ON THE PARASPECTRUM AND THE CONTINUITY  
OF THE SPECTRUM IN ALGEBRA OF OPERATORS

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Received January 30, 2001; revised April 17, 2001

Abstract. In this paper some conditions are given for the continuity of the spectrum using the paraspectrum of operators. Also, Luecke’s theorem for $G_1$-operators is given as a simple consequence of those conditions.

1. Introduction

Let $X$ be a complex infinite-dimensional Banach space and let $B(X)$ denotes a Banach algebra of all bounded operators on $X$. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of $T$. For $A, B \in B(X)$ we define the $*$-prominance of $A$ by $B$, $* \in \{\alpha, \beta, \gamma\}$, by

$$\text{prom}_a(A; B) = \{ \lambda \notin \sigma(A) : \|(A - \lambda)^{-1}\| : \|A - B\| \geq 1 \} ;$$
$$\text{prom}_\beta(A; B) = \{ \lambda \notin \sigma(A) : \|(A - \lambda)(A - B)\| \geq 1 \} ;$$
$$\text{prom}_\gamma(A; B) = \{ \lambda \notin \sigma(A) : \|A - B\| \geq d(\lambda, \sigma(A)) \} .$$

The $*$-paraspectrum of $A$ by $B$ is the set

$$\sigma_*(A; B) = \text{prom}_*(A; B) \cup \sigma(A) , \quad * \in \{\alpha, \beta, \gamma\} .$$

It has been introduced in [3] in the case where $X$ is a Hilbert space.

An operator $A \in B(X)$ is a $G_1$-operator if $A$ satisfies the growth condition [4]

$$\|(A - \lambda)^{-1}\| \leq \frac{1}{d(\lambda, \sigma(A))} , \quad \lambda \notin \sigma(A) .$$

The continuity of spectra for $G_1$-operators on a Hilbert space has been discussed by several authors [2,3,4,6]. To discuss it for arbitrary operators on a Banach space, we need the distances $d_1$ and $d_2$ among compact subsets in the complex plane. Let $M$ and $N$ be a compact subsets in the complex plane. We define the distances $d_1(M, N)$ and $d_2(M, N)$ between $M$ and $N$ by

$$d_1(M, N) = \sup_{n \in N} \inf_{m \in M} |m - n| = \sup_{n \in N} \text{dist } (n, M)$$
$$d_2(M, N) = \inf_{m \in M} \sup_{n \in N} |m - n| = \sup_{n \in N} \text{dist } (m, N) .$$

AMS Subject Classification (1991): 47A10, 47A53

Keywords and Phrases: Paraspectrum, continuity of the spectrum
It is well-known that the distance $d(M, N)$ define by

$$d(M, N) = \max\{d_1(M, N), d_2(M, N)\}$$

is the Hausdorff distance between compact subsets $M$ and $N$.

A mapping $p$, defined on $B(X)$ whose values are compact subset of $\mathbb{C}$, is said to be upper (lower) semi-continuous at $A$, provided that if $A_n \to A$ then

$$d_1(p(A), p(A_n)) \to 0 \quad (d_2(p(A), p(A_n)) \to 0), \quad n \to \infty.$$

If $p$ is both upper and lower semi-continuous at $A$, then it is said to be continuous at $A$ and in this case $\lim p(A_n) = p(A)$.

In this paper we consider the spectral variation inequality

$$(1, i.) \quad d_i(\sigma(A), \sigma(B)) \leq \|A - B\|, \quad i = 1, 2$$

and we discuss a continuity of the spectrum of $A$ using the $*$-paraspectrum of $A$ by $B$.

2. Variation of spectrum

Directly from the definition of the $*$-paraspectrum follows that $\sigma(A) \subset \sigma_*(A; B)$, $* \in \{\alpha, \beta, \gamma\}$, for every $B \in B(X)$. Also, by [3] we get $\sigma(B) \subset \sigma_\alpha(A; B)$ and $\sigma_\gamma(A; B) \subset \sigma_\beta(A; B) \subset \sigma_\alpha(A; B)$ for every $A, B \in B(X)$.

If $(\tau_n)$ is a sequence of compact subsets of $\mathbb{C}$, then its limit inferior is

$$\lim\inf \tau_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\}$$

and its limit superior is

$$\lim\sup \tau_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_{n_k} \in \tau_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}.$$

It is well-known that a mapping $p$ which maps $B(X)$ into the family of compact subset of $\mathbb{C}$ is upper (lower) semi-continuous at $A$ if for every sequence $\{A_n\}$ in $B(X)$ such that $A_n \to A$ holds

$$\lim\sup p(A_n) \subset p(A) \quad (p(A) \subset \lim\inf p(A_n)).$$

**Theorem 1.** Let $A \in B(X)$ and let $\{A_n\}$ be a sequence in $B(X)$ such that $A_n \to A$. Then the next conditions are equivalent:

1. $\lim \sigma(A_n) = \sigma(A)$;
2. $\bigcap_{n=1}^\infty \sigma_\alpha(A; A_n) \subset \lim\inf \sigma(A_n)$;
3. $\bigcap_{n=1}^\infty \sigma_\alpha(A_n; A) \subset \lim\inf \sigma(A_n)$.

**Proof.** (1) $\Rightarrow$ (2) Let $\lim \sigma(A_n) = \sigma(A)$ and suppose that (2) is not true. Then there exists a $\lambda \in \left(\bigcap_{n=1}^\infty \sigma_\alpha(A; A_n)\right) \setminus (\lim\inf \sigma(A_n))$. For this $\lambda$ we get:

(i) $\lambda \in \sigma_\alpha(A; A_n)$, for every $n \in \mathbb{N}$;
(ii) $\lambda \notin \lim\inf \sigma(A_n)$ and so $\lambda \notin \sigma(A)$ by (1);

By (i) and (ii) it follows $\lambda \in \text{proj}_0(A; A_n)$, i.e.

$$\|(A - \lambda)^{-1}\|^{-1} \leq \|A - A_n\|, \quad \text{for every } n \in \mathbb{N}.$$
If \( n \to \infty \), then \( \| (A - \lambda)^{-1} \| \to 0 \). Hence it is a contradiction.

(2) \( \Rightarrow \) (3) Let the condition (2) holds and suppose that (3) does not hold. Then there exists a \( \lambda \in \left( \bigcap_{n=1}^{\infty} \sigma_a(A_n; A) \right) \setminus (\lim \inf \sigma(A_n)) \). For this \( \lambda \) we get:

(i) \( \lambda \in \sigma_a(A_n; A) \), for every \( n \in \mathbb{N} \);
(ii) there exists a \( n_0 \in \mathbb{N} \) such that \( \lambda \notin \sigma(A_n) \) for every \( n \geq n_0 \).

From (i) and (ii) it follows that \( \lambda \in \text{pr}_{\alpha}(A_n, A) \), i.e.

\[ (*) \quad \| (A_n - \lambda)^{-1} \|^{-1} \leq \| A_n - A \| \to 0, \quad n \to \infty. \]

Suppose that \( \lambda \in \sigma(A) \). Then \( \lambda \in \sigma_a(A; A_n) \), for every \( n \in \mathbb{N} \), i.e. \( \lambda \in \bigcap_{n=1}^{\infty} \sigma_a(A; A_n) \subset \lim \inf \sigma(A_n) \) and this is a contradiction. Hence \( \lambda \notin \sigma(A) \).

Since \( A_n - \lambda \to A - \lambda \) and \( \lambda \notin \sigma(A) \) it follows that \( (A_n - \lambda)^{-1} \to (A - \lambda)^{-1} \) (by the continuity of the function \( T \mapsto T^{-1} \) [1, Theorem 50.7]). But, by (\( *\)), we get that \( \| (A_n - \lambda)^{-1} \| \to \infty, \quad n \to \infty \), i.e. \( (A_n - \lambda)^{-1} \) converges to a noninvertible operator. Hence it is a contradiction.

(3) \( \Rightarrow \) (1) Suppose that \( \bigcap_{n=1}^{\infty} \sigma_a(A_n; A) \subset \lim \inf \sigma(A_n) \). Let \( \lambda \in \sigma(A) \). Then \( \lambda \in \sigma_a(A_n; A) \) for every \( n \in \mathbb{N} \) [3], i.e.

\[ \lambda \in \bigcap_{n=1}^{\infty} \sigma_a(A_n; A) \subset \lim \inf \sigma(A_n). \]

Hence we have \( \sigma(A) \subset \lim \inf \sigma(A_n) \).

Now, since \( \sigma \) is always upper semi-continuous [5, Theorem 1], it follows
\[ \lim \sigma(A_n) = \sigma(A). \quad \Box \]

Next necessary and sufficient conditions for the continuity of spectrum by means of \( \alpha \)-paraspectrum is an easy consequence of the previous theorem.

**Corollary 2.** Let \( A \in B(X) \). Then the spectrum is continuous at \( A \) if and only if for every sequence \( \{ A_n \} \) such that \( A_n \to A \) one of the following equivalent conditions is satisfied:

1. \( \bigcap_{n=1}^{\infty} \sigma_a(A_n; A) \subset \lim \inf \sigma(A_n); \)
2. \( \bigcap_{n=1}^{\infty} \sigma_a(A_n; A) \subset \lim \inf \sigma(A_n). \)

**Theorem 3.** If for \( A, B \in B(X) \) is \( \sigma_\gamma(A; B) = \sigma_a(A; B) \), then the spectral variation inequality (1.1) holds for \( A \) and \( B \).

**Proof.** Let \( \sigma_\gamma(A; B) = \sigma_a(A; B) \). Since
\[ d_1(\sigma(A), \sigma(B)) = \sup_{\lambda \in \sigma(B) \mu \in \sigma(B)} \inf |\lambda - \mu| \]
and \( \sigma(B) \subset \sigma_a(A; B) = \sigma_\gamma(A; B) \) we have that
\[ d_1(\sigma(A), \sigma(B)) \leq ||A - B||, \quad \text{for every} \quad \lambda \in \sigma(B), \]
we have that the spectral variation inequality (1.1) holds for \( A \) and \( B \). \( \Box \)
**Corollary 4.** If for $A \in B(X)$ $\sigma_n(B;A) = \sigma_n(B;A)$ holds that for every $B \in B(X)$, then
the spectrum is continuous at $A$.

**Proof.** Let $\{A_n\}$ be a sequence in $B(X)$ such that $A_n \to A$. Since $d_1(\sigma(A_n),\sigma(A)) = d_2(\sigma(A),\sigma(A_n))$, Theorem 3 implies

$$d_2(\sigma(A),\sigma(A_n)) \leq ||A - A_n|| \to 0,$$

i.e. the spectrum is lower semi-continuous at $A$. Then it follows from [5, Theorem 1] that the spectrum is continuous at $A$. \qed

**Remark.** Recall that $\sigma_n(A;B) = \sigma_n(A;B)$ for every $A \in B(X)$ is not a necessary condition for the continuity of the spectrum at $B$. An example can be constructed by using [3, Example 4 (1)].

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be the matrix acting on a two-dimensional Hilbert space and $B = 2A^*A^*$. Then by [3, Example 4] it follows $\sigma_n(A;B) \neq \sigma_n(A;B)$. Since $\sigma(B)$ is totally disconnected, the spectrum is continuous at $B$ by [5, Theorem 3]. \[\square\]

It is well-known that the spectrum is a continuous function on the set of $G_1$-operators [3,4]. Now we can get it as an easy consequence of Theorem 3 and Corollary 4.

**Corollary 5.** If $A_n \in B(X)$ are $G_1$-operators and $A_n \to A$, then $\lim_n \sigma(A_n) = \sigma(A)$.

**Proof.** By [3, Theorem 3] we have $\sigma_n(A_n;A) = \sigma_n(A_n;A)$ for every $n \in \mathbb{N}$ and by Corollary 4 we have $\lim_n \sigma(A_n) = \sigma(A)$. \[\square\]

**Acknowledgement.** The author is grateful to the referee for helpful suggestions concerning the original version of the paper.

**References**


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