LÖWNER-HEINZ THEOREM AND OPERATOR MEANS

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Abstract. Based on the Kubo-Ando theory of operator means, we give a proof of the well-known Löwner-Heinz theorem which asserts that for bounded linear operators $A$ and $B$, if $A \geq B \geq 0$ then $A^p \geq B^p$ for $0 \leq p \leq 1$. A key fact for the proof of the theorem is its special case for $p = 1/2$: if $A \geq B \geq 0$ then $A^{1/2} \geq B^{1/2}$, which says that the geometric mean $X^{1/2}$ of the identity operator $1$ and a positive operator $X$ is monotone. We give a short proof of this fact, using the arithmetic-harmonic mean defined by J. I. Fujii.

1. Throughout this note, a capital letter means a (bounded linear) operator on a Hilbert space $H$. An operator $A$ is said to be positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. Then it induces the order $A \geq B$ for selfadjoint operators $A$ and $B$. The following result called Löwner-Heinz theorem [8], [12] is well-known:

Theorem A. Let $A$ and $B$ be positive operators. Then

$$A \geq B \implies A^p \geq B^p \quad \text{for} \quad 0 \leq p \leq 1.$$  

This theorem says that the function $t \mapsto t^p$ with $0 \leq p \leq 1$ is operator monotone on $[0, \infty)$. Recently Furuta [7] presented an exquisite extension of the inequality (1), called Furuta inequality, which enjoys the great worth of the theorem.

The proof of the theorem was initiated by Löwner [12] in the complete description of operator monotone functions, later a clear expression of the theorem was given by Heinz [8], and a completely operator theoretic proof was given by Kato [10]. A lot of authors since then gave proofs of the theorem ([1], [3], [9], [13], etc.). Among them it is noted that Pedersen [13] gave a proof, using fundamental properties of the spectral radius of an operator, and that Ando [1] obtained the theorem from operator monotonicity of the geometric mean defined on positive operators.

A proof of the theorem is to take full advantage of the following reduced inequality for $p = 1/2$, that is,

Theorem B. Let $A$ and $B$ be positive operators. Then

$$A \geq B \implies A^{1/2} \geq B^{1/2}.$$  

Several authors ([2], [6], [9], [13], [14], etc.) have already indicated, explicitly or implicitly, that Theorem B is equivalent to Theorem A.

In this note, we first give an elementary proof of Theorem B, using a technique due to J. I. Fujii [4], [5], and next show a proof of Theorem A based on the Kubo-Ando theory of operator means [11].

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2. Following [11], we recall the basic three operator means: For $A$, $B \geq 0$,

(A) arithmetic mean 
\[ A \nabla B = \frac{1}{2} (A + B), \]

(H) harmonic mean 
\[ A \!^{-1} B = \left( \frac{1}{2} (A^{-1} + B^{-1}) \right)^{-1} = 2 (A + B)^{-1} B, \]

and

(G) geometric mean 
\[ A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}. \]

In the above definitions (H) and (G), both $A$ and $B$ (or at least one of them) must be assumed to be invertible. Without any assumption they are well-defined as the (strong operator) limits of $(A + \varepsilon I) \!^{-1} (B + \varepsilon I)$ and $(A + \varepsilon I) \# (B + \varepsilon I)$ as $\varepsilon \downarrow 0$ respectively. (1 is the identity operator.) For simplicity of discussions, from now on we assume that all positive operators are invertible.

Among the three operator means (A), (H) and (G), the following fundamental inequalities hold (cf. [11]):

(3) 
\[ A \nabla B \geq A \# B \geq A \!^{-1} B. \]

(It is very easy to obtain these inequalities, in particular, if $A$ and $B$ commute.) For the monotone property of those means, we can easily see that if $A \geq B$ and $C \geq D$ then

(4) 
\[ A \nabla C \geq B \nabla D \quad \text{and} \quad A \!^{-1} C \geq B \!^{-1} D. \]

However, it need some device to obtain

(5) 
\[ A \# C \geq B \# D. \]

Since $A \# 1 = A^{1/2}$, we see that (2) of Theorem B is nothing but a special case of (5). Now at the present, we prove the special case, employing the arithmetic-harmonic mean technique presented by J. I. Fujii [4], [5].

**Proof of Theorem B.** First let $X_0 = 1$, $Y_0 = A$, and following [4], [5], define

\[ X_n = X_{n-1} \nabla Y_{n-1} \quad \text{and} \quad Y_n = X_{n-1} \!^{-1} Y_{n-1} \quad (n = 1, 2, \ldots). \]

Then we see that $X_n$ and $Y_n$ commute for all $n$, and that

\[ X_n Y_n = X_{n-1} Y_{n-1} = \cdots = X_0 Y_0 = A. \]

Since $X_n \geq Y_n$ for $n \geq 1$ by (3) we can see that

\[ X_1 \geq \cdots \geq X_n \geq Y_n \geq \cdots \geq Y_1. \]

Furthermore, $2(X_n - X_{n+1}) = X_n - Y_n \geq 0$, so that we obtain $A^{1/2} = 1 \# A$ as the common limit of $\{X_n\}$ and $\{Y_n\}$, which is nothing but the arithmetic-harmonic mean of $X_0 = 1$ and $Y_0 = A$. Next in the same manner as before, putting $Z_0 = 1, W_0 = B$,

\[ Z_n = Z_{n-1} \nabla W_{n-1} \quad \text{and} \quad W_n = Z_{n-1} \!^{-1} W_{n-1} \quad (n = 1, 2, \ldots), \]
we can similarly obtain $B^{1/2}$ as the common limit of $\{Z_n\}$ and $\{W_n\}$. It is also not difficult
to see that $X_n \geq Z_n$ and $Y_n \geq W_n$ for $n = 1, 2, \ldots$. Taking the limits, we then obtain
$A^{1/2} \geq B^{1/2}$. □

From the definition (G) and Theorem B we now obtain the following fact, a little weaker
than (5). (Afterwords we shall mention of (5) again.)

(6) \quad A \geq B \quad \text{and} \quad C \geq 0 \quad \text{imply} \quad C\#A \geq C\#B.

In fact, since $DAD \geq DBD$ for any $D \geq 0$, we have, applying Theorem B,

$$C\#A = C^{1/2}(C^{-1/2}AC^{-1/2})^{1/2}C^{1/2} \geq C^{1/2}(C^{-1/2}BC^{-1/2})^{1/2}C^{1/2} = C\#B.$$  

At this stage we give

**Proof of Theorem A. (cf. [6, Lemma 1])** By norm continuity of $p \mapsto A^p$, we may show
that (1) holds for every $p$ such that $p = m/2^k$, $k = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^k$ for each $k$.
We take the mathematical induction with respect to $k$. For the first step, (1) clearly holds
for $k = 1$. For the next step, assuming that (1) holds for $k = n$, we may show it for $k = n+1$. We then consider the two cases (i) $1 \leq m \leq 2^n$ and (ii) $2^n + 1 \leq m \leq 2^{n+1}$. For (i), since
$A^{m/2^n} \geq B^{m/2^n}$ by assumption, we have $A^{m/2^{n+1}} = 1\#A^{m/2^n} \geq 1\#B^{m/2^n} = B^{m/2^{n+1}}$.
For (ii),

$$A^{m/2^{n+1}} = B^{1/2} \\{ (B^{-1/2}A^{m/2^{n+1}}B^{-1/2})^{1/2} \} B^{1/2} = B\#(A^{m/2^{n+1}}A^{-1}A^{m/2^{n+1}}) \quad \text{(by $B^{-1} \geq A^{-1}$ and (6))} = B\#A^{m/2^n - 1} \quad \text{(by $A^{m/2^n - 1} \geq B^{m/2^n - 1}$ and (6))} = B^{m/2^{n+1}}.$$  

□

**Remark 1.** By uniqueness of the square root of a positive operator we can see that
$X = A\#B$ if and only if $XA^{-1}X = B$ or $(A^{-1/2}XA^{-1/2})^2 = A^{-1/2}BA^{-1/2}$ for $X \geq 0$.
Since $XA^{-1}X = B$ is equivalent to $XB^{-1}X = A$, we have

(7) \quad A\#B = B\#A.

With this identity (7) and the inequality (6) we now obtain the desired (5) as follows : if
$A \geq B$ and $C \geq D$, then

$$A\#C \geq A\#D = D\#A \geq D\#B = B\#D.$$  

In [1], Ando defined $A\#B$ by

$$A\#B = \max \left\{ X \geq 0 \left[ \begin{array}{cc} A & X \\ X & B \end{array} \right] \geq 0 \right\},$$

which is equivalent to the definition (G). From this definition he obtained (5) immediately.
Remark 2. Using (5), we can give another proof of Theorem A (cf. [1, Corollary I. 2. 2]): Let $\Delta = \{ p \in [0, 1]; A^p \geq B^p \}$. Then by norm continuity of $p \mapsto A^p$, $\Delta$ is closed, so that we may only show that $\Delta$ is convex. Let $q, r \in \Delta$, that is, $A^q \geq B^q$ and $A^r \geq B^r$. Then
\[ A^{(q+r)/2} = A^q \# A^r \geq B^q \# B^r = B^{(q+r)/2}, \]
which implies $(q + r)/2 \in \Delta$, that is, convexity of $\Delta$.

References

[7] T. Furuta, *A \geq B \geq 0 assures $(B^p A^q B^r)^{1/(p+q+r)} \geq B^{(p+2r)/3}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$*, Proc. Amer. Math. Soc., 101 (1987), 85-88.

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