ESTIMATES FOR MODULI OF COEFFICIENTS OF POSITIVE TRIGONOMETRIC POLYNOMIALS

Dedicated to Professor Tsuyoshi Ando on his seventieth birthday

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ABSTRACT. Suppose that a trigonometric polynomial

$$
\tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \alpha_k e^{i k \theta}, \quad \theta \in [0, 2\pi],
$$

is positive, \(\alpha_{N-1} \neq 0\), \(N \geq 2\). Then a classical matter due to Fejér asserts that the estimate

$$
|\alpha_1| \leq \alpha_0 \cos \frac{\pi}{N+1}
$$

for the modulus \(|\alpha_1|\) of \(\alpha_1\) holds and that the equality occurs only for the polynomial

$$
\alpha_0 \tau_N(e^{i(\theta-\varphi)}),
$$

where

$$
\tau_N(e^{i\theta}) = \frac{2}{N+1} \sum_{k=0}^{N-1} \left( \sin \left( \frac{(k+1)\pi}{N+1} \right) \right) e^{i k \theta}, \quad \theta \in [0, 2\pi],
$$

and \(\varphi \in [0, 2\pi)\). In this paper, we will show that the corresponding estimate

$$
|\alpha_n| \leq \alpha_0 \cos \frac{\pi}{\lfloor N/n \rfloor + 1}
$$

for the modulus \(|\alpha_n|\) of \(\alpha_n\) is true, \(1 \leq n \leq N-1\), \(\lfloor N/n \rfloor\) the minimum integer not smaller than \(N/n\), and that the equality for \(n = n_0\) occurs only for the polynomial \(\tau\) of the form

$$
\tau(e^{i\theta}) = \sigma(e^{i\theta})\tau_{N/n_0}(e^{i(n_0\theta-\varphi)}), \quad \theta \in [0, 2\pi],
$$

where \(\sigma\) is a positive trigonometric polynomial and \(\varphi \in [0, 2\pi)\).

1. Introduction.

Let \(S_N\), where \(N \geq 2\), be the \(N \times N\) shift matrix, i.e.,

$$
S_N = \begin{pmatrix}
0 & 1 & 0 \\
1 & \ddots & 1 \\
0 & \ddots & 1 \\
1 & & 1
\end{pmatrix}.
$$

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Then it is known by Davidson and Holbrook [1], Corollary 2, that for $n$ with $1 \leq n \leq N - 1$, $\lfloor N/n \rfloor$ denoting the minimum integer not smaller than $N/n$ (which in fact is $\lfloor (N - 1)/n \rfloor + 1$), the numerical radius

$$w((S_N)^n) = \sup_{\|\zeta\| = 1} |(S_N^n)^n \zeta, \zeta|$$

of the power $(S_N)^n$ of $S_N$ coincides with $\cos \left( \frac{\pi}{\lfloor N/n \rfloor + 1} \right)$. But, in the case when $n = 1$, Haagerup and de la Harpe [3], Proposition 1 (and T. Yoshino [5], Lemmas 6 and 7, p.134, also) proves that, given a unit vector $\zeta \in C^N$, the equality

$$\langle S_N \zeta, \zeta \rangle = \cos \left( \frac{\pi}{N + 1} \right)$$

holds if and only if

$$\zeta = e^{i\varphi} \zeta_1 \quad \text{for some} \quad \varphi \in [0, 2\pi),$$

where $\zeta_1$ is the vector in $C^N$ of which $m$th coordinate is

$$\left( \frac{2}{N + 1} \right)^{1/2} \sin \frac{m\pi}{N + 1}, \quad 1 \leq m \leq N.$$

Haagerup and de la Harpe observed further that this serves to lead us to the classical matter due to Fejér ([2]; [4], 8.4) which asserts that if a trigonometric polynomial

$$\tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \alpha_k e^{ik\theta}, \quad \theta \in [0, 2\pi),$$

is positive, namely

$$\tau(e^{i\theta}) \geq 0 \quad \text{for any} \quad \theta \in [0, 2\pi),$$

and not identically zero (or equivalently $\alpha_0 > 0$) with $\alpha_{N-1} \neq 0$, then one has the estimate

$$|\alpha_1| \leq \alpha_0 \cos \frac{\pi}{N + 1}$$

for the modulus of $\alpha_1$, and the equality occurs only for the polynomial $\alpha_0 \tau_N(e^{i(\theta-\varphi)})$, where

$$\tau_N(e^{i\theta}) = \frac{2}{N + 1} \sum_{k=0}^{N-1} \left( \sin \frac{(k + 1)\pi}{N + 1} \right) e^{ik\theta}, \quad \theta \in [0, 2\pi).$$

It is easy for us to give the corresponding estimates for the moduli $|\alpha_n|$ of the $n$th coefficients $\alpha_n$ of $\tau$, $-N + 1 \leq n \leq N - 1$ (but for the case $n = 0$ we give an appropriate understanding). In fact, By the Fejér-Riesz theorem (See [2], [4]), there exists a polynomial

$$\sigma(e^{i\theta}) = \sum_{k=0}^{N-1} \gamma_k e^{ik\theta}$$

such that

$$\tau(e^{i\theta}) = |\sigma(e^{i\theta})|^2 = \sum_{k,l=0}^{N-1} \gamma_k \gamma_l e^{i(k-l)\theta}.$$
(So it is immediate that $\alpha_{-n} = \bar{\alpha}_n$, $-N + 1 \leq n \leq N - 1$). Let $\zeta$ be the vector in $C^N$ of which $k$th coordinate are $\gamma_{k-1}$, $1 \leq k \leq N$. Then we have

$$\alpha_0 = ||\zeta||^2 \quad \text{and} \quad \alpha_n = \langle (S_N)^n \zeta, \zeta \rangle, \quad 1 \leq n \leq N - 1.$$  

Therefore, by [1], Corollary 2, it actually follows that

$$|\alpha_n| = ||\zeta||^2 \left| \left( \frac{(S_N)^n \zeta}{\| \zeta \|} \right) \right| \leq \alpha_0 \cos \frac{\pi}{|N/n| + 1}.$$  

We will devote ourselves in the following two sections to determining the polynomial $\tau$ for which the equality

$$|\alpha_n| = \alpha_0 \cos \frac{\pi}{|N/n| + 1}, \quad 1 \leq n \leq N - 1,$$

occurs. In the last section, an application will be given to positive "operator-valued" trigonometric polynomials.

**2. Unit vectors which attain the numerical radius of $(S_N)^n$.**

For the sake of convenience, we identify, through the canonical manner, the space $C^N$ with a subspace of the space $C^{[N/n]} \otimes C^n$, and accordingly the power $(S_N)^n$ of $S_N$ with the operator $P_n(S_{[N/n]} \otimes I_n)\{C^N\}$ which restricts the operator $P_n(S_{[N/n]} \otimes I_n)$ on $C^N$, $I_n$ the $n \times n$ unit matrix, $P_n$ the orthogonal projection from $C^{[N/n]} \otimes C^n$ onto $C^N$.

Let $\xi_k \in C^{[N/n]}$ be the unit vector of which $m$th coordinate is

$$\left( \sum_{\nu=1}^{[N/n]} \sin^2 \frac{k \nu \pi}{|N/n| + 1} \right)^{-1/2} \sin \frac{k m \pi}{|N/n| + 1}, \quad 1 \leq m \leq [N/n],$$

and $\iota_l \in C^n$ the unit vector of which $l$th coordinate is 1 and others 0. Then the vectors $\xi_k \otimes \iota_l$, $1 \leq k \leq [N/n]$, $1 \leq l \leq n$, make an orthonormal basis for $C^{[N/n]} \otimes C^n$.

**Lemma 1** Let $1 \leq n \leq N - 1$, and let $\zeta \in C^N$ be a unit vector. Then

$$\langle (S_N)^n \zeta, \zeta \rangle = \cos \frac{\pi}{|N/n| + 1}$$

occurs if and only if $\zeta \in C^N$ is of the form

$$\zeta = P_n(\xi_1 \otimes \eta),$$

where $\eta = \sum_{l=1}^{r} \beta_l \iota_l$ with $\sum_{l=1}^{r} |\beta_l|^2 = 1$, $r = N - ([N/n] - 1) n$.

**Proof.** First assume that $n$ divides $N$, that $\zeta \in C^N$ is a unit vector and that

$$\langle (S_N)^n \zeta, \zeta \rangle = \cos \frac{\pi}{N/n + 1}.$$
Put
\[ \zeta = \sum_{1 \leq k \leq N/n, \ 1 \leq t \leq n} \beta_{k,t} \xi_k \otimes \iota_t, \quad \text{with} \quad \sum_{1 \leq k \leq N/n, \ 1 \leq t \leq n} |\beta_{k,t}|^2 = 1. \]

Then, since
\[ \text{Re}(S_{N/n})\xi_k = \left( \cos \frac{k\pi}{N/n + 1} \right) \xi_k, \ 1 \leq k \leq N/n, \]
we have
\[
\cos \frac{\pi}{N/n + 1} = \langle (S_{N/n} \otimes I_n)\zeta, \zeta \rangle = \sum_{k_l \leq t_l,} \beta_{k,t} \beta_{k',t'} \left( \text{Re}(S_{N/n})\xi_k, \xi_{k'} \right) \langle \iota_t, \iota_{t'} \rangle \\
= \sum_{k_l \leq t_l,} \beta_{k,t} \beta_{k',t'} \left( \cos \frac{k\pi}{N/n + 1} \right) \xi_k, \xi_{k'} \langle \iota_t, \iota_{t'} \rangle \\
= \sum_{k_l \leq t_l,} |\beta_{k,t}|^2 \cos \frac{k\pi}{N/n + 1}.
\]

This shows that \( \beta_{k,t} = 0 \) for \( k \geq 2 \). So, putting \( \beta_t = \beta_{1,t} \), we have
\[ \eta = \sum_{t=1}^n \beta_{1,t} \quad \text{and} \quad \sum_{t=1}^n |\beta_t|^2 = 1. \]

Next assume that \( n \) does not divide \( N \), and that a unit vector \( \zeta \in C^N \) satisfies
\[ \langle (S_N)^n \zeta, \zeta \rangle = \cos \frac{\pi}{N/n} \]
Then we have \( \langle (S_{N/n})^n \otimes I_n)\zeta, \zeta \rangle = \cos \frac{\pi}{N/n + 1} \). It follows that \( \zeta \) is of the form
\[ \zeta = \xi_1 \otimes \sum_{t=1}^n \beta_{1,t} \]
with \( \sum_{t=1}^n |\beta_t|^2 = 1 \). But one has \( \beta_t = 0 \) if \( l > r \), since \( \zeta \) is in \( C^N \).

3. Positive polynomial for which the modulus of \( \alpha_n \) attains the bound.

Now we will show the aimed theorem in this paper:

Theorem 2 Suppose that a trigonometric polynomial
\[ \tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} \alpha_k e^{ik\theta}, \quad \theta \in [0, 2\pi), \]
is positive and such that $\alpha_{N-1} \neq 0$, $N \geq 2$. If $1 \leq n_0 \leq N - 1$, and the equality

$$|\alpha_{n_0}| = \alpha_0 \cos \frac{\pi}{[N/n_0] + 1}$$

holds, then $\tau$ is of the form

$$\tau(e^{i\theta}) = \sigma(e^{i\theta})\tau_{[N/n_0]}(e^{in_0(\theta - \varphi)}), \quad \theta \in [0, 2\pi),$$

where $\sigma$ is a positive trigonometric polynomial of degree $\tau_0 - 1$, $\tau_0 = N - ([N/n_0] - 1)n_0$, $\tau_{[N/n_0]}$ the trigonometric polynomial already introduced and $\varphi \in [0, 2\pi)$. Moreover, for any $n \neq n_0$, $1 \leq n \leq N - 1$, one has

$$|\alpha_n| < \alpha_0 \cos \frac{\pi}{[N/n] + 1}$$

Conversely, for the polynomial $\sigma(e^{i\theta})\tau_{[N/n_0]}(e^{in_0(\theta - \varphi)})$, the modulus $|\alpha_{n_0}|$ of $\alpha_{n_0}$ is equal to $\alpha_0 \cos \frac{\pi}{[N/n_0] + 1}$.

**Proof.** By the Fejér-Riesz theorem one has a polynomial

$$\sigma(e^{i\theta}) = \sum_{k=0}^{N-1} \gamma_k e^{ik\theta}$$

such that

$$\tau(e^{i\theta}) = |\sigma(e^{i\theta})|^2 = \sum_{k,l=0}^{N-1} \gamma_k \gamma_l e^{i(k-l)\theta}.$$

Assume that the equality

$$|\alpha_{n_0}| = \alpha_0 \cos \frac{\pi}{[N/n_0] + 1}$$

holds for $n_0, 1 \leq n_0 \leq N - 1$.

First we let $\alpha_0 = 1$ and $\alpha_{n_0} \geq 0$. The vector $\zeta$ of which $k$th coordinate is $\gamma_{k-1}$ ($1 \leq k \leq N$) achieves the numerical radius $\nu((S_N)^{n_0})$ of the matrix $(S_N)^{n_0}$, so, by Lemma 1, $\zeta$ is of the form

$$\zeta = P_{n_0} \left( \xi_1 \oplus \sum_{l=1}^{n_0} \beta_l \iota_l \right), \quad \sum_{l=1}^{n_0} |\beta_l|^2 = 1,$$

where $P_{n_0}$ is the orthogonal projection from $C^{[N/n_0]} \otimes C^{n_0}$ onto $C^N$, $\xi_1$ the unit vector in $C^{[N/n_0]}$ of which $k$th coordinate is

$$\left( \frac{2}{[N/n_0] + 1} \right)^{1/2} \sin \frac{k\pi}{[N/n_0] + 1}, \quad 1 \leq k \leq [N/n_0],$$

$\iota_l$ the unit vector in $C^{n_0}$ of which $l$th coordinate is 1 and others 0, $1 \leq l \leq n_0$. Then we have

$$\gamma_k = \beta_l \left( \frac{2}{[N/n_0] + 1} \right)^{1/2} \sin \frac{(j + 1)\pi}{[N/n_0] + 1}$$
if \( k = l + n_0j - 1, \ 1 \leq l \leq r_0, \ 0 \leq j \leq \lceil N/n_0 \rceil - 1, \) and \( \gamma_k = 0 \) otherwise. Therefore, we have

\[
\tau(e^{i\theta}) = \left| \sum_{k=0}^{N-1} \gamma_k e^{ik\theta} \right|^2
\]

\[
= 2 \left( \frac{1}{n_0} \sum_{j=0}^{\lceil N/n_0 \rceil - 1} \sum_{l=1}^{r_0} \beta_l \sin \left( \frac{(j + 1)\pi}{n_0} \right) e^{i(l-1+n_0j)\theta} \right)^2
\]

\[
= \left( \sum_{l=1}^{r_0} \beta_l e^{i(l-1)\theta} \right)^2 \left( \frac{2}{n_0} \sum_{j=1}^{\lceil N/n_0 \rceil - 1} \sin \left( \frac{(j + 1)\pi}{n_0} \right) e^{in_0j\theta} \right)^2,
\]

and \( \beta_{r_0} \neq 0. \) Therefore, putting

\[
\sigma(e^{i\theta}) = \left| \sum_{l=1}^{r_0} \beta_l e^{i(l-1)\theta} \right|^2, \quad \theta \in [0, 2\pi),
\]

which in fact is positive, we have

\[
\tau(e^{i\theta}) = \sigma(e^{i\theta}) \tau_{\lceil N/n_0 \rceil}(e^{in_0\theta}), \quad \theta \in [0, 2\pi).
\]

Assume that

\[
|\alpha_{n_1}| = \cos \frac{\pi}{n_1 + 1}
\]

holds for \( n_1, 1 \leq n_1 \leq N - 1. \) Then \( \zeta \) is of the form

\[
\zeta = \Pi_{n_1}^\perp (\xi_1^1 \otimes \sum_{l=1}^{r_1} \beta_l^l),
\]

where \( \Pi_{n_1}^\perp \) the projection from \( C^{[N/n_1]} \circ C^{m_1} \) onto \( C^N, \) \( \xi_1^1 \) the vector in \( C^{[N/n_1]} \) of which \( k \)th coordinate is

\[
e^{i\psi_k} \left( \frac{2}{[N/n_1] + 1} \right)^{1/2} \sin \frac{k\pi}{[N/n_1] + 1}, \quad \psi_k \in [0, 2\pi), \ 1 \leq k \leq \lceil N/n_1 \rceil,
\]

\( \xi_1^l \) the vector in \( C^{n_1} \) of which \( l \)th coordinate is 1 and others 0 and \( r_1 = N - (\lceil N/n_1 \rceil - 1) n_1. \)

Therefore, we have

\[
\Pi_{n_1}^\perp (\xi_1^1 \otimes \sum_{l=1}^{r_1} \beta_l^l) = \Pi_{n_1}^\perp (\xi_1^1 \otimes \sum_{l=1}^{r_0} \beta_l^l).
\]

But it occurs only when \( r_1 = r_0 \) and \( n_1 = n_0. \)

Now we turn to the general case. We apply the foregoing argument to the positive trigonometric polynomial

\[
\tau(e^{i\theta}) = \tau(e^{i(\theta - \varphi)})/\alpha_0, \quad \theta \in [0, 2\pi),
\]

\( \varphi = \arg \alpha_{n_0}/n_0. \) Then we have the desired conclusion.

Conversely, let

\[
\tau(e^{i\theta}) = \sigma(e^{i\theta}) \tau_{\lceil N/n_0 \rceil}(e^{in_0(\theta - \varphi)}), \quad \theta \in [0, 2\pi),
\]
where $\sigma$ is a positive trigonometric polynomial, then we can easily have the equality
\[ |\alpha_{n_0}| = \alpha_0 \cos \frac{\pi}{|N/n_0| + 1}. \]

QED

4. An application to operator-valued trigonometric polynomials.

Theorem 2 yields the estimates for numerical radii of operators which are coefficients of positive operator-valued trigonometric polynomials:

**Corollary 3** Let $A_k$ be bounded operators on a Hilbert space $H$, $-N+1 \leq k \leq N-1$, $N \geq 2$. Suppose that
\[ \tau(e^{i\theta}) = \sum_{k=-N+1}^{N-1} A_k e^{ik\theta} \geq O \]
for any $\theta \in [0,2\pi)$. Then, $A_0 \geq O$ and one has
\[ \nu(A_n) \leq \|A_0\| \cos \frac{\pi}{|N/n| + 1}, \quad 1 \leq n \leq N-1. \]

**Proof.** Let $\zeta \in H$ and $||\zeta|| = 1$. Then
\[ \tau(\zeta e^{i\theta}) = \sum_{k=-N+1}^{N-1} \langle A_k \zeta, \zeta \rangle e^{ik\theta}, \quad \theta \in [0,2\pi), \]
is a positive trigonometric polynomial. So it follows that $A_0 \geq O$. If $\langle A_0 \zeta, \zeta \rangle > 0$, then we know that the inequality
\[ |\langle A_n \zeta, \zeta \rangle| \leq \langle A_0 \zeta, \zeta \rangle \cos \frac{\pi}{|N/n| + 1} \]
holds. If $\langle A_0 \zeta, \zeta \rangle = 0$, then we have $\langle A_n \zeta, \zeta \rangle = 0$, $1 \leq n \leq N-1$, and so, we know that the above inequality turns out to be trivial. Hence we have
\[ \nu(A_n) \leq \|A_0\| \cos \frac{\pi}{|N/n| + 1}. \]

QED

**References**


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