CONTROLLED CONVERGENCE THEOREM FOR BANACH-VALUED HL INTEGRALS

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Abstract. Henstock’s strongly variational integral for Banach-valued functions is called the HL integral, which is in the form of Henstock’s Lemma. In this paper, we shall prove a controlled convergence theorem for such integrals.

The Henstock integral for Banach-valued functions has been discussed in [1-8, 10, 12, 17-23]. However Henstock’s Lemma may not hold for such integral [2, 17-20]. The stronger version (see Definition 1.2) [2, 12], using Henstock’s Lemma as a definition of an integral, has richer properties. For example, it has differentiation and measurability properties [2, 4, 6, 21, 23]. On the other hand, the Denjoy-Dunford, Denjoy-Pettis and Denjoy-Bochner integrals have been discussed in [7, 9, 11, 16, 24]. In [24], a controlled convergence theorem is claimed to be true without proof, for the Denjoy-Bochner integral. In this note, following the idea in [14], we shall prove a controlled convergence theorem for the HL integral. We remark that we do not follow the idea in [13, p40], since in [13, p40, line 17], we do not know whether the primitive function is differentiable a.e.

1 HL integral and AC*(X) In this section, we shall define the HL integral and discuss properties of AC*(X).

Definition 1.1. Let δ be a positive function on a closed interval [a, b]. A division \(D = \{(u, v), \xi\}\) of [a, b] is said to be Henstock δ-fine if \(\xi \in [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))\) for every \((u, v), \xi\) \(\in D\).

In the following, we always use partial divisions instead of divisions. \(D = \{(u, v), \xi\}\) is said to be a partial division of \([a, b]\) if \(\{(u, v)\}\) is a collection of nonoverlapping subintervals of \([a, b]\). The union of \([u, v]\) in \(D\) may not equal to \([a, b]\).

Definition 1.2. Let \((B, \|\|)\) denote a Banach space with norm \(\|\|\). A function \(f : [a, b] \to (B, \|\|)\) is HL integrable on \([a, b]\) if there exists a function \(F : [a, b] \to (B, \|\|)\) satisfying the following property: for every \(\epsilon > 0\), there exists a positive function \(\delta(\xi)\) on \([a, b]\) such that if \(D = \{(u, v), \xi\}\) is a Henstock δ-fine partial division of \([a, b]\), we have

\[
(\text{D}) \sum \|f(\xi)(v - u) - F(u)\| < \epsilon
\]

where \(F(u, v) = F(v) - F(u)\).

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Henceforth, a Banach-valued function shall be referred to as a function with values in $(B, \|\|)$.

**Definition 1.3.** A Banach-valued function $F$ is said to be *absolutely continuous* on $[a, b]$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$, with $\sum |b_i - a_i| < \eta$ we have
\[
\sum_i \|F(a_i, b_i)\| < \epsilon
\]
where $F(a_i, b_i) = F(b_i) - F(a_i)$.

**Definition 1.4.** Let $X \subset [a, b]$. A Banach valued function $F$ defined on $X$ is said to be $AC(X)$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ satisfying $\sum_i |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all $i$, we have
\[
\sum_i \|F(a_i, b_i)\| < \epsilon
\]
where the endpoints $a_i, b_i \in X$ for all $i$.

**Definition 1.5.** A Banach-valued function $F$ defined on $X \subset [a, b]$ is said to be $AC^*(X)$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ satisfying $\sum_i |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all $i$, we have
\[
\sum_i \omega(F; [a_i, b_i]) < \epsilon
\]
where $\omega$ denotes the oscillation of $F$ over $[a_i, b_i]$, i.e.,
\[
\omega(F; [a_i, b_i]) = \sup \{\|F(x, y)\|; x, y \in [a_i, b_i]\}.
\]

**Definition 1.6.** A Banach-valued function $F$ is said to be $ACG^*$ on $X$ if $X$ is the union of a sequence of closed sets $\{X_i\}$ such that on each $X_i$, $F$ is $AC^*(X_i)$.

Following ideas in [13, pp.27-28], we can prove

**Lemma 1.7.** Let $X$ be a closed set in $[a, b]$ and $(a, b) \setminus X$ be the union of $(c_k, d_k)$ for $k = 1, 2, \ldots$ Suppose a Banach-valued function $F$ is continuous on $[a, b]$. Then the following statements are equivalent:
(i) $F$ is $AC^*(X)$
(ii) $F$ is $AC(X)$ and $\sum_{k=1}^{\infty} \omega(F; [c_k, d_k]) < \infty$
(iii) Definition 1.4 holds with $a_i$ or $b_i$ belonging to $X$ for every $i$.

To justify that $X$ is closed in Definition 1.6, we shall prove the following lemma.

**Lemma 1.8.** Let $X \subset [a, b]$. If $F$ is $AC^*(X)$ and continuous on $[a, b]$, then $F$ is $AC^*(\overline{X})$, where $\overline{X}$ is the closure of $X$. 
**Proof.** Suppose $F$ is $AC^* (X)$. Then for every $\epsilon > 0$, there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in X$ and $\sum_i |b_i - a_i| < \eta$, we have

$$\sum_i \|F(b_i) - F(a_i)\| < \epsilon.$$

Now, let $\{[c_i, d_i]\}$ be any finite or infinite sequence of non-overlapping intervals with $c_i, d_i \in \overline{X}$ and $\sum_i |d_i - c_i| < \eta$. For each $i$, there exist $u_i, v_i \in X$ with $u_i < v_i$ and $\sum_i |v_i - u_i| < \eta$ such that

$$\|F(u_i) - F(c_i)\| < \frac{\epsilon}{2^i} \quad \text{and} \quad \|F(v_i) - F(d_i)\| < \frac{\epsilon}{2^i}.$$

Observe that $\{[u_i, v_i]\}$ may not be non-overlapping intervals. However, we can divide $\{[c_i, d_i]\}$ into two parts, wherein intervals in each part are disjoint, so that we can choose $\{[u_i, v_i]\}$ to be disjoint. Hence we may assume $\{[u_i, v_i]\}$ to be non-overlapping. As a result, we have

$$\sum_i \|F(d_i) - F(c_i)\| \leq \sum_i \|F(d_i) - F(v_i)\| + \sum_i \|F(v_i) - F(u_i)\|$$

$$+ \sum_i \|F(u_i) - F(c_i)\|$$

$$< \epsilon + \epsilon + \epsilon.$$

Therefore, $F$ is $AC^*(\overline{X})$. \qed

**Remark 1.9.** Similarly, we can prove that if the statement (iii) in Lemma 1.7 holds for $X$, then it also holds for $\overline{X}$. Hence, when referring to $AC^*(X)$, we may assume that $X$ is closed.

**Definition 1.10.** A sequence $\{f_n\}$ of Banach-valued functions is said to be control convergent to $f$ on $[a, b]$ if the following conditions are satisfied:

(i) $f_n(x) \xrightarrow{a.e.} f(x)$ in $[a, b]$ as $n \to \infty$ where each $f_n$ is $HL$ integrable in $[a, b]$;

(ii) the primitives $F_n$ of $f_n$ are $ACG^*$ uniformly in $n$, i.e., $[a, b]$ is the union of a sequence of closed sets $X_i$ such that on each $X_i$, the functions $F_n$ are $AC^*(X_i)$ uniformly in $n$;

(iii) the primitives $F_n$ converge uniformly on $[a, b]$. 

2 Properties of HL integral Most of the theorems that we will be using in proving our main theorem shall be discussed in this section.

Lemma 2.1. If \( f(x) = 0 \) a.e. in \([a, b]\), i.e., for all \( x \in [a, b] \) except perhaps on a set \( X \) of measure zero, then \( f \) is HL integrable to 0 on \([a, b]\).

The proof is standard [13, p6].

Theorem 2.2. If \( f \) is HL integrable on \([a, b]\), then its primitive \( F \) is continuous on \([a, b]\).

Proof. See [13, p12].

Theorem 2.3. If \( f \) is HL integrable on \([a, b]\), then its primitive \( F \) is \( ACG^* \) on \([a, b]\).

Proof. The proof is standard. However we shall give the detail here.

For every \( \epsilon > 0 \), there is a function \( \delta(\xi) > 0 \) such that for any Henstock \( \delta \)-fine partial division \( D = \{ [u, v]; \xi \} \) in \([a, b]\), we have

\[
(D) \sum \| F(u, v) - f(\xi)(v - u) \| < \epsilon.
\]

We may assume that \( \delta(\xi) \leq 1 \). Let

\[
X_{ni} = \{ x \in [a, b] : \| f(x) \| \leq n; \frac{1}{n} < \delta(x) \leq \frac{1}{n-1} \text{ and } x \in [a + \frac{i}{n}, a + \frac{i+1}{n}] \}
\]

for \( n = 2, 3, \ldots, i = 1, 2, \ldots \). Fix \( X_{ni} \) and let \( \{ [a_k, b_k] \} \) be any finite sequence of non-overlapping intervals with \( a_k, b_k \in X_{ni} \) for all \( k \). Then \( \{ [a_k, b_k], a_k \} \) is a Henstock \( \delta \)-fine partial division of \([a, b]\). Furthermore, if \( a_k \leq u_k \leq v_k \leq b_k \), then \( \{ [a_k, u_k], a_k \} , \{ [a_k, b_k], b_k \} \) are Henstock \( \delta \)-fine partial divisions of \([a, b]\). Thus,

\[
\sum_k \| F(u_k, v_k) \| \leq \sum_k \| F(a_k, u_k) \| + \sum_k \| F(v_k, b_k) \| + \sum_k \| F(a_k, b_k) \|
\]

\[
\leq 3\epsilon + \sum_k \| f(a_k)(u_k - a_k) \| + \sum_k \| f(b_k)(b_k - v_k) \|
\]

\[
\quad + \sum_k \| f(a_k)(b_k - a_k) \|
\]

\[
\leq 3\epsilon + 3n \sum_k (b_k - a_k).
\]

Choose \( \eta \leq \frac{\epsilon}{3n} \) and \( \sum_k (b_k - a_k) < \eta \). Then

\[
\sum_k \omega(F; [a_k, b_k]) \leq 3\epsilon + \epsilon.
\]

Therefore, \( F \) is \( ACG^*(X_{ni}) \) and also \( ACG^*(X_{ni}) \). Consequently, \( F \) is \( ACG^* \) on \([a, b]\). \( \square \)
Theorem 2.4. If $f$ is HL integrable on $[a, b]$, then its primitive $F$ is differentiable a.e. and $F'(x) = f(x)$ a.e. on $[a, b]$.

Proof. See [13, p21].

Theorem 2.5. Let $(B, ||||)$ be a Banach space and $f : [a, b] \to (B, ||||)$. Suppose there exists a function $F : [a, b] \to B$ which is continuous and $ACG^*$ on $[a, b]$ such that $F'(x) = f(x)$ a.e. in $[a, b]$. Then $f$ is HL integrable on $[a, b]$ with primitive $F$.

Proof. See [13, p31].

The following is a special version of Egoroff’s theorem for Banach-valued functions.

Lemma 2.6. If $f_n(x) \to f(x)$ a.e. in $[a, b]$ as $n \to \infty$ where each $f_n$ is HL integrable then for every $\eta > 0$ there exists an open set $G$ with $|G| < \eta$ such that $f_n$ converges uniformly to $f$ on $[a, b] \setminus G$.

Theorem 2.7. Suppose
(i) $f_n(x) \to f(x)$ a.e. in $[a, b]$ as $n \to \infty$ where each $f_n$ is HL integrable on $[a, b]$;
(ii) the primitives $F_n$ of $f_n$ are uniformly absolutely continuous.

Then for every $\epsilon > 0$ there exists a positive integer $N$ such that for every partial partition $D = \{[u, v]\}$ of $[a, b]$ we have

$$(D) \sum \|F_n(u, v) - F_m(u, v)\| < \epsilon$$

whenever $n, m \geq N$.

Proof. See [13, pp 37 - 38].

Theorem 2.8. Suppose
(i) $f_n(x) \to f(x)$ a.e. in $[a, b]$ as $n \to \infty$, where each $f_n$ is HL integrable on $[a, b]$;
(ii) the primitives $F_n$ of $f_n$ are uniformly absolutely continuous.

Then $f$ is HL integrable on $[a, b]$ and

$$\int_a^b f_n \to \int_a^b f \quad \text{as} \quad n \to \infty.$$ 

Proof. See [13, p38].

Theorem 2.9. Let $\{f_n\}$ be a sequence of Banach-valued functions on $[a, b]$ which is control convergent to $f$ on $[a, b]$. Then for each $X_i$ and for every $\epsilon > 0$, there exists a positive integer $N$ such that for every partial partition $D = \{[u, v]\}$ of $[a, b]$ with $u, v \in X_i$, we have

$$(D) \sum \omega(F_n - F_m; [u, v]) < \epsilon$$

whenever $n, m \geq N$.

Proof. Fix $X_i$ and let $X = X_i$. Assume that $a, b \in X$. Define $G_n(x) = F_n(x)$ when $x \in X$ and linear elsewhere in $[a, b]$. More precisely, let $(a, b) \setminus X = \bigcup_k (a_k, b_k)$ and define
\[ G_n(x) = \begin{cases} 
F_n(x) & \text{if } x \in X 
\frac{b_k - x}{b_k - a_k} F_n(a_k) + \frac{x - a_k}{b_k - a_k} F_n(b_k) & \text{if } x \in (a_k, b_k), k = 1, 2, \ldots 
\end{cases} \]

Observe that if \([u_i, v_i]\) is a finite or infinite sequence of non-overlapping intervals contained in \((a_k, b_k)\), then

\[
\sum_i \|G_n(u_i, v_i)\| = \sum_i \left\| \left( \frac{b_k - u_i}{b_k - a_k} F_n(a_k) + \frac{v_i - a_k}{b_k - a_k} F_n(b_k) \right) 
- \left( \frac{b_k - u_i}{b_k - a_k} F_n(a_k) + \frac{u_i - a_k}{b_k - a_k} F_n(b_k) \right) \right\| 
= \frac{1}{b_k - a_k} \sum_i \left\| (v_i - u_i)(F_n(b_k) - F_n(a_k)) \right\| 
= \frac{\|F_n(b_k) - F_n(a_k)\|}{b_k - a_k} \sum_i |v_i - u_i|. \tag{2.1} \]

On the other hand, \(\sum_k \omega(F_n; [a_k, b_k])\) converges uniformly in \(n\), due to the fact that \(F_n\) is \(AC^* (X)\) uniformly in \(n\). Hence, by (2.1), we need only to consider the first finite number of intervals \([a_k, b_k], k = 1, 2, \ldots, m\). It is clear from (2.1) that the functions \(G_n\) are absolutely continuous on each \([a_k, b_k]\). Consequently, the functions \(G_n\) are uniformly absolutely continuous on \([a, b]\) in view of the fact that \(G_n(x) = F_n(x)\) on \(X\).

Now, define

\[
g_n(x) = \begin{cases} 
f_n(x) & \text{if } x \in X 
\frac{f_n(x) - F_n(b_k)}{b_k - a_k} & \text{if } x \in (a_k, b_k). 
\end{cases} \]

Then \(g_n\) converges a.e. on \([a, b]\) and \(g_n(x) \to f(x)\) a.e. on \(X\). We shall now use Definition 1.2 to prove that each \(g_n\) is \(HL\) integrable on \([a, b]\) and the primitive of \(g_n\) is \(G_n\). First, note that if \(x \in (a_k, b_k)\), we can choose \(\delta(x) > 0\) such that whenever \((u, v, \xi)\) is \(\delta\)-fine, we have \([u, v] \subset (a_k, b_k)\). By linearity of \(G_n\) on \((a_k, b_k)\) and definition of \(g_n\), we have

\[
\|g_n(\xi)(u - v) - G_n(u, v)\| = 0. \]

Secondly if \(\xi \in X\), we consider interval-point pairs of the form \(([u, \xi], \xi)\) or \((\xi, v], \xi)\). For the case \(u, v \in X\), we observe that

\[
\|g_n(\xi)(\xi - u) - G_n(u, \xi)\| = \|f_n(\xi)(\xi - u) - F_n(u, \xi)\|. \]

Similarly for the case \((\xi, v], \xi\). Hence we need only to consider the case when \(u, v \notin X\). Now suppose \(u \in (a_k, b_k)\) for some \(k\). Then

\[
\|g_n(\xi)(\xi - u) - G_n(u, \xi)\|
\leq \|f_n(\xi)(\xi - b_k) - G_n(b_k, \xi)\| + \|f_n(\xi)(b_k - u) - G_n(u, b_k)\|
= \|f_n(\xi)(\xi - b_k) - F_n(b_k, \xi)\| + \|f_n(\xi)(b_k - u) - G_n(u, b_k)\|. \]
Therefore finally we need only to consider

$$\| f_n(\xi)(b_k - u) - G_n(u, b_k) \|.
$$

Let $X_\delta = \{ \xi \in X; q - 1 \leq \| f_n(\xi) \| < q \}, \ q = 1, 2, \ldots$. Let $q$ be fixed. Given $\epsilon > 0$, we first choose $\ell$ such that

$$\sum_{k=\ell}^{\infty} |b_k - a_k| < \frac{\epsilon}{q \cdot 2^v} \quad \text{and} \quad \sum_{k=\ell}^{\infty} \omega(F_n; [a_k, b_k]) < \frac{\epsilon}{2^v}.
$$

Let $\xi \in X_\delta$ and $\xi \neq a_k, b_k$ for all $k$. Now we choose $\delta(\xi) > 0$ such that when $[a_k, b_k] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$, we have $k \geq \ell$. Hence, if $D = \{(u, v), \xi\}$ is a $\delta$-fine partial division with $\xi \in X_\delta$, we have

$$\begin{align*}
(D) \sum \| f_n(\xi)(b_k - u) - G_n(u, b_k) \| &\leq (D) \sum \| f_n(\xi)(b_k - u) \| + (D) \sum \| G_n(u, b_k) \| \\
&< \frac{\epsilon}{2^v} + \frac{\epsilon}{2^v}.
\end{align*}
$$

When $\xi \in X_\delta$ and $\xi = a_p$ or $b_p$ for some $p$, in view of the continuity of $G_n$ at $\xi$, we can choose $\delta(\xi) > 0$ such that when $(u, v), \xi$ in $\delta$-fine, we have

$$\begin{align*}
\| g_n(\xi)(v - u) - G_n(u, v) \| &= \| f_n(\xi)(v - u) - G_n(u, v) \| \\
&\leq \| f_n(\xi)(v - u) \| + \| G_n(u, v) \| \\
&< \epsilon/2^p + \epsilon/2^p.
\end{align*}
$$

From the above analysis, $g_n$ is HL integrable on $[a, b]$ with primitive $G_n$. By Theorem 2.7 we get the required result, without oscillation. To get the required result with oscillation, we observe that, for each $n$, there exist $p_k, q_k \in [a_k, b_k]$ such that

$$\omega(F_n - F; [a_k, b_k]) = \| (F_n - F)(p_k, q_k) \|.
$$

However, $p_k$ and $q_k$ depend on $n$. Now we do some adjustment. Let $c_k, d_k \in (a_k, b_k)$ with $c_k < d_k$ and fixed, independent of $n$. Define $H_n(x) = F_n(x) - F(x)$ if $x \in X$ and linearly on $[a_k, c_k], [c_k, d_k]$ and $[d_k, b_k]$ with $H_n(a_k) = (F_n - F)(a_k); H_n(c_k) = (F_n - F)(b_k); H_n(d_k) = (F_n - F)(p_k, q_k) + H_n(c_k)$ and $H_n(b_k) = H_n(c_k)$. Hence \( \| H_n(d_k) - H_n(c_k) \| = \| F_n - F; [a_k, b_k] \| \), and the oscillation of $H_n$ over $[a_k, b_k]$ is equal to that of $F_n - F$ over $[a_k, b_k]$.

As in the proof of the first part with $G_n(x)$ replaced by $H_n(x)$ and $\{[a_k, b_k]\}$ replaced by $\{[a_k, c_k], [c_k, d_k], [d_k, b_k]\}$ by Theorem 2.7, given any $\epsilon > 0$, there exists a positive integer $N$ such that for any partial partition $D = \{[u, v]\}$ of $[a, b]$, we have

$$\begin{align*}
(D) \sum \| H_n(u, v) - H_m(u, v) \| &< \epsilon
\end{align*}
$$

whenever $n, m \geq N$. Note that the limit of the sequence $H_n(x)$ exists as $n \to \infty$ for each $x$ and in view of (iii) of Definition 1.10, it is zero. Thus, the above inequality implies that
(2.2) \[
(D) \sum\|H_n(u, v)\| < \epsilon
\]
whenever \(n \geq N\). Observe that if \(u, v \in X\) and \([p, q] \subset (u, v)\) with \(p, q \notin X\), we can divide \([p, q]\) into three sub-intervals, where two of them are in \(\bigcup_{k} [a_k, b_k]\) and another with endpoints in \(X\), namely \([p, q] = [p, s] \cup [s, t] \cup [t, q]\), where \(p \in [a_i, b_i]\), \(q \in [a_j, b_j]\) and \(s, t \in X\). Then

\[
\| (F_n - F)(p, q) \| \leq \| (F_n - F)(p, s) \| + \| (F_n - F)(s, t) \| + \| (F_n - F)(t, q) \|
\]
\[
\leq w(F_n - F; [a_i, b_i]) + \| (F_n - F)(s, t) \| + w(F_n - F; [a_j, b_j])
\]
\[
= \| H_n(c_i, d_i) \| + \| H_n(s, t) \| + \| H_n(c_j, d_j) \|
\]

Hence, by (2.2), for any partial partition \(D = \{[u, v]\}\) of \([a, b]\) and any \([p, q] \subset [u, v]\), we have

\[
(D) \sum\| (F_n - F)(p, q) \| < \epsilon
\]
whenever \(n \geq N\). Note that \([p, q]\) is any subinterval of \([u, v]\). Thus

\[
(D) \sum w(F_n - F; [u, v]) \leq \epsilon
\]
whenever \(n \geq N\). Consequently

\[
(D) \sum w(F_n - F_m; [u, v]) \leq 2\epsilon
\]
whenever \(n, m \geq N\).

3 Main Result Theorem 3.1. Controlled Convergence Theorem

If a sequence of Banach-valued functions \(\{f_n\}\) is control convergent to \(f\) on \([a, b]\), then \(f\) is also \(HL\) integrable on \([a, b]\) and

\[
\int_{a}^{b} f_n(x)dx \longrightarrow \int_{a}^{b} f(x)dx \quad \text{as} \quad n \to \infty.
\]

Proof. In view of Lemma 2.1, we may assume \(f_n(x) \to f(x)\) everywhere in \([a, b]\) as \(n \to \infty\). Since each \(f_n\) is \(HL\) integrable on \([a, b]\), with primitive \(F_n\), then given \(\epsilon > 0\) there exists \(\delta_n(\xi) > 0\) such that for any Henstock \(\delta_n\)-fine partial division \(D = \{[u, v]; \xi\}\) of \([a, b]\), we have

\[
(D) \sum\| f_n(\xi)(v - u) - F_n(u, v) \| < 2^{-n}.
\]

Since \(f_n(x) \to f(x)\), there exists a positive integer \(m = m(\epsilon, \xi)\) such that

\[
\| f_m(\xi) - f(\xi) \| < \epsilon.
\]

By the hypothesis, we also have

\[
\lim_{n \to \infty} F_n(u, v) = F(u, v) \quad \text{exists}
\]
for any sub-interval \([u, v]\) of \([a, b]\).

From the definition of control convergence, \([a, b]\) is the union of a sequence of closed sets \(X_i\) such that on each \(X_i\), the functions \(F_n\) are \(AC^\alpha(X_i)\) uniformly in \(n\). By Theorem 2.9, it follows that, for each \(i\), there exists a positive integer \(N(i)\) such that for any partial partition \(D = \{[u, v]\}\) of \([a, b]\) with \(u, v \in X_i\), we have

\[
(D) \sum w(F_n - F; [u, v]) < \epsilon
\]

whenever \(n \geq N(i)\).

Hence, for each \(i\), there exists a subsequence \(\{F_{n(i,j)}\}_{j=1}^\infty\) of \(\{F_n\}_{n=1}^\infty\) such that

\[
(D) \sum w(F_{n(i,j)} - F; [u, v]) < \epsilon 2^{-i-j}
\]

(3.4)

for any partial partition \(D = \{[u, v]\}\) of \([a, b]\) with \(u, v \in X_i\). We may assume that for each \(i > 1\), \(\{F_{n(i,j)}\}_{j=1}^\infty\) is a subsequence of \(\{F_{n(i-1,j)}\}_{j=1}^\infty\). From now onwards, \((n, j)\) is denoted by \((m(j), j)\), and we only consider subsequences \(\{f_{m(j)}\}\) and \(\{F_{m(j)}\}\). Now we shall define \(\delta(\xi)\) on \([a, b]\). If \(\xi \in Y_i = X_i \setminus (X_1 \cup X_2 \cup \cdots \cup X_{i-1})\), where \(X_0 = \emptyset\), then we choose \(m(j) > m(i)\) such that \(\|f_{m(j)}(\xi) - f(\xi)\| < \epsilon\). Note that \(m(j)\) depends on \(\xi\). We denote \(m(j)\) by \(m(\xi)\). Define \(\delta(\xi) = \delta_{m(\xi)}(\xi)\). Let \(D = \{([u, v]; \xi)\}\) be any Henstock \(\delta\)-fine partial division of \([a, b]\), we shall prove that

\[
(D) \sum \|f(\xi)(v - u) - F(u, v)\| < \epsilon (b - a) + 2\epsilon.
\]

(3.5)

First

\[
(D) \sum \|f(\xi)(v - u) - F(u, v)\| \leq (D) \sum \|f(\xi) - f_{m(\xi)}(\xi)\|(v - u)
\] + \( (D) \sum \|f_{m(\xi)}(\xi)(v - u) - F_{m(\xi)}(u, v)\|
\] + \( (D) \sum \|F_{m(\xi)}(u, v) - F(u, v)\|\)

The first sum on the right side of the above inequality is less than \(\epsilon (b - a)\). The second sum can be written as

\[
\sum_{j=1}^\infty (D_j) \sum \|f_{m(\xi)}(\xi)(v - u) - F_{m(\xi)}(u, v)\|
\]

where \(D_j = \{([u, v], \xi)\}\) is a subset of \(D\) and each \(\xi\) in \(D_j\) induces the same \(m(j)\) i.e. \(m(\xi) = m(j)\) for all \(\xi\) in \(D_j\). Hence the second sum is less than

\[
\epsilon \sum_{j=1}^\infty 2^{-m(j)}\]

by (3.1)

Consequently it is less than \(\epsilon\). Now we shall handle the third sum. For convenience, we may assume that \(a, b \in X_i\), for all \(i\). For any \([u, v], \xi\) in \(D\), \([u, v] = [u, \xi] \cup [\xi, v]\). Suppose \(\xi \in Y_i = X_i \setminus (X_1 \cup X_2 \cup \cdots \cup X_{i-1})\). Then either \(u \in X_i\) or \([u, \xi]\) lies in an interval with endpoints in \(X_i\). On the other hand, the third sum can be written as
\[
\sum_{i} \sum_{j} \sum_{\xi \in X_{i}, m(\xi) = m(j)} \| F_{m(\xi)}(u, v) - F(u, v) \|.
\]

Recall that \( m(\xi) = m(j) = n(j, j) > n(i, i) \). Thus \( j > i \). Hence \( \{ n(j, k) \}_{k=1}^{\infty} \) is a subsequence of \( \{ n(i, k) \}_{k=1}^{\infty} \). So \( m(j) = n(j, j) = n(i, k(j)) \) for some \( k(j) \). Hence, by (3.4),
\[
\sum_{\xi \in X_{i}, m(\xi) = m(j)} \| F_{m(\xi)}(u, v) - F(u, v) \| \leq \varepsilon 2^{-i-k(j)}.
\]

Note that \( \{ n(j + 1, k) \}_{k=1}^{\infty} \) is a subsequence of \( \{ n(j, k) \}_{k=1}^{\infty} \). We may choose \( \{ n(j + 1, k) \}_{k=1}^{\infty} \) such that \( k(j) \) is strictly increasing. Thus the third sum is less than \( \varepsilon \). Consequently, (3.5) holds. With (3.5) and (3.3), the proof is complete.

**Remark.** In general, a Banach-valued function \( F \) which is \( ACG^{+} \) may not be differentiable a.e. From the result of proof, we know that \( F \) is differentiable a.e. if it satisfies the conditions of Theorem 3.1, however the ideas in [13, p40] does not work for proving the above theorem, since in the proof, we use the result “\( F \) is differentiable a.e.”.

**References**


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