ON FUZZY QUOTIENT BCI-ALGEBRAS INDUCED BY FUZZY IDEALS

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Received March 1, 2002

Abstract. We define fuzzy quotient BCI-algebras induced by fuzzy ideals and study the relation between fuzzy quotient BCI-algebras and fuzzy ideals. We establish isomorphism theorem.

1. Introduction

For the general development of BCI-algebras, the (fuzzy) ideal theory plays an important role. Of course, the quotient structure by (fuzzy) ideal plays an important role also. In general, the relation "" in a BCI-algebra X defined by x y if and only if x y A and y x A is used, where x, y X and A is an ideal of X, to constructing quotient structure of BCI-algebra induced by an ideal. F. L. Zhang [8] gave an equivalent relation on a BCI-algebra by using a different method, and constructed the corresponding quotient structures. S. M. Hong and Y. B. Jun [1] fuzzified the equivalence relation obtained by Zhang's way, and established a quotient BCI-algebra which is induced by a fuzzy ideal. In this paper, we consider another fuzzification of the equivalence relation given by F. L. Zhang, and construct fuzzy quotient BCI-algebras induced by fuzzy ideals. We establish an isomorphism theorem, and give a characterization for a quotient BCI-algebra induced by a fuzzy ideal to be commutative (positive implicative).

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Recall that a BCI-algebra is an algebra (X, *, 0) of type (2, 0) satisfying the following axioms for every x, y, z X,

(a1) ((x y) * (x z)) * (z y) = 0,
(a2) (x * (x y)) * y = 0,
(a3) x * x = 0,
(a4) x y = 0 and y x = 0 imply x = y.

A partial ordering ≤ on X can be defined by x y if and only if x y = 0. In a BCI-algebra X, the following hold:

(b1) x = 0 = x,
(b2) (x y) * z = (x z) * y,
(b3) 0 * (x y) = (0 x) * (0 y),
(b4) x y implies x z y z and z y ≤ z x.

2000 Mathematics Subject Classification. 06F35, 03G25, 03B52.

Keywords and phrases. Fuzzy (commutative, positive implicative) ideal, fuzzy quotient BCI-algebra induced by fuzzy ideal.
A mapping \( f : X \to Y \) of BCI-algebras is called a homomorphism if \( f(x \cdot y) = f(x) \cdot f(y) \) for all \( x, y \in X \). An ideal of a BCI-algebra \( X \) is defined to be a subset \( A \) of \( X \) containing \( 0 \) such that if \( x \cdot y \in A \) and \( y \in A \) then \( x \in A \). If \( x \) is an element of an ideal \( A \) of a BCI-algebra \( X \) and \( y \leq x \), then \( y \in A \). For any elements \( x \) and \( y \) of a BCI-algebra \( X \) and \( n \in \mathbb{N} \), let us write \( x \cdot y^n \) instead of \( (((x \cdot y) \cdot y) \cdot \cdots) \cdot y \) in which \( y \) occurs \( n \) times.

**Proposition 2.1.** (Huang [2]) For any elements \( x \) and \( y \) of a BCI-algebra \( X \) and \( n \in \mathbb{N} \), we have \( 0 \cdot (x \cdot y)^n = (0 \cdot x^n) \cdot (0 \cdot y^n) \).

We now review some fuzzy logic concepts. Let \( X \) be a set. A fuzzy set in \( X \) is a mapping from \( X \) to \([0, 1]\). In the sequel, we place a bar over a symbol to denote a fuzzy set so \( \bar{A}, \bar{B}, \ldots \) all represent fuzzy sets in \( X \). A fuzzy ideal of a BCI-algebra \( X \) is defined to be a fuzzy set \( \bar{A} \) in \( X \) such that

(F1) \( \bar{A}(0) \geq \bar{A}(x) \) for all \( x \in X \),
(F2) \( \bar{A}(x) \geq \min\{\bar{A}(x \cdot y), \bar{A}(y)\} \) for all \( x, y \in X \).

Note that every fuzzy ideal \( \bar{A} \) of a BCI-algebra \( X \) is order reversing, i.e., if \( x \leq y \) then \( \bar{A}(x) \geq \bar{A}(y) \). A fuzzy ideal \( \bar{A} \) of a BCI-algebra \( X \) is said to be closed if \( \bar{A}(0 \cdot x) \geq \bar{A}(x) \) for all \( x \in X \). A fuzzy set \( \bar{A} \) in a BCI-algebra \( X \) is called a fuzzy commutative ideal if it satisfies (F1) and

(F3) \( \bar{A}(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))), \bar{A}(z)) \) for all \( x, y, z \in X \).

A fuzzy set \( \bar{A} \) in a BCI-algebra \( X \) is called a fuzzy positive implicative ideal if it satisfies (F1) and

(F4) \( \bar{A}(x \cdot z) \geq \min\{\bar{A}((x \cdot z) \cdot x) \cdot (y \cdot z), \bar{A}(y)\} \) for all \( x, y, z \in X \).

### 3. Quotient Structures

Let \( A \) be an ideal of a BCI-algebra \( X \) and let \( n \in \mathbb{N} \). We define a relation “\( \sim \)” on \( X \) as follows:

\[ x \sim y(A) \text{ if and only if } 0 \cdot (x \cdot y)^n \in A \text{ and } 0 \cdot (y \cdot x)^n \in A. \]

Then “\( \sim \)” is a congruence relation on \( X \) (see [8] and [1]).

Let \( X \) be a BCI-algebra and denote by \( A_x \) the equivalence class containing \( x \in X \), and by \( X/A \) the set of all equivalence classes of \( X \) with respect to “\( \sim \)”, that is,

\[ A_x := \{y \in X \mid x \sim y(A)\} \text{ and } X/A := \{A_x \mid x \in X\}. \]

Define a binary operation “\( \circ \)” on \( X/A \) by \( A_x \circ A_y = A_{x \cdot y} \) for all \( A_x, A_y \in X/A \). Then \( (X/A; \circ, A_0) \) is a BCI-algebra (see [8]).

**Theorem 3.1.** If \( A \) is an ideal of a BCI-algebra \( X \), then the mapping \( \phi : X \to X/A \) given by \( \phi(x) = A_x \) is an epimorphism with kernel \( A \).

**Proof.** The map \( \phi : X \to X/A \) is clearly surjective and since

\[ \phi(x \cdot y) = A_{x \cdot y} = A_x \circ A_y = \phi(x) \circ \phi(y), \]

\( \phi \) is an epimorphism. Now

\[ \ker \phi = \{x \in X \mid \phi(x) = A_x = A_0\} = \{x \in X \mid x \in A\} = A. \]

This completes the proof.

**Theorem 3.2.** Let \( f : X \to Y \) be an epimorphism of BCI-algebras. If \( Y \) satisfies the implication \( 0 \cdot x^n = 0 \cdot y^n \Rightarrow x = y \) for every \( x, y \in Y \) and \( n \in \mathbb{N} \), then the quotient algebra \( X/\ker f \) is isomorphic to \( Y \).
Proof. Obviously, \( \text{Ker} f \) is an ideal of \( X \). Let \( x, y \in X \) be such that \( f(x) = f(y) \). Then
\[
f(0 \ast (x \ast y)^n) = f(0) \ast f((x \ast y)^n) = 0 \ast f(x \ast y)^n = 0 \ast (f(x) \ast f(y))^n = 0.
\]
Similarly, \( f(0 \ast (y \ast x)^n) = 0 \) and so \( 0 \ast (x \ast y)^n \in \text{Ker} f \) and \( 0 \ast (y \ast x)^n \in \text{Ker} f \). Hence \( x \sim y(\text{Ker} f) \). This means that \( x \) and \( y \) belong to a class of \( X/\text{Ker} f \). Conversely if \( x \sim y(\text{Ker} f) \), then \( 0 \ast (x \ast y)^n \in \text{Ker} f \) and \( 0 \ast (y \ast x)^n \in \text{Ker} f \), which imply that
\[
0 = f(0 \ast (x \ast y)^n) = f((0 \ast x^n) \ast (0 \ast y^n)) = f(0 \ast x^n) \ast f(0 \ast y^n) = (0 \ast f(x^n)) \ast (0 \ast f(y^n))
\]
and \( 0 \ast f(y^n) \ast 0 \ast f(x^n) = 0 \) by the similar way. It follows from (a4) that \( 0 \ast f(x^n) = 0 \ast f(y^n) \) so from the hypothesis that \( f(x) = f(y) \). Therefore \( X/\text{Ker} f \cong (\text{Ker} f)_x \leftrightarrow f(x) \in Y \) is a one-to-one correspondence between \( X/\text{Ker} f \) and \( Y \). Moreover \( (\text{Ker} f)_x \circ (\text{Ker} f)_y = (\text{Ker} f)_{x,y} \) implies \( f(x) \ast f(y) = f(x \ast y) \). Hence the above correspondence gives the required isomorphism.

Let \( \tilde{A} \) be a fuzzy ideal of a \( BCI \)-algebra \( X \). Define a binary relation "\( \tilde{A} \)" on \( X \) as follows:
\[
ex \sim y(\tilde{A}) \quad \text{if and only if} \quad \tilde{A}(0 \ast (x \ast y)^n) = \tilde{A}(0) = \tilde{A}(0 \ast (y \ast x)^n)
\]
for all \( x, y \in X \) and \( n \in \mathbb{N} \).

**Lemma 3.3.** The binary relation "\( \tilde{A} \)" is an equivalence relation on a \( BCI \)-algebra \( X \).

**Proof.** Obviously, "\( \tilde{A} \)" is reflexive and symmetric. Let \( x, y, z \in X \) be such that \( x \sim y(\tilde{A}) \) and \( y \sim z(\tilde{A}) \). Then
\[
\tilde{A}(0 \ast (x \ast y)^n) = \tilde{A}(0) = \tilde{A}(0 \ast (y \ast x)^n) \quad \text{and} \quad \tilde{A}(0 \ast (y \ast z)^n) = \tilde{A}(0) = \tilde{A}(0 \ast (z \ast y)^n)
\]
for every \( n \in \mathbb{N} \). On the other hand,
\[
(0 \ast (x \ast z)^n) \ast (0 \ast (x \ast y)^n) = ((0 \ast x^n) \ast (0 \ast z^n)) \ast ((0 \ast x^n) \ast (0 \ast y^n))
\]
\[
\leq (0 \ast y^n) \ast (0 \ast z^n) = 0 \ast (y \ast z)^n.
\]
Since \( \tilde{A} \) is order reversing, it follows that
\[
\tilde{A}((0 \ast (x \ast z)^n) \ast (0 \ast (x \ast y)^n)) \geq \tilde{A}(0 \ast (y \ast z)^n)
\]
so from (F2) that
\[
\tilde{A}(0 \ast (x \ast z)^n) \geq \min\{\tilde{A}(0 \ast (x \ast z)^n) \ast (0 \ast (x \ast y)^n)), \tilde{A}(0 \ast (y \ast z)^n))
\]
\[
\geq \min\{\tilde{A}(0 \ast (y \ast z)^n), \tilde{A}(0 \ast (x \ast y)^n))
\]
\[
= \tilde{A}(0).
\]
Clearly \( \tilde{A}(0 \ast (x \ast z)^n) \leq \tilde{A}(0) \) by (F1), and so \( \tilde{A}(0 \ast (x \ast z)^n) = \tilde{A}(0) \). Similarly, we obtain \( \tilde{A}(0 \ast (y \ast z)^n) = \tilde{A}(0) \). Hence \( x \sim z(\tilde{A}) \), which proves the transitivity of "\( \tilde{A} \). This completes the proof.

**Lemma 3.4.** For any elements \( x, y \) and \( z \) of a \( BCI \)-algebra \( X \), \( x \sim y(\tilde{A}) \) implies \( x \sim z(\tilde{A}) \) and \( z \sim x \sim z \sim z(\tilde{A}) \).

**Proof.** If \( x \sim y(\tilde{A}) \), then \( \tilde{A}(0 \ast (x \ast y)^n) = \tilde{A}(0) = \tilde{A}(0 \ast (y \ast x)^n) \) for every \( n \in \mathbb{N} \). Note that
\[
(0 \ast ((x \ast z) \ast (y \ast z))^n) \ast (0 \ast (x \ast y)^n)
\]
\[
= ((0 \ast (x \ast z)^n) \ast (0 \ast (y \ast z)^n)) \ast (0 \ast (x \ast y)^n)
\]
\[
= ((0 \ast x^n) \ast (0 \ast z^n)) \ast ((0 \ast y^n) \ast (0 \ast z^n)) \ast (0 \ast (x \ast y)^n)
\]
\[
\leq (0 \ast x^n) \ast (0 \ast y^n) \ast (0 \ast (x \ast z)^n)
\]
\[
= (0 \ast (x \ast y)^n) \ast (0 \ast (x \ast z)^n)
\]
\[
= 0.
\]
Since \( \tilde{A} \) is order reversing, it follows that
\[
\tilde{A}((0 \ast ((x \ast z) \ast (y \ast z))^n) \ast (0 \ast (x \ast y)^n)) \geq \tilde{A}(0)
\]
so from (F2) that

\[
\begin{align*}
\Delta(0 \ast ((x \ast z) \ast (y \ast z))^n) \\
\geq \min \{\Delta((0 \ast ((x \ast z) \ast (y \ast z))^n) \ast (0 \ast (x \ast y))^n), \Delta(0 \ast (x \ast y)^n)\}
\end{align*}
\]

Obviously, \(\Delta(0 \ast ((x \ast z) \ast (y \ast z))^n) \leq \Delta(0)\) by (F1). Hence

\[\Delta(0 \ast ((x \ast z) \ast (y \ast z))^n) = \Delta(0).\]

Similarly, we get \(\Delta(0 \ast ((y \ast z) \ast (x \ast z))^n) = \Delta(0)\), and therefore \(x \ast z \approx y \ast z(\Delta)\). Similar argument induces \(z \ast x \approx z \ast y(\Delta)\). This completes the proof. \(\square\)

Using Lemma 3.4 and the transitivity of \(\approx\), we have the following lemma.

**Lemma 3.5.** If \(x \approx u(\Delta)\) and \(y \approx v(\Delta)\) in a BCI-algebra \(X\), then \(x \ast y \approx u \ast v(\Delta)\).

Let \(X\) be a BCI-algebra and denote by \(\Delta_x\) the equivalence class containing \(x \in X\), and by \(X/\Delta\) the set of all equivalence classes of \(X\) with respect to \(\approx\), that is,

\[\Delta_x := \{y \in X \mid x \approx y(\Delta)\}\] and \(X/\Delta := \{\Delta_x \mid x \in X\}\).

Define a binary operation “\(\circ\)” on \(X/\Delta\) by \(\Delta_x \circ \Delta_y = \Delta_{x \ast y}\) for all \(\Delta_x, \Delta_y \in X/\Delta\). We first verify that the operation “\(\circ\)” is well defined. Let \(x, y, u, v \in X\) be such that \(\Delta_x = \Delta_u\) and \(\Delta_y = \Delta_v\). Then \(x \approx u(\Delta)\) and \(y \approx v(\Delta)\), which imply that \(x \ast y \approx u \ast v(\Delta)\) by Lemma 3.5. Let \(w \in \Delta_x \ast \Delta_y\). Then \(w \approx x \ast y \approx u \ast v(\Delta)\), and so \(w \in \Delta_{u \ast v} = \Delta_u \ast \Delta_v\). Now if \(z \in \Delta_u \ast \Delta_v\), then \(z \approx u \ast v \approx x \ast y(\Delta)\), and thus \(z \in \Delta_{x \ast y} = \Delta_x \ast \Delta_y\). Therefore \(\Delta_x \ast \Delta_y = \Delta_u \ast \Delta_v\), that is, “\(\circ\)” is well defined. Next we shall show that \((X/\Delta; \circ, \Delta_0)\) is a BCI-algebra. Let \(\Delta_x, \Delta_y, \Delta_z \in X/\Delta\). Then

\[
\begin{align*}
((\Delta_x \circ \Delta_y) \circ (\Delta_x \circ \Delta_z)) \circ (\Delta_z \circ \Delta_y) \\
= (\Delta_{x \ast y} \circ \Delta_{x \ast z}) \circ \Delta_{z \ast y} \\
= \Delta_{(x \ast y) \ast (x \ast z) \ast (z \ast y)} \\
= \Delta_{0},
\end{align*}
\]

which shows that \(X/\Delta\) satisfies the condition (a1). Similarly, we can deduce the conditions (a2) and (a3). Let \(x, y \in X\) be such that \(\Delta_x \circ \Delta_y = \Delta_0\) and \(\Delta_y \circ \Delta_x = \Delta_0\). Then \(\Delta_{x \ast y} = \Delta_0 = \Delta_{y \ast x}\), and so \(x \ast y \approx 0 \approx y \ast x(\Delta)\). It follows from (b1) that

\[\Delta(0 \ast ((x \ast y) \ast 0)^n) = \Delta(0)\]

and

\[\Delta(0 \ast ((y \ast x) \ast 0)^n) = \Delta(0)\]

so that \(x \approx y(\Delta)\). Hence \(\Delta_x = \Delta_y\). We shall state this as a theorem.

**Theorem 3.6.** If \(\Delta\) is a fuzzy ideal of a BCI-algebra \(X\), then \((X/\Delta; \circ, \Delta_0)\) is a BCI-algebra.

We then call \(X/\Delta\) fuzzy quotient BCI-algebra of \(X\) induced by the fuzzy ideal \(\Delta\).

**Lemma 3.7.** (Xi [7]) Let \(f : X \rightarrow Y\) be an epimorphism of BCI-algebras. If \(\Delta\) is a fuzzy ideal of \(Y\), then the homomorphic preimage of \(\Delta\) under \(f\), denoted by \(f^{-1}(\Delta)\), is a fuzzy ideal of \(X\).

**Theorem 3.8.** (Isomorphism theorem) Let \(f : X \rightarrow Y\) be an epimorphism of BCI-algebras and let \(\Delta\) be a fuzzy ideal of \(Y\). Then \(X/\Delta\) is isomorphic to \(Y/\Delta\), where \(\Delta = f^{-1}(\Delta)\).
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Proof. Note that $X/\bar{A}$ and $Y/\bar{B}$ are BC1-algebras (see Theorem 3.6 and Lemma 3.7). Let
$\Phi : X/\bar{A} \rightarrow Y/\bar{B}$ be a mapping defined by $\Phi(\bar{A}_x) = \bar{B}_{f(x)}$, where $x \in X$. Let $x, y \in X$ be
such that $\bar{A}_x = \bar{A}_y$. Then
$$
\bar{B}(0) = \bar{B}(f(0)) = f^{-1}(\bar{B})(0) = \bar{A}(0) = \bar{A}(0 \ast (x \ast y)^n)
$$
$$
= f^{-1}(\bar{B})(0 \ast (x \ast y)^n) = \bar{B}(f(0 \ast (x \ast y)^n)) = \bar{B}(0 \ast (f(x) \ast f(y))^n).
$$
Similarly $\bar{B}(0 \ast (f(y) \ast f(x))^n) = \bar{B}(0)$. Hence $f(x) \approx f(y)(\bar{B})$, that is, $\bar{B}_{f(x)} = \bar{B}_{f(y)}$.
Therefore $\Phi$ is well defined. For any $\bar{A}_x, \bar{A}_y \in X/\bar{A}$, we have
$$
\Phi(\bar{A}_x \odot \bar{A}_y) = \Phi(\bar{A}_{x \ast y}) = \bar{B}_{f(x \ast y)} = \bar{B}_{f(x) \ast f(y)} = \bar{B}_{f(x) \odot f(y)} = \Phi(\bar{A}_x) \odot \Phi(\bar{A}_y).
$$
Hence $\Phi$ is a homomorphism. Now let $x, y \in X$ be such that $\bar{B}_{f(x)} = \bar{B}_{f(y)}$. Then $f(x) \approx f(y)(\bar{B})$, and so
$$
\bar{A}(0) = f^{-1}((\bar{B})(0) = f^{-1}(\bar{B})(0) = \bar{B}(0 \ast (x \ast y)^n)) = \bar{B}(f(0 \ast (x \ast y)^n)) = \bar{A}(0 \ast (x \ast y)^n),
$$
and $\bar{A}(0 \ast (y \ast x)^n) = \bar{A}(0)$ by the same way. Thus $x \approx y(\bar{A})$, that is, $\bar{A}_x = \bar{A}_y$. This shows
that $\Phi$ is injective. Clearly $\Phi$ is surjective, and the proof is complete. \qed

Lemma 3.9. (Meng and Xin [6]) A BC1-algebra $X$ is positive implicative if and only if it satisfies $x \ast y = ((x \ast y) \ast y) \ast (0 \ast y)$ for all $x, y \in X$.

Lemma 3.10. (Liu and Meng [4]) A fuzzy ideal $\bar{A}$ of a BC1-algebra $X$ is fuzzy positive implicative if and only if it satisfies $\bar{A}(x \ast y) = \bar{A}(((x \ast y) \ast y) \ast (0 \ast y))$ for all $x, y \in X$.

Theorem 3.11. Let $\bar{A}$ be a fuzzy ideal of a BC1-algebra $X$. Then the fuzzy quotient BC1-algebra $X/\bar{A}$ of $X$ induced by $\bar{A}$ is positive implicative if and only if $\bar{A}$ is a fuzzy positive implicative ideal of $X$.

Proof. Assume that the quotient algebra $X/\bar{A}$ is positive implicative. Then
$$
\bar{A}_{x \ast y} = \bar{A}_x \odot \bar{A}_y = ((\bar{A}_x \odot \bar{A}_y) \odot \bar{A}_y) \odot (\bar{A}_0 \odot \bar{A}_y) = \bar{A}_{((x \ast y) \ast y) \ast (0 \ast y)},
$$
that is, $x \ast y \approx ((x \ast y) \ast y) \ast (0 \ast y)(\bar{A})$. It follows from (F1) and (F2) that
$$
\bar{A}(x \ast y) \geq \min\{\bar{A}(((x \ast y) \ast y) \ast (0 \ast y), \bar{A}(((x \ast y) \ast y) \ast (0 \ast y)) = \bar{A}(((x \ast y) \ast y) \ast (0 \ast y)).
$$
Obviously $\bar{A}(x \ast y) \leq \bar{A}(((x \ast y) \ast y) \ast (0 \ast y))$ because $((x \ast y) \ast y) \ast (0 \ast y) \leq x \ast y$ by (a1),
(b1) and (b2) and $\bar{A}$ is order reversing. Hence $\bar{A}(x \ast y) = \bar{A}(((x \ast y) \ast y) \ast (0 \ast y))$, and thus
$\bar{A}$ is a fuzzy positive implicative ideal of $X$. Conversely suppose that $\bar{A}$ is a fuzzy positive
implicative ideal of $X$. Using (b2) and Lemma 3.10, we have
$$
\bar{A}(x \ast y) \ast ((x \ast y) \ast y) \ast (0 \ast y))
$$
$$
= \bar{A}(x \ast (((x \ast y) \ast y) \ast (0 \ast y))) \ast y
$$
$$
= \bar{A}((((x \ast y) \ast y) \ast (0 \ast y))) \ast y
$$
$$
= \bar{A}(0).
$$
Since $(((x \ast y) \ast y) \ast (0 \ast y)) \ast (x \ast y) = 0$, it follows that
$$
\bar{A}(((x \ast y) \ast y) \ast (0 \ast y)) \ast (x \ast y)) = \bar{A}(0).
$$
Hence $x \ast y \approx ((x \ast y) \ast y) \ast (0 \ast y)(\bar{A})$, and so
$$
\bar{A}_x \odot \bar{A}_y = \bar{A}_{x \ast y} = \bar{A}_{((x \ast y) \ast y) \ast (0 \ast y)} = ((\bar{A}_x \odot \bar{A}_y) \odot \bar{A}_y) \odot (\bar{A}_0 \odot \bar{A}_y).
$$
It follows from Lemma 3.9 that $X/\bar{A}$ is a positive implicative BC1-algebra. \qed

Lemma 3.12. (Meng and Xin [5]) A BC1-algebra $X$ is commutative if and only if it satisfies $x \ast (x \ast y) = y \ast (y \ast (x \ast y)))$ for all $x, y \in X$. 

Lemma 3.13. (Jun and Meng [3]) Let $\tilde{A}$ be a closed fuzzy ideal of a $BCI$-algebra $X$. Then
$\tilde{A}$ is fuzzy commutative if and only if it satisfies $\tilde{A}(x * (y * (x * y))) \geq \tilde{A}(x * y)$ for all $x, y \in X$.

Theorem 3.14. Let $\tilde{A}$ be a closed fuzzy ideal of a $BCI$-algebra $X$. Then the fuzzy quotient $BCI$-algebra $X/\tilde{A}$ of $X$ induced by $\tilde{A}$ is commutative if and only if $\tilde{A}$ is fuzzy commutative.

Proof. Assume that $\tilde{A}$ is a closed fuzzy commutative ideal of $X$. Then, by Lemma 3.13, (b2) and (a3), we have
\[ \tilde{A}((x * (x * y)) * (y * (x * (x * y)))) \geq \tilde{A}((x * (x * y)) * (x * y)) = \tilde{A}(0). \]
On the other hand, note that
\[ \tilde{A}((y * (x * (x * y))) * (x * (x * y))) = \tilde{A}((y * (x * (x * y))) * (y * (x * (x * y)))) = \tilde{A}(0) \]
by (b2) and (a3). Hence $x * (x * y) \cong y * (x * (x * y))(\tilde{A})$, which implies that
\[ \tilde{A}(x \odot (\tilde{A} x \odot \tilde{A} y)) = \tilde{A}(y \odot (\tilde{A} y \odot (\tilde{A} x \odot \tilde{A} y))). \]
It follows from Lemma 3.12 that $X/\tilde{A}$ is commutative. Conversely let $\tilde{A}$ be a closed fuzzy ideal of $X$ such that $X/\tilde{A}$ is commutative. Then
\[ A_{x * (x y)} = A_x \odot (A_x \odot A_y) = A_y \odot (A_y \odot (A_x \odot A_y)) = A_{y * (x * (x y))}, \]
and hence $x * (x * y) \cong y * (x * (x * y))(\tilde{A})$. It follows from (b2) and (F1) that
\[ \tilde{A}((y * (x * (x * y))) * (x * y)) = \tilde{A}((x * (x * y)) * (y * (x * (x * y)))) = \tilde{A}(0) \geq \tilde{A}(x * y), \]
so from (F2) that
\[ A(x * (y * (x * (x * y)))) \geq \min\{ \tilde{A}(x * (y * (x * (x * y)))) \odot (x * y), \tilde{A}(x * y) \} \]
\[ = \tilde{A}(x * y). \]
Using (a1), (b2) and (a3), we get
\[ (x * (y * (x * y))) * (x * (y * (x * (x * y)))) \leq 0 * (x * y). \]
Since $\tilde{A}$ is order reversing, it follows from (F2) and its closedness that
\[ \tilde{A}(x * (y * (x * y))) \geq \min\{ \tilde{A}(x * (y * (x * y))) \odot (x * (y * (x * (x * y)))), \tilde{A}(x * (x * y)) \} \]
\[ = \tilde{A}(x * y). \]
Hence, by Lemma 3.13, $\tilde{A}$ is fuzzy commutative.

Acknowledgements. This work was supported by Korea Research Foundation Grant (KRF-2001-005-D00002).

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