2.3, we have to compute the maximum or an upper bound of \( T_p(a, b) = T(a, b, p) \) at points \((a, b)\) such that

\[
a = (m_1, \ldots, m_1, M_1, \ldots, M_1), \quad \text{and} \quad b = (m_2, \ldots, m_2, M_2, \ldots, M_2),
\]

\[
a = (m_1, \ldots, m_1, M_1, \ldots, M_1), \quad \text{and} \quad b = (m_2, \ldots, m_2, M_2, \ldots, M_2),
\]
where \( s \) and \( t \) are integers in \( I_k \).

We may again assume that \( M_1 = M_2 = 1 \), so that \( m_1 = \alpha \) and \( m_2 = \beta \). It is essential to consider the problem when \( \beta < \alpha \). Now the first case is

Case 1: \( 0 \leq t \leq s \leq n \).

\[
\begin{align*}
\phi(s) &= (\alpha, \ldots, \alpha, \frac{t}{s}, \frac{s-t}{n-s}, \frac{n-s}{n-s}) \quad \text{and} \quad \psi(t) = (\beta, \ldots, \beta, \frac{t}{s}, \frac{s-t}{n-s}, \frac{n-s}{n-s}).
\end{align*}
\]

Then by the similar argument as in Lemma 4.1 (cf. (17)), we have

\[
T(\phi(s), \psi(t); p) = \alpha^2 (1 - \beta)^2 P_s (P_t - P_s) + (\alpha - \beta)^2 P_s (1 - P_s) + (1 - \alpha)^2 (P_t - P_s)(1 - P_t) = AP_t (P_s - P_t) + BP_t (1 - P_s) + C (P_s - P_t) (1 - P_t).
\]

First note that \( A, B, C > 0 \) (cf. \( \beta < \alpha \)) and by definition

\[
\hat{A} = B + C - A = (\alpha - \beta)^2 + (1 - \alpha)^2 - \alpha^2 (1 - \beta)^2 = (1 - \alpha) \left\{ 1 + \beta^2 - (1 + 2 \beta - \beta^2) \alpha \right\},
\]

so that \( \hat{A} > 0 \) if (and only if) \( 1 + \beta^2 - (1 + 2 \beta - \beta^2) \alpha > 0 \), or equivalently

\[
\alpha < \overline{\alpha} = \frac{1 + \beta^2}{1 + 2 \beta - \beta^2}.
\]

Here, it is not difficult to see \( \beta < \overline{\alpha} < 1 \). Next we have

\[
\hat{B} = C + A - B = (1 - \alpha)(1 - \beta) \{(1 + \alpha)\beta + 1 - \alpha \} > 0
\]

and

\[
\hat{C} = A + B - C = (1 - \beta) \{(1 - \beta)\alpha^2 + 2 \alpha - (1 + \beta) \},
\]

so that \( \hat{C} > 0 \) if (and only if) \( (1 - \beta)\alpha^2 + 2 \alpha - (1 + \beta) > 0 \), or equivalently

\[
(1 >) \alpha > \underline{\alpha} = \frac{1 + \sqrt{2 - \beta^2}}{1 - \beta}.
\]

Here, by an elementary computation we can see \( \underline{\alpha} < \overline{\alpha} < 1 \), so that we have (23). Now from Lemma 2.4 we have the following three subcases.

I-(1): If \( \beta < \alpha \leq \underline{\alpha} \), then \( \hat{A}, \hat{B}, \hat{C} > 0 \), \( \hat{C} \leq 0 \), so that

\[
T(a, \phi, \psi; p) \leq C \left( \frac{1}{4} - \nu^2 \right) \leq C_1 \left( \frac{1}{4} - \nu^2 \right).
\]

I-(2): If \( \underline{\alpha} < \alpha < \overline{\alpha} \), then \( \hat{A}, \hat{B}, \hat{C} > 0 \), so that

\[
T(a, \phi, \psi; p) \leq \frac{ABC}{D} - C \nu^2 - \frac{D}{4C} \lambda^2.
\]

Here, by an elementary computation we can see that

\[
D = (1 + \alpha)(1 + \beta)(1 - \alpha)^2 (1 - \beta)^2 \left\{ (3 - \beta)\alpha - (1 + \beta) \right\}
\]
and
\[
\frac{ABC}{D} = \frac{\alpha^2(\alpha - \beta)^2}{(1 + \alpha)(1 + \beta)(3 - \beta)\alpha - (1 + \beta)}.
\]

I-(3): If \( \pi \leq \alpha < 1 \), then \( \hat{A} \leq 0 \), \( \hat{B} > 0 \) and \( \hat{C} > 0 \), so that
\[
T(a, b; p) \leq A \left( \frac{1}{4} - \nu^2 \right) \leq C_1 \left( \frac{1}{4} - \nu^2 \right).
\]

Case II: \( 0 \leq s \leq t \leq n \). Let
\[
a^{(s)} = (\alpha, \ldots, \alpha, 1, \ldots, 1, \ldots, 1) \quad \text{and} \quad b^{(t)} = (\beta, \ldots, \beta, \beta, \ldots, \beta, 1, \ldots, 1).
\]

Then similarly as in Case I, we have
\[
T(a^{(s)}, b^{(t)}; p) = \beta^2(1 - \alpha)^2 P_0 + (\alpha - \beta)^2 P_1 + (1 - \beta)^2 (P_0 - P_1)(1 - P_i)
\]
\[
= A_1 P_0 + B_1 P_1 + C_1 (P_0 - P_1)(1 - P_i).
\]

For the signs of the constants \( \hat{A}_1 \), \( \hat{B}_1 \) and \( \hat{C}_1 \), we have
\[
\hat{A}_1 = B_1 + C_1 - A_1 = (1 - \beta) \left\{ 1 + \alpha^2 - \beta(1 + 2\alpha - \alpha^2) \right\}
\]
\[
\geq (1 - \beta) \left\{ 1 + \alpha^2 - \alpha(1 + 2\alpha - \alpha^2) \right\}
\]
\[
= (1 - \beta)(1 + \alpha)(1 - \alpha^2) > 0,
\]

\[
\hat{B}_1 = C_1 + A_1 - B_1 = (1 - \alpha)(1 - \beta)^2 > 0.
\]

and
\[
\hat{C}_1 = A_1 + B_1 - C_1 = (1 - \alpha) \left\{ -1 + 2\beta + \beta^2 - \alpha(1 + \beta^2) \right\}
\]
\[
\leq (1 - \alpha) \left\{ -1 + 2\beta + \beta^2 - \beta(1 + \beta^2) \right\}
\]
\[
= -(1 - \alpha)(1 - \beta)(1 - \beta^2) \leq 0.
\]

Hence by Lemma 2.4 we have
\[
T(a, b; p) \leq C_1 \left( \frac{1}{4} - \nu^2 \right).
\]

Summarizing Cases I and II, we obtain the desired facts in the theorem. \( \square \)

**Theorem 5.2** With the same notations and the same assumptions as in Theorem 5.1,
\[
T(a, b; p) \leq \frac{M_1^2 M_2^2 C_1}{4} = \frac{M_1^2 M_2^2 (1 - \beta)^2}{4}.
\]

If there is an integer \( t = t_0 \) such that \( P_{t_0} = 1/2 \), then
\[
T_{\max}(= \text{the maximum of } T(a, b; p)) = \frac{M_1^2 M_2^2 C_1}{4},
\]

which is attained at \((a', b')\) such that
\[
a' = (M_1, \ldots, M_1) \quad \text{and} \quad b' = (m_2, \ldots, m_2, M_2, \ldots, M_2).
\]
Proof. By Theorem 5.1, we have only to show that if \( \alpha < \alpha < \pi \), (or if \( \tilde{A}, \tilde{B} \) and \( \tilde{C} > 0 \)) then

\[
\frac{ABC}{D} < \frac{C_1}{4},
\]

or \( \frac{ABC}{D} < \frac{B + C}{4} \) because

\[
B + C = (\alpha - \beta)^2 + (1 - \alpha)^2 < (1 - \beta)^2 = C_1.
\]

Since

\[
\frac{B + C}{4} - \frac{ABC}{D} = \frac{(B + C)(4BC - \tilde{A}^2) - 4(B + C - \tilde{A})BC}{4D},
\]

we have to show \( (B + C)(4BC - \tilde{A}^2) - 4(B + C - \tilde{A})BC > 0 \). Note that \( D = 4BC - \tilde{A}^2 \) and \( A = B + C - \tilde{A} \), so that we have

\[
(B + C)(4BC - \tilde{A}^2) - 4(B + C - \tilde{A})BC = \tilde{A}\{A(B + C) - (B - C)^2\} \geq \tilde{A}\{A^2 - (B - C)^2\} \quad (\text{cf. } B + C > A)
\]

\[
= \tilde{A}\tilde{B}\tilde{C} > 0.
\]

\[\square\]

**Remark 5.3** Related to Theorem 5.2 (and also Theorem 4.3), we ask if the value \( T_p(a'', b''; p) = T(a'', b''; p) = \frac{M^2M^2_{ABC}}{D} \) at the point \( (a'', b'') \) with

\[
a'' = (m_1, \ldots, m_1, M_1, \ldots, M_1) \quad \text{and} \quad b'' = (m_2, \ldots, m_2, M_2, \ldots, M_2)
\]

is the maximum of \( T_p(a, b) \), whenever \( \tilde{A}, \tilde{B}, \tilde{C} > 0 \) and there are integers \( s = s_0, t = t_0 \) satisfying

\[
P_{s_0} = \frac{CC}{D} \quad \text{and} \quad P_{s_0} - P_{t_0} = \frac{B \tilde{B}}{D}.
\]

Unfortunately, this is not true. In fact, if \( P_{s_0} = \frac{CC}{D} \) is 'sufficiently near' to \( 1/2 \), then for the point \( (a', b') \) with

\[
a' = (M_1, \ldots, M_1) \quad \text{and} \quad b' = (m_2, \ldots, m_2, M_2, \ldots, M_2),
\]

we have

\[
T_p(a', b') = M^2_1 M^2_2 T(a^{(n)}_{1}, b^{(t_0)}; p) = C_1 P_{s_0} (1 - P_{t_0})
\]

\[
= C_1 \left\{ \frac{1}{4} - (\frac{1}{2} - P_{t_0})^2 \right\} = \frac{C_1}{4} - C_1 \epsilon^2 > \frac{ABC}{D} \quad \left( \epsilon = \frac{1}{2} - P_{t_0} \right)
\]

by the inequality (25).

Concerning the preceding remark, as a numerical example, let \( M_1 = M_2 = 1, \quad m_1 = \alpha = \frac{7}{16} \) and \( m_2 = \beta = \frac{1}{2} \), then \( A = \frac{48}{100}, \quad B = \frac{1}{25}, \quad C = \frac{8}{100}, \quad C_1 = \frac{1}{4}, \quad D = \frac{2295}{100}, \ldots \) If we put
A WEIGHTED VERSION OF OZEKI’S INEQUALITY

\[ n = 3 \text{ and } p = (p_1, p_2, p_3) = \left( \frac{CC}{D}, \frac{BC}{D}, \frac{CB}{D} \right) = \left( \frac{1044}{2295}, \frac{1104}{2295}, \frac{1417}{2295} \right), \text{ then for } s_0 = 2, t_0 = 1, \]
that is, for \( a'' = (\frac{1}{3}, \frac{1}{3}, 1) \), \( b' = (\frac{1}{3}, 1, 1) \), we have
\[
T(a'', b''; p) = \frac{ABC}{D} = \frac{196}{6375} = 0.0307...
\]

On the other hand, for \( s_0 = 0, t_0 = 1 \), that is, for \( a' = (1, 1, 1) \), \( b' = (\frac{1}{3}, 1, 1) \), we have
\[
T(a', b'; p) = C_1 P_1 (1 - P_1) = \frac{4031}{65025} = 0.0619... > \frac{ABC}{D}.
\]

**Corollary 5.4** With the same notations and the same assumptions as in Theorem 5.1, in particular, if the weight \( p = (p_1, \ldots, p_n) \) is uniform, that is, \( p_1 = \cdots = p_n = 1/n \), and if \( n \) is even, then
\[
T_{\text{max}} = \frac{M_1^2 M_2^2 (1 - \beta)^2}{4}
\]

### 6 A concluding remark
We can show corresponding continuous or measurable versions of all results in this paper. For example, corresponding to Theorem 3.2, we obtain the following:

**Theorem 6.1** Let \( f \) and \( g \) be positive measurable functions on a finite measure space \((\Omega, \mu)\) with \( \mu(\Omega) = 1 \). Assume that \( m_1 \leq f \leq M_1, m_2 \leq g \leq M_2, 0 < m_1 < M_1 \) and \( 0 < m_2 < M_2 \). Further assume that \( \alpha = m_1/M_1 \geq m_2/M_2 = \beta \). Then
\[
\int_{\Omega} f^2 d\mu \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} fg d\mu \right)^2
\leq M_1^2 M_2^2 \sup_{X \subseteq \Omega} \left\{ \frac{(1 - \alpha\beta)^2}{4} (1 - \mu(X))^2 + (1 - \beta)^2 \mu(X) \left( 1 - \mu(X) \right) \right\}
\leq \frac{(M_1 M_2 - m_1 m_2)^2}{3}.
\]

To sketch the proof, let \( \{X_1, \ldots, X_n\} \) be a decomposition of measurable sets in \( \Omega \) and let \( x \in X_k \) \( (k = 1, \ldots, n) \). Then from Theorem 3.2 we have
\[
\sum_{k=1}^{n} f(x_k)^2 \mu(X_k) \sum_{k=1}^{n} g(x_k)^2 \mu(X_k) - \left( \sum_{k=1}^{n} f(x_k)g(x_k) \mu(X_k) \right)^2
\leq M_1^2 M_2^2 \sup_{X \subseteq \Omega} \left\{ \frac{(1 - \alpha\beta)^2}{4} (1 - \mu(X))^2 + (1 - \beta)^2 \mu(X) \left( 1 - \mu(X) \right) \right\}.
\]

Taking the limit of the decomposition we obtain the desired inequality.

**Acknowledgment.** The authors would like to express their thanks to Professor M. Fujii and Professor J. I. Fujii for their valuable comments.
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