ON THE LATTICE OF IDEALS OF AN MV-ALGEBRA

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Received January 24, 2002

Abstract. For an MV-algebra \((A, +, \cdot, 0)\) we denote by \(I(A)\) the set of all ideals of \(A\). For \(I_1, I_2 \in I(A)\) we define \(I_1 \cap I_2 = I_1 \cap I_2, I_1 \vee I_2 = \) the ideal generated by \(I_1 \cup I_2\), and for \(I \in I(A), I^* = \{a \in A : a \wedge x = 0 \text{ for every } x \in I\} \).

The aim of this paper is to prove that \((I(A), \vee, \wedge, \cdot, \{0\}, A)\) is a Boolean lattice iff \(A\) is a finite Boolean lattice relative to the natural order on \(A\) (Theorem 2.8.)

1 Definitions and preliminaries

Definition 1.1 [2, 3]. An MV-algebra is an algebra \((A, +, \cdot, 0)\) of type \((2, 1, 0)\) satisfying the following equations:

\[
\begin{align*}
MV_1 & : x + (y + z) = (x + y) + z \\
MV_2 & : x + y = y + x \\
MV_3 & : x + 0 = x \\
MV_4 & : x^{**} = x \\
MV_5 & : x + 0^* = 0^* \\
MV_6 & : (x^* + y)^* + y = (y^* + x)^* + x.
\end{align*}
\]

MV-algebras were originally introduced by Chang in [2] in order to give an algebraic counterpart of the Łukasiewicz many valued logic (MV=many valued). Note that axioms \(MV_1-MV_5\) state that \((A, +, 0)\) is an abelian monoid; following tradition, we denote an MV-algebra \((A, +, \cdot, 0)\) by its universe \(A\).

Remark 1 If in \(MV_6\) we put \(y = 0\) we obtain \(x^{**} = 0^{**} + x\), so, if \(0^{**} = 0\) then \(x^{**} = x\) for every \(x \in A\). Hence, the axiom \(MV_4\) is equivalent with \(MV_6\) \(0^{**} = 0\).

Examples:

\(E_1\) A singleton \(\{0\}\) is a trivial example of an MV-algebra; an MV-algebra is said nontrivial provided its universe has more that one element.

\(E_2\) Let \((G, \oplus, - , 0, \leq)\) an l-group. For each \(u \in G, u > 0\), let

\[
[0, u] = \{x \in G : 0 \leq x \leq u\}
\]

and for each \(x, y \in [0, u]\), let \(x + y = x \wedge (x \oplus y)\) and \(x^* = u - x\). Then \(([0, u], +, \cdot, 0)\) is an MV-algebra. In particular, if consider the real unit interval \([0, 1]\) and for all \(x, y \in [0, 1]\) we define \(x \oplus y = \min\{1, x + y\}\) and \(x^* = 1 - x\), then \(([0, 1], \oplus, \cdot, 0)\) is an MV-algebra.

\(E_3\) If \((A, \vee, \wedge, \cdot, 0, 1)\) is a Boolean lattice, then \((A, \vee, ^*, 0)\) is an MV-algebra.

\textit{2000 Mathematics Subject Classification.} 06D35, 03G25.

\textit{Key words and phrases.} MV-algebra, Boolean lattice, ideal.
$E_1$) The rational numbers in $[0,1]$, and, for each integer $n \geq 2$, the $n$-element set $L_n = \left\{ \frac{0}{(n-1)\frac{1}{n-1}}, \frac{1}{(n-1)\frac{1}{n-1}}, \ldots, \frac{n-2}{(n-1)\frac{1}{n-1}}, 1 \right\}$ yield examples of subalgebras of $[0,1]$.

$E_2$) Given an $MV$-algebra $A$ and a set $X$, the set $A^X$ of all functions $f : X \to A$ becomes an $MV$-algebra if the operations $+$, and $\cdot$ and the element 0 are defined pointwise. The continuous functions from $[0,1]$ into $[0,1]$ form a subalgebra of the $MV$-algebra $[0,1]$.$^{[0,1]}$.

In the rest of this paper, by $A$ we denote an $MV$-algebra.

On $A$ we define the constant 1 and the operations $\cdot$ and $\cdot$ as follows: $1 = 0^*$, 

\[ x \cdot y = (x^* + y^*)^* \quad \text{and} \quad x - y = x \cdot y^* = (x^* + y)^* \] (we consider the $\cdot$ operation more binding than any other operation, and the $\cdot$ more binding that $+ \text{ and } -$).

**Lemma 1.2** $^{[3,4]}$ For $x, y \in A$, the following conditions are equivalent:

(i) $x^* + y = 1$
(ii) $x \cdot y^* = 0$
(iii) $y = x + (y - x)$
(iv) There is an element $z \in A$ such that $x + z = y$.

For any two elements $x, y \in A$ let us agree to write $x \leq y$ iff $x$ and $y$ satisfy the equivalent conditions (i)-(iv) in the above lemma. So, $\leq$ is a partial order relation on $A$ (which is called the natural order on $A$).

**Theorem 1.3** $^{[3,4]}$ If $x, y, z \in A$ then the following hold:

$c_1$) $1^* = 0$
$c_2$) $x + y = (x^* \cdot y^*)^*$
$c_3$) $x + 1 = 1$
$c_4$) $(x - y) + y = (y - x) + x$
$c_5$) $x + x^* = 1$
$c_6$) $x - 0 = x, 0 - x = 0, x - x = 0, 1 - x = x^*, x - 1 = 0$
$c_7$) $x + x = x$ iff $x \cdot x = x$
$c_8$) $x \leq y$ iff $y^* \leq x^*$
$c_9$) If $x \leq y$, then $x + z \leq y + z$ and $x \cdot z \leq y \cdot z$
$c_{10}$) If $x \leq y$, then $x - z \leq y - z$ and $z - y \leq z - x$
$c_{11}$) $x - y \leq x, x - y \leq y^*$
$c_{12}$) $(x + y) - x \leq y$
$c_{13}$) $x \cdot z \leq y$ iff $z \leq x^* + y$
$c_{14}$) $x + y + x \cdot y = x + y$
Remark 2 [3,4] On $A$, the natural order determines a bounded distributive lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements $x$ and $y$ are given by:
\[
x \vee y = (x - y) + y = (y - x) + x
\]
\[
x \wedge y = (x \vee y)^*\]
Clearly, $x \cdot y \leq x \wedge y \leq x \leq x \vee y \leq x + y$.

For each $x \in A$, we let $0 \cdot x = 0$, and for each integer $n \geq 0$, $(n + 1)x = nx + x$.

Theorem 1.4 [3,4] If $x, y, z, (x_i)_{i \in I}$ are elements of $A$, then the following hold:
\[c_{15}\] $x \cdot y = (x \vee y) \cdot (x \wedge y)$
\[c_{16}\] $x \cdot y = (x \vee y) \cdot (x \wedge y)$
\[c_{17}\] $x + \left( \bigcup_{i \in I} x_i \right) = \bigcup_{i \in I} (x + x_i)$
\[c_{18}\] $x + \left( \bigcap_{i \in I} x_i \right) = \bigcap_{i \in I} (x + x_i)$
\[c_{19}\] $x \cdot \left( \bigcup_{i \in I} x_i \right) = \bigcup_{i \in I} (x \cdot x_i)$
\[c_{20}\] $x \cdot \left( \bigcap_{i \in I} x_i \right) = \bigcap_{i \in I} (x \cdot x_i)$
\[c_{21}\] $x \wedge \left( \bigcup_{i \in I} x_i \right) = \bigcup_{i \in I} (x \wedge x_i)$
\[c_{22}\] $x \vee \left( \bigcap_{i \in I} x_i \right) = \bigcap_{i \in I} (x \vee x_i)$ (if all the suprema and infima exist).

Lemma 1.5 For every $x, y, z \in A$ we have
\[c_{23}\] $(x + y) - z \leq (x - z) + (y - z)$.

Proof. We have $(x + y - z)^* + (x - z) + (y - z) = (x + y)^* + (y - z) = (x + y)^* + (x + (y - z)) = (x + y)^* + (x \vee (y - z)) = (x + y)^* + (x \vee (y - z)) \leq (x + y)^* + (x \vee (y - z)) = (x + y)^* + (((x + (y - z)) \vee y) \vee z) = (x + y)^* + (((x + (y - z)) \vee (y - z)) \vee z) = (x + y)^* + (((x + (y - z)) \vee (y - z)) \vee z) = (x + y)^* + (((x \vee (y - z)) \vee (y - z)) \vee z) = (x + y)^* + (((x \vee y) + (y - z)) \vee z) = (x + y)^* + (((x \vee y) + (y - z)) \vee z) = (x + y)^* + (((x \vee y) + (y - z)) \vee z)$.

So, to prove $c_{23}$ it suffices to prove $x + y \leq (x \vee y) + y \leq (x + y) + y$ which result from $c_9$ (since $x \leq x \vee y$, hence $x + y \leq (x \vee y) + y \leq (x + y) + y$).

2 The lattice of ideals of an MV-algebra

Definition 2.1 A ideal of an MV-algebra $A$ is a non-empty subset $I$ of $A$ satisfying the following conditions:

$I_1\) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$

$I_2\) If $x, y \in I$ then $x + y \in I$. 
We denote by $I(A)$ the set of all ideals of $A$. For $M \subseteq A$ we denote by $(M)$ the ideal of $A$ generated by $M$ (that is $(M) = \cap \{ I \in I(A) | M \subseteq I \}$). If $M = \{a\}$ with $a \in A$, we denote by $(a)$ the ideal generated by $\{a\}$ which is called principal.

**Proposition 2.2** [3,4]

(i) If $M \subseteq A$, then $(M) = \{ x \in A : x \leq x_1 + \ldots + x_n \text{ for some } x_1, \ldots, x_n \in M \}$.

(ii) If $I_1, I_2 \in I(A)$, then $I_1 \cap I_2 = \{ a \in A : a \leq x_1 + x_2 \text{ for some } x_1 \in I_1 \text{ and } x_2 \in I_2 \}$.

(iii) If $x, y \in A$, then $(x) \cap (y) = (x \wedge y)$ (see [4, p.112]).

For $I \in I(A)$ and $a \in A \setminus I$ we denote by $I(a) = (a) \cup I = (I \cup \{a\})$.

**Remark 3** [3,4] For $I(a)$ we have the next characterization:

$I(a) = \{ x \in A : x \leq y + (na) \text{ for some } y \in I \text{ and integer } n \geq 0 \}$.

**Proposition 2.3** For $a \in A \setminus I$, $I(a) = \{ x \in A : x - a \in I \}$

**Proof.** Let $I_a = \{ x \in A : x - a \in I \}$. Since $a - a = 0 \in I$ we deduce that $a \in I_a$. Since for $x \in I$, $x - a \leq x$ (by c14) we deduce that $x - a \in I$, hence $I \subseteq I_a$. To prove $I_a \in I(A)$ we observe that $0 - a = 0 \in I$, hence $0 \in I_a$. If $x \leq y$ and $y \in I_a$, then $x - a \leq y - a$ (c10) and $y - a \in I$ we deduce $x - a \in I$, hence $x \in I_a$. Let $x, y \in I_a$, that is $x - a, y - a \in I$. From Lemma 1.5, we have $(x + y) - a \leq (x - a) + (y - a)$, hence $(x - y - a) \in I$ that is $x + y \in I_a$. From $a \in I_a$, $I \subseteq I$ and $I_a \in I(A)$ we deduce $I(a) \subseteq I_a$. Let now $J \in I(A)$ such that $a \in J$ and $I \subseteq J$. If $x \in I_a$, then $x - a \in I \subseteq J$, hence $x \vee a = (x - a) + a \in J$.

Since $a \leq x \vee a$ we deduce $x \in J$ that is $I_a \subseteq J$, hence $I_a \subseteq I \cup J = I(a)$. From $I(a) \subseteq I_a$ and $I_a \subseteq I(a)$ we deduce $I_a = I(a)$.

**Corollary 2.4** If $x, y \in A$ then $(x) \cup (y) = (x + y)$.

**Proof 1:** By Proposition 2.3, we have

$$(x) \cup (y) = (y)(x) = \{ a \in A : a - x \in (y) \}.$$ 

Since by c12 $(x + y) - x \leq y$, we deduce $x + y \in (x) \cup (y)$, hence $(x + y) \subseteq (x) \cup (y)$. Since the inclusion $(x) \cup (y) \subseteq (x + y)$ is obviously, we obtain the equality $(x) \cup (y) = (x + y)$.

**Proof 2:** It is sufficient to show the inclusion $(x + y) \subseteq (x) \cup (y)$. If $z \in (x + y)$ then $z \leq n(x + y)$ for some integer $n \geq 0$. But $n(x + y) = (nx) + (ny)$ and so $z \leq (nx) + (ny)$. Since $nx \in (x)$ and $ny \in (y)$ we deduce that $z \in (x) \cup (y)$, that is $(x + y) \subseteq (x) \cup (y)$.

For $I_1, I_2 \in I(A)$, we put $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2 = (I_1 \cup I_2)/I_1 \rightarrow I_2 = \{ a \in A : (a \cap I_1 \subseteq I_2 \}$.

Then $(I(A), \wedge, \wedge, \{0\}, A)$ is a complete Brouwerian lattice ([4, p.114]); we recall that a complete lattice is Brouwerian if it satisfies the identity $a \wedge \bigg( \bigvee_{i \in I} b_i \bigg) = \bigvee_{i \in I} (a \wedge b_i)$.

**Lemma 2.5** If $I_1, I_2 \in I(A)$, then

(i) $I_1 \rightarrow I_2 \in I(A)$

(ii) If $I \in I(A)$, then $I \cap I \subseteq I$ if $I \subseteq I \rightarrow I_2$ (that is, $I_1 \rightarrow I_2 = \sup \{ I \in I(A) : I_1 \cap I \subseteq I_2 \}$).
Proof (i) Since \((0) \cap I_1 \subseteq I_2\) we deduce that \(0 \in I_1 \rightarrow I_2\). If \(x, y \in A, x \leq y\) and \(y \in I_1 \rightarrow I_2\), then \([y] \cap I_1 \subseteq I_1\). Since \([x] \subseteq [y]\) we deduce that \([x] \cap I_1 \subseteq [y] \cap I_1 \subseteq I_2\), hence \(x \in I_1 \rightarrow I_2\). Let now \(x, y \in I_1 \rightarrow I_2\); then \((x) \cap I_1 \subseteq I_2\) and \((y) \cap I_1 \subseteq I_2\). We deduce \(((x) \cap I_1) \lor \((y) \cap I_1) \subseteq I_2\) hence \((x \lor y) \cap I_1 \subseteq I_2\), so \((x + y) \cap I_1 \subseteq I_2\) (by Corollary 2.4), that is \(x + y \in I_1 \rightarrow I_2\).

(ii) \(\iff\) Let \(I \in I(A)\); then \(I_1 \cap I \subseteq I_2\). If \(x \in I\) then \((x) \cap I_1 \subseteq I \cap I_1 \subseteq I_2\) hence \(x \in I_1 \rightarrow I_2,\) that is \(I \subseteq I_1 \rightarrow I_2\).

(\(\iff\)) We suppose \(I \subseteq I_1 \rightarrow I_2\) and let \(x \in I_1 \cap I\); then \(x \in I\), hence \(x \in I_1 \rightarrow I_2\), that is \(x \cap I_1 \subseteq I_2\). Since \(x \in \{x \cap I \mid I \in I(A)\}\), then \(x \in I_2\), that is \(I_1 \cap I_1 \subseteq I_2\).

Remark 4 From Lemma 2.5, we deduce that \((I(A), \lor, \land, \rightarrow, \{0\}), A\) is a Heyting algebra; for \(I \in I(A), I^* = I \rightarrow \{0\} = \{x \in A : (x) \cap I = \{0\}\}.

Corollary 2.6 (i) For every \(I \in I(A)\), \(I^* = \{x \in A : x \land y = 0 \text{ for every } y \in I\}\) (see [4, p.11]).

(ii) For any \(x \in A\), \((x)^* = \{y \in A : (y) \cap (x) = \{0\}\} = \{y \in A : x \land y = 0\}\) (by Proposition 2.2. (iii)).

We recall that for a bounded distributive lattice \(L\), following tradition, by \(B(L)\) we denoted the Boolean lattice of complemented elements in \(L\).

For an \(MV\)-algebra \((A, +, *, 0, 1)\) we shall denote by \(B(A)\) the Boolean lattice associated with the bounded distributive lattice \((A, \lor, \land, 0, 1)\).

Proposition 2.7 [4, p. 127] For every \(x \in A\), the following conditions are equivalent:

(i) \(x \in B(A)\)

(ii) \(x + x = x\)

(iii) \(x \cdot x = x\)

(iv) \(x \land x^* = 0\)

(v) \(x \lor x^* = 1\).

Theorem 2.8 If \(A\) is an \(MV\)-algebra, then the following conditions are equivalent:

(i) \((I(A), \lor, \land, \rightarrow, \{0\}), A\) is a Boolean lattice

(ii) \((A, \lor, \land, *, 0, 1)\) is a finite Boolean lattice.

Proof (i) \(\iff\) (ii). Let \(x \in A\); since \(I(A)\) is a Boolean lattice then \((x) \lor (x)^* = A\). By Proposition 2.3. and Corollary 2.6. (ii), we have \((x) \lor (x)^* = (x)^* = \{y \in A : y \leq x \in (x)^*\} = \{y \in A : (y - x) \land x = 0\}\). Since \((x) \lor (x)^* = A\), then \(1 \in (x) \lor (x)^*\), hence \((1 - x) \land x = 0\). We obtain that \(x^* \land x = 0\), hence \(x \in B(A)\) (by Proposition 2.7. (iii)), that is \((A, \lor, \land, *, 0, 1)\) is a Boolean lattice. To show that \(A\) is finite it suffices to prove that every ideal of \(A\) is principal ([5, p. 77]). If \(I \in I(A)\), because \(I(A)\) is supposed Boolean lattice then \(I \cap I^* = A\), hence \(1 \in I \cap I^*\) . By Proposition 2.2. (i), \(a + b = a + b\) for every \(a \in I\). By Corollary 2.6. (i), \(x \land b = 0\) for every \(x \in I\). So \((x) \lor (b)^* = 0 \iff x^* \lor b = 1 \iff (x + b)^* \land b = 1 \iff x + b \leq b \iff x + b = b\) for every \(x \in I\). Since \(a + b = 1\) we obtain \(b^* \leq a\) hence \(x + b = b^* \leq a\) for every \(x \in I\). Finally, we obtain \(x \leq x + b \leq a\), hence \(x \leq a\) for every \(x \in I\), that is \(I = (a)\).

(ii) \(\iff\) (i). Suppose \((A, \lor, \land, *, 0, 1)\) is a finite Boolean lattice. By Remark 4, \(I(A)\) is a Heyting algebra. To prove \(I(A)\) is a Boolean lattice we must show \(I^* = \{0\}\) only for \(I = A\) ([1, p. 179]). Since in finite Boolean lattice every ideal is principal, then \(I = (a)\) for some \(a \in A\). By Corollary 2.6. (iii), \(I^* = (a)^* = \{x \in A : x \land a = 0\}\). Since \(I^* = \{0\}\) and \(a^* \land a = 0\), then \(a^* = 0\), hence \(a = 1\) so \(I = (1) = A\).
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