UNCOUNTABLE LEVEL SETS OF LIPSCHITZ FUNCTIONS AND ANALYTIC SETS

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Abstract. We show that a subset of the interval $[0, 1]$ is an analytic set with Lebesgue measure zero if and only if it coincides with the set of values of uncountable order of some Lipschitz function from $[0, 1]$ into $[0, 1]$.

1. Introduction

Let $f : [0, 1] \rightarrow \mathbb{R}$. A value $y$ of the function $f$ is said to be of uncountable order if the set $f^{-1}([y, 1])$ is uncountable. The characterization of the set of points where level sets of continuous functions are uncountable is a very old result of S. Mazurkiewicz and W. Sierpinski ([5], [4]). Their characterization is as follows.

Theorem 1.1. A subset of the interval $[0, 1]$ is analytic if and only if it coincides with the set of values of uncountable order of some continuous function from $[0, 1]$ into $[0, 1]$.

Recently, in a joint paper with U. B. Darji, we have characterized the set of points where level sets of $C^1$ functions are uncountable [3]. Our result is as follows.

Theorem 1.2. A subset of the interval $[0, 1]$ coincides with the set of values of uncountable order of some $C^1$ function $f : [0, 1] \rightarrow [0, 1]$ if and only if it is analytic and its closure has Lebesgue measure zero.

In this paper we characterize such sets for Lipschitz functions. Our characterization is as follows.

Theorem 1.3. Let $M$ be a subset of $[0, 1]$. Then $M$ is equal to $\{y : f^{-1}([y, 1])$ is uncountable $\}$ for some Lipschitz function $f : [0, 1] \rightarrow [0, 1]$ if and only if $M$ is an analytic set with Lebesgue measure zero.

2. Uncountable Level Sets

We proceed towards the goal of this paper. We first need few definitions and terminology. Throughout, $\lambda$ denotes the Lebesgue measure, and $\pi_1$ and $\pi_2$ denote coordinate projections.

Definition 2.1. Let $f$ be a Lipschitz function on a closed interval $I$. By $U_I$, $D_I$ and $\hat{U}_{(f, I)}$, we denote the sets

$$\{y : f^{-1}([y]) \text{ is uncountable}\}.$$
\{x : f \text{ is differentiable at } x\}
and
\{x \in D_f : f^{(1)}(x) = 0\},
respectively.

**Theorem 2.2.** ([2]: Lemma 1) If \( f \) is a continuous function of bounded variation on \([0,1]\), there exists a homeomorphism \( h \) of \([0,1]\) onto itself such that \( f \circ h \) is a Lipschitz function.

**Definition 2.3.** A box is a set of the form \( I \times J \) where \( I, J \) are compact intervals.

**Definition 2.4.** Let \( I \) be a closed interval. Then, we use \( I_L, I_M, I_R \) to denote the left third, middle third and the right third intervals of \( I \), respectively. If \( B = I \times J \) is a box, then \( B_L = I_L \times J, B_M = I_M \times J, \text{ and } B_R = I_R \times J \). We call \( B_L, B_M, B_R \) the vertical splitting of \( B \).

**Definition 2.5.** Let \( B \) be a box. A continuous function \( f \) is diagonal to \( B \) if the restriction of \( f \) to \( B \) is a linear function which passes through the diagonal corners of \( B \).

**Definition 2.6.** A continuous function \( f \) is said to be jagged inside box \( B \) if \( f \) is diagonal to each of \( B_L, B_M, B_R \).

Henceforth, we shall denote by \( CBV \) continuous functions of bounded variation, and, given a \( CBV \) function \( f \), we shall denote by \( V(f) \) its variation.

**Lemma 2.7.** Let \( I = [a, b], \ J = [c, d], B = I \times J \) and \( \epsilon > 0 \). Let \( \{C_i\}_{i \in \mathbb{N}} \) be a sequence of closed subsets of \( J \), let \( A \) be a subset of \( J \) with \( \lambda (A) = 0 \) and \( A \cap C_i \neq \emptyset \) for every \( i \). Then, there is a \( CBV \) function \( f \) from \( I \) onto \( J \) and there is a sequence \( \{G_i\}_{i \in \mathbb{N}} \) of countable collections of boxes contained in \( B \) such that
1. the variation of \( f \) on \( I \) is less than \( \lambda (J) + \epsilon \),
2. \( f^{-1}(\{y\}) \) is countable for all \( y \in J \),
3. \( f(a) = c, f(b) = d \),
4. \( f \) is linearly jagged in each \( B' \in \bigcup_{i=1}^{\infty} G_i \),
5. if \( i \neq j \), then \( G_i \cap G_j = \emptyset \) and \( \bigcup_{i=1}^{\infty} G_i \) is a pairwise disjoint collection,
6. \( A \cap C_i \subseteq \bigcup_{i=1}^{\infty} G_i \) and \( \bigcap_{i=1}^{\infty} G_i \cap C_i \neq \emptyset \) for all \( B' \in G_i \), and
7. \( \text{diam}(B') < \epsilon \) for all \( B' \in G_i \).

**Proof.** We will construct a sequence \( \{f_k\}_{k \in \mathbb{N}} \) of \( CBV \) functions whose uniform limit is the desired function.

Let \( f_0 : I \rightarrow J \) be a linear function which satisfies Condition 3 of the Lemma. Let \( J_1^1, J_2^1, \ldots, J_n^1, \ldots \) be a sequence of non-overlapping closed intervals contained in \( J \) with the following properties:

a. \( A \cap C_1 \subseteq \bigcup_{i=1}^{\infty} J_i^1, A \cap C_1 \cap J_i^1 \neq \emptyset \),
b. \( \sum_{i=1}^{\infty} \lambda (J_i^1) < \frac{\epsilon}{2} \) and \( \lambda (f_0^{-1}(J_1^1)) < \frac{\epsilon}{2} \).

Let \( J_i^1 = f_0^{-1}(J_1^1) \). In each of \( J_i^1 \), replace \( f_0 \) by an appropriate continuous function which is jagged in \( (I_i^1 \times J_i^1)_L \), diagonal to \( (I_i^1 \times J_i^1)_M \) and diagonal to \( (I_i^1 \times J_i^1)_R \). Let \( f_1 \) be the resulting continuous piecewise linear function and let \( G_1 = \{(I_i^1 \times J_i^1)_L : i \in \mathbb{N}\} \). Then, at the end of stage 1, the following properties are satisfied:

i. \( f_1 \) is a continuous function linearly jagged inside each \( B' \in G_1 \) with \( f_1(a) = c \) and \( f_1(b) = d \),
ii. \( \left| f_1^{-1}(\{y\}) \right| \leq 5 \) for all \( y \in J \),
(iii) $f_1$ is a CBV function and

$$V(f_1) \leq V(f_0) + 5 \sum_{i=1}^{\infty} \lambda(J^1_i)$$

$$< V(f_0) + 5 \cdot \frac{\varepsilon}{5 \cdot 2}$$

$$= \lambda(J) + \frac{\varepsilon}{2}.$$ 

(iv) $A \cap C_1 \subseteq \pi_2(\bigcup G_1)$ and $\pi_2(B') \cap A \cap C_1 \neq \emptyset$ for all $B' \in G_1,$

(v) $\|f_i - f_0\|_0 \leq \sum_{i=1}^{\infty} \lambda(J^i_i) < \frac{\varepsilon}{5 \cdot 2}.$ 

Let us now construct $f_2.$ Let $J^2_1, J^2_2, \ldots, J^2_n, \ldots$ be a sequence of non-overlapping closed intervals contained in $J$ such that

a. either there is $i'$ such that $f^2_i \subseteq J^2_i$ or $J^2_i$ does not overlap with any $J^1_s, s \in \mathbb{N},$

b. $A \cap C_2 \subseteq \bigcup_{i=1}^{\infty} f^2_i, A \cap C_2 \cap J^2_i \neq \emptyset,$

c. $\sum_{i=1}^{\infty} \lambda(J^2_i) < \frac{\varepsilon}{5 \cdot 2}$ and $\lambda(f^{-1}_1(J^1_i)) < \frac{\varepsilon}{2}.$

Now, if there is $i'$ such that $J^2_i \subseteq J^1_i,$ we let $I^2 = f^{-1}_1(J^1_i) \cap (I^1_i)_R.$ otherwise we let $I^2 = f^{-1}_1(J^1_i).$ Let $f_2$ be the modification of $f_1$ on $\bigcup I^2_i \times J^2_i$ as earlier and $G_2 = \{ (I^2_i \times J^2_i) : i \in \mathbb{N} \}.$ Then, at the end of stage 2, the following properties are satisfied:

(i) $f_2$ is a continuous function linearly jagged inside each $B' \in G_2$, with $f_2(a) = c$ and $f_2(b) = d,$

(ii) $|f_2^{-1}(\{y\})| \leq 5 + (2 - 1) \cdot 4$ for all $y \in J,$

(iii) $f_2$ is a CBV function and

$$V(f_2) \leq V(f_1) + 5 \sum_{i=1}^{\infty} \lambda(J^2_i)$$

$$< V(f_1) + 5 \cdot \frac{\varepsilon}{5 \cdot 2},$$

$$= V(f_1) + \frac{\varepsilon}{2 \cdot 2}.$$ 

(iv) $A \cap C_2 \subseteq \pi_2(\bigcup G_2)$ and $\pi_2(B') \cap A \cap C_2 \neq \emptyset$ for all $B' \in G_2,$

(v) $\|f_2 - f_1\|_0 \leq \sum_{i=1}^{\infty} \lambda(J^2_i) < \frac{\varepsilon}{5 \cdot 2}.$

Now let us assume that we are at stage $k > 1,$ $f_k$ and $G_k$ have been constructed so that the following properties are satisfied:

(i) $f_k$ is a continuous function linearly jagged inside each $B' \in G_k$, with $f_1(a) = c$ and $f_k(b) = d,$

(ii) $|f_k^{-1}(\{y\})| \leq 5 + (k - 1) \cdot 4$ for all $y \in J,$

(iii) $f_k$ is a CBV function and

$$V(f_k) < V(f_{k-1}) + \frac{\varepsilon}{2^k},$$

(iv) $A \cap C_k \subseteq \pi_2(\bigcup G_k)$ and $\pi_2(B') \cap A \cap C_k \neq \emptyset$ for all $B' \in G_k,$

(v) $\|f_k - f_{k-1}\|_0 \leq \sum_{i=1}^{\infty} \lambda(J^k_i) < \frac{\varepsilon}{5 \cdot 2^k}.$

Let us now construct $f_{k+1}.$ Let $J^{k+1}_1, J^{k+1}_2, \ldots, J^{k+1}_n, \ldots$ be a sequence of non-overlapping closed intervals contained in $J$ such that

a. either there is $i'$ such that $J^{k+1}_i \subseteq J^k_i$ or $J^{k+1}_i$ does not overlap with any $J^k_s, s \in \mathbb{N},$

b. $A \cap C_{k+1} \subseteq \bigcup_{i=1}^{\infty} J^{k+1}_i, A \cap C_{k+1} \cap J^{k+1}_i \neq \emptyset,$

c. $\sum_{i=1}^{\infty} \lambda(J^{k+1}_i) < \frac{\varepsilon}{5 \cdot 2^k}$ and $\lambda(f^{-1}_k(J^{k+1}_i)) < \frac{\varepsilon}{2^k}$.
Now, if there is $i'$ such that $J_{i'}^{k+1} \subseteq J_i^k$, we let $I_i^{k+1} = f_{i'}^{-1}((J_i^k) \cap (I_i^k))_R$, otherwise we let $I_i^{k+1} = f_{i'}^{-1}(J_i^{k+1})$. Let $f_{k+1}$ be the modification of $f_k$ on $\cup_i I_i^{k+1} \times J_i^{k+1}$ as earlier and $G_{k+1} = \{(I_i^{k+1} \times J_i^{k+1})_R : i \in \mathbb{N}\}$.

Now, it is easy to verify that $f_{k+1}$ satisfies all induction hypotheses of stage $k + 1$ except (iii). In order to prove (iii) we notice that

$$V(f_{k+1}) \leq V(f_k) + 5 \cdot \sum_i \lambda(J_i^k)$$

$$\leq V(f_k) + 5 \cdot \frac{\epsilon}{5 \cdot 2^{k+1}}$$

$$= V(f_k) + \frac{\epsilon}{2^{k+1}}$$

By (v), we have that $\{f_k\}$ converges uniformly to some continuous function $f$. By (iii) we have that

$$V(f) \leq \lambda(J) + \epsilon.$$ 

Hence, $f$ is of bounded variation. Clearly, $f$ satisfies the required conditions. \hfill \qed

**Definition 2.8.** We will use $\tau, \sigma$ etc. to denote an element of $\mathcal{N}^{<\mathcal{N}}$ (= finite sequences of elements of $\mathcal{N}$) or $\mathcal{N}^{\mathcal{N}}$. We use $|\tau|$ to denote the length of $\tau$ and, if $|\tau| > k$, then $\sigma[k]$ to denote the restriction of $\sigma$ to the first $k$ coordinates, and $\sigma(k)$ to denote the $k$-th coordinate of $\sigma$. If $\sigma$ is a finite string and $k$ is a positive integer, then $\sigma[k]$ denotes the concatenation of $\sigma$ followed by $k$.

**Proposition 2.9.** Let $A \subseteq [0,1]$ be an analytic set with $\lambda(A) = 0$. Then, there is a CBV function $f : [0,1] \rightarrow [0,1]$ such that

(i) $f^{-1}(\{y\})$ is uncountable for all $y \in A$, and

(ii) $f^{-1}(\{y\})$ is countable for all $y \notin A$.

**Proof.** As $A$ is an analytic subset of $[0,1]$, we may obtain a Suslin scheme $[1] \{C_{\tau}\}_{\tau \in \mathcal{N}^{<\mathcal{N}}}$ such that

(a) each of $C_{\tau}$ is a closed subset of $[0,1]$,

(b) for each $\sigma \in \mathcal{N}^{\mathcal{N}}$, $C_{\sigma[k+1]} \subseteq C_{\sigma[k]}$, and $\text{diam}(C_{\sigma[k]}) < \frac{1}{2^k}$,

(c) $A = \bigcup_{\sigma \in \mathcal{N}^{<\mathcal{N}}} \bigcap_{k=1}^{\infty} C_{\tau[k]}$.

We will construct the desired $f$ as the uniform limit of a sequence of continuous functions $\{f_k\}_{k \in \mathcal{N}}$. Let $f_0 : [0,1] \rightarrow [0,1]$ be the identity map and let $G_0 = \{[0,1] \times [0,1], \{C_{\tau}\}_{\tau \in \mathcal{N}^{<\mathcal{N}}}, A \text{ and } \epsilon = \frac{1}{2}, \}$, we obtain a function $f_1$ and a sequence of countable collections of boxes $\{\mathcal{H}_i\}_{i \in \mathcal{N}}$ which satisfy the conclusion of Lemma 2.7. Let $G_1 = \{B^i, B^i_{\sigma} : B^i \in \mathcal{H}_i \text{ for some } i\}$. For each $B^i \in \mathcal{H}_i$, define $\phi_i(B^i) = \phi_i(B^i_{\sigma}) = i$. Note that $\phi_1$ is well-defined as $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$ for $i \neq j$. Then, $f_1, \phi_1$ and $G_1$ satisfy the following conditions:

1. $f_1$ is a CBV function with $V(f_1) < 1 + \frac{\epsilon}{2}$,

2. $f_1 = f_0$ outside $\pi_1(\mathcal{G}_1)$,

3. $f_1^{-1}(\{y\})$ is countable for all $y$,

4. $G_1$ is a pairwise disjoint collection,

5. $f_1$ is diagonal to each $B^i \in G_1$,

6. for each $\sigma \in \mathcal{N}^{\mathcal{N}}$, $\phi_1^{-1}(\{\sigma\})$ is a countable collection of boxes and $A \cap C_{\tau} \subseteq \tau_2(\mathcal{G}_1(\{\sigma\}))$, and if $B^i \in \phi_1^{-1}(\{\sigma\})$, then $\tau_2(\mathcal{G}_1) \cap A \cap C_{\tau} \neq \emptyset$, and

7. for each $B^i \in G_1$, $\text{diam}(B^i) < \frac{1}{4}$.
Now, suppose that we are at stage \( k > 1 \) and a function \( f_k \), a countable collection of boxes \( \mathcal{G}_k \) contained in the unit square and a function \( \phi_k : \mathcal{G}_k \to \mathbb{N} \) have been constructed such that:

1. \( f_k \) is a CBV function with \( V(f_k) < 1 + \sum_{i=1}^{k} \frac{1}{2^i} \),
2. \( f_k^{-1}(\{y\}) \) is countable for all \( y \),
3. \( f_k = f_{k-1} \) outside \( \pi_1(\mathcal{U}\mathcal{G}_k) \),
4. \( \mathcal{G}_k \) is a pairwise disjoint collection,
5. \( f_k \) is diagonal to each \( B' \in \mathcal{G}_k \),
6. for each \( \sigma \in \mathbb{N}^k \), \( \phi_k^{-1}(\{\sigma\}) \) is a countable collection of boxes and \( A \cap C_\sigma \subseteq \pi_2(\cup \phi_k^{-1}(\{\sigma\})) \),
   and if \( B' \subseteq \phi_k^{-1}(\{\sigma\}) \), then \( \pi_2(B') \cap A \cap C_\sigma \neq \emptyset \),
7. for each \( \sigma \in \mathbb{N}^k \), \( \cup \phi_k^{-1}(\{\sigma\}) \subseteq \cup \phi_k^{-1}(\{\sigma|k-1\}) \),
8. for each \( B' \in \mathcal{G}_k \), \( d\text{iam}(B') < \frac{1}{10^k} \), and
9. for each \( B' \in \mathcal{G}_{k-1} \) with \( \tau = \phi_{k-1}(B') \) and for positive integer \( m \), we have that if \( \pi_2(B') \cap C_{\tau m} \neq \emptyset \), then for each \( y \in A \cap C_{\tau m} \cap \pi_2(B') \), there are two disjoint boxes \( B_1, B_2 \in \mathcal{G}_k \) with \( B_1 \cup B_2 \subseteq B' \) such that \( y \in \pi_2(B_1) \cap \pi_2(B_2) \) and \( \phi_k(B_1) = \phi_k(B_2) = \tau m \).

Let us now define \( f_{k+1} \). Enumerate \( \mathcal{G}_k \) as \( B_1, B_2, \ldots \). Let \( l \geq 1 \) and \( \sigma = \phi_k(B_l) \). If there is no \( i \) so that \( \pi_2(B_i) \cap A \cap C_\sigma \neq \emptyset \), then we let \( g_l = f_k \) on \( \pi_1(B_i) \). Otherwise, we apply Lemma 2.7 to \( B_i, \pi_2(B_i) \cap A \cap C_\sigma \), \( i = 1, 2, \ldots \) (listing only the non-empty ones), \( A \cap \pi_2(B_l) \) and \( \sigma = \phi_k(B_l) \), and obtain a function \( g_l \) and a sequence of countable collections of boxes \( \{H_i^l\}_{i \in \mathbb{N}} \) which satisfy the conclusion of Lemma 2.7. For each \( B' \in H_i^l \), define \( \phi_{k+1}^l(B'_i) = \phi_{k+1}^l(B_i') = \sigma i \). We do this for each \( l \) and let \( f_{k+1} = f_k \) outside \( \cup_{i=1}^\infty \pi_1(B_i) \) and \( f_{k+1} = g_l \) on \( \pi_1(B_i) \). We let \( \mathcal{G}_{k+1} = \{B_i^l, B_i'^l : B' \in H_i^l \text{ for some } i, l\} \) and let \( \phi_{k+1} \) be the union of all the partial \( \phi_{k+1}^l \). These \( f_{k+1}, \mathcal{G}_{k+1}, \phi_{k+1} \) satisfy the induction hypotheses. As \( f_{k+1} \) is continuous and modified only inside boxes of stage \( k \) and these boxes have diameters less than \( \frac{1}{10^k} \), we have that \( \{f_k\}_{k \in \mathbb{N}} \) converges uniformly to some continuous function \( f \). By induction hypothesis 1, we have that \( f \) is also a CBV function.

Let us now show that \( f^{-1}(\{y\}) \) is uncountable for \( y \in A \) and countable otherwise. We shall prove that \( y \in A \) if and only if \( f^{-1}(\{y\}) \) is uncountable.

(\( \Rightarrow \)) Let \( y \in A \). Let \( \sigma \in \mathbb{N}^N \) be such that \( y \in \cap_{n=1}^\infty C_{\sigma|n} \). Applying induction hypothesis 9 at stage \( k = 1 \) with \( B = [0,1] \times [0,1] \), we may obtain two disjoint boxes \( B_0^y \) and \( B_1^y \) in \( \mathcal{G}_1 \) such that \( y \in \pi_2(B_0^y) \cap \pi_2(B_1^y) \) and that \( \phi_1(B_0^y) = \phi_1(B_1^y) = \sigma|1 \). Now suppose that \( k \geq 1 \) and we have \( 2^k \) many pairwise disjoint boxes \( B_0^y, B_1^y, \ldots, B_{2^k-1}^y \in \mathcal{G}_k \) with \( B_0^y \cap B_1^y = \emptyset \) for each \( B_0^y \in \mathcal{G}_k \). Applying induction hypothesis 9 at stage \( k+1 \) to each \( B_0^y \), for \( \sigma \in \{0,1\}^k \) and \( m = \sigma(k+1) \), we obtain an analogous appropriate collection of boxes at stage \( k+1 \). Now, it is easy to verify that the Cantor set \( \cap_{n=0}^\infty \pi_1(B_0^y|n) \) maps to \( y \) under \( f \).

(\( \Leftarrow \)) Let \( f^{-1}(\{y\}) \) be uncountable. As \( f = f_1 \) outside \( \pi_1(\mathcal{U}\mathcal{G}_1) \) and \( f_1^{-1}(\{y\}) \) is countable, we have that there is \( B_1 \in \mathcal{G}_1 \) such that \( B_1 \) contains uncountably many points of the graph of \( f \) whose second coordinate is \( y \). Let \( l_1 = \phi_1(B_1) \). By a similar argument, there is to be \( B_2 \in \mathcal{G}_2 \) such that \( B_2 \) contains uncountably many points of the graph of \( f \) whose second coordinate is \( y \) and \( B_2 \subseteq B_1 \). By induction hypotheses 4 and 7, we have that \( \phi_2(B_2) = (l_1, l_2) \) for some \( l_2 \). Continuing in this fashion, we obtain a sequence of boxes \( \{B_k\}_{k \in \mathbb{N}} \) and a sequence of integers \( \{l_k\}_{k \in \mathbb{N}} \) such that \( y \in \pi_2(B_k), B_k \in \mathcal{G}_l, B_{k+1} \subseteq B_k \) and \( \phi_k(B_k) = \sigma k \) where \( \sigma = (l_1, l_2, \ldots) \). From Condition 6 we have that \( \pi_2(B_k) \cap A \cap C_{\sigma|k} \neq \emptyset \), and from Condition 8 that \( \text{iam}(B_k) \rightarrow 0 \) as \( k \rightarrow \infty \). Hence, \( y \in \cap_{n=1}^\infty C_{\sigma|n} \). Therefore, \( y \in A \). \( \square \)
Proposition 2.10. Let $A$ be an analytic subset of $[0,1]$ with $\lambda(A) = 0$. Then, there is a Lipschitz function $f : [0,1] \to [0,1]$ such that

(i) $f^{-1}(\{y\})$ is uncountable for all $y \in A$, and
(ii) $f^{-1}(\{y\})$ is countable for all $y \notin A$.

Proof. By Proposition 2.9 there exists a CBV function $g : [0,1] \to [0,1]$ such that $g^{-1}(\{y\})$ is uncountable for all $y \in A$ and countable otherwise. Applying Theorem 2.2 we obtain a homeomorphism $h$ from $[0,1]$ onto $[0,1]$ such that $g \circ h$ is a Lipschitz function. Now, $f = g \circ h$ is the desired function. $\square$

Proposition 2.11. Suppose that $f : [0,1] \to \mathbb{R}$ is a Lipschitz function. Then, the set of points where level sets are uncountable is an analytic set with Lebesgue measure zero.

Proof. By a very old result of S. Mazurkiewicz and W. Sierpinski [5] $U_f$ is an analytic set. Let $U_f^1 = \{y \in U_f : \exists x \in D_f \text{ such that } f(x) = y\}$. As $f$ is Lipschitz, it follows that $\lambda(U_f^1 \setminus U_f^2) = 0$. As for every $y \in U_f^1 \setminus U_f^2$ there is a perfect set $y \in \tilde{Z}(f)$, we obtain that $f^{-1}(\{y\})$ is uncountable, it contains a perfect set and hence it is clear that for every $y \in U_f^1$ and $x \in \tilde{Z}(f)$ we have $y = x$. Let $U_f^2$ be the set of all points in $U_f^1$ which are local extremum of $f$. As $U_f^2 \setminus U_f^3$ is at most countable, it has Lebesgue measure equal to zero. Let $\epsilon > 0$. For every $y \in U_f^3$ choose a sequence $\{p_{y,k}\}_{k \in \mathbb{N}}$ converging to $x_y$ such that, for every $k$,

(i) the image under $f$ of the semi-open interval $(x_y, y)$ containing $x_y$ and having as endpoints $x_y$ and $p_{y,k}$ is a non-degenerate interval, and
(ii) $|f(x_y) - f(p_{y,k})| < \epsilon \cdot |x_y - p_{y,k}|$.

Let, for every $y \in U_f^3$ and for every $k$, $f(J_{y,k}) = [f(a_{y,k}), f(b_{y,k})]$. Now, $V_y = \{[f(a_{y,k}), f(b_{y,k})]\}_{k \in \mathbb{N}}$ is a family of non-degenerate intervals containing $y$ and with diameters going to zero. Let $V = \bigcup_{y \in U_f^3} V_y$. Clearly, $V$ is a Vitali covering of $U_f^3$. By the Vitali covering theorem ([1]), there exists a countable sub-collection of pairwise disjoint intervals $\{[f(a_{y,k}), f(b_{y,k})]\}_{i \in \mathbb{N}}$ such that $\lambda(U_f^3 \setminus \bigcup_{i=1}^{\infty} [f(a_{y_i,k_i}), f(b_{y_i,k_i})]) = 0$. The collection is pairwise disjoint since so is $\{[f(a_{y_i,k_i}), f(b_{y_i,k_i})]\}_{i \in \mathbb{N}}$.

Moreover, we have that

\[
\begin{align*}
\left| f(a_{y_i,k_i}) - f(b_{y_i,k_i}) \right| \\ \leq \left| f(a_{y_i,k_i}) - f(x_{y_i}) \right| + \left| f(x_{y_i}) - f(b_{y_i,k_i}) \right| \\ < \epsilon \cdot |a_{y_i,k_i} - x_{y_i}| + |x_{y_i} - b_{y_i,k_i}|. 
\end{align*}
\]

Therefore,

\[
\begin{align*}
\sum_{i=1}^{\infty} \left| f(a_{y_i,k_i}) - f(b_{y_i,k_i}) \right| \\ \leq \sum_{i=1}^{\infty} \left( \left| f(a_{y_i,k_i}) - f(x_{y_i}) \right| + \left| f(x_{y_i}) - f(b_{y_i,k_i}) \right| \right) \\ \leq \epsilon \cdot \sum_{i=1}^{\infty} \left| a_{y_i,k_i} - x_{y_i} \right| + \left| x_{y_i} - b_{y_i,k_i} \right| \\ \leq 2 \cdot \epsilon.
\end{align*}
\]

Hence, $\lambda(U_f) = \lambda(U_f^3) \leq 2 \cdot \epsilon$, for every $\epsilon$. $\square$
Theorem 2.12. Let $M \subseteq [0,1]$. Then the following are equivalent:

1. $M$ is an analytic set with $\lambda(M) = 0$,
2. there is a Lipschitz function $f$ from $[0,1]$ into $[0,1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.

Proof. (1) $\Rightarrow$ (2) This is Proposition 2.10.
(2) $\Rightarrow$ (1) This is Proposition 2.11.

References


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