FUZZIFICATIONS OF $a$-$\mathcal{I}$-IDEALS IN IS-ALGEBRAS

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Abstract. The fuzzification of $a$-$\mathcal{I}$-ideals in IS-algebras is considered. Relations between fuzzy $p$-$\mathcal{I}$-ideals and fuzzy $a$-$\mathcal{I}$-ideals are stated. Characterizations of fuzzy $a$-$\mathcal{I}$-ideals are given. Extension property for fuzzy $a$-$\mathcal{I}$-ideal is established.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In 1993, Y. B. Jun et al. [5] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Y. B. Jun et al. [8] renamed the BCI-semigroup (resp. BCI-monoid and BCI-group) as the IS-algebra (resp. IM-algebra and IG-algebra) and studied further properties of these algebras (see [7] and [8]). In [9], E. H. Roh et al. introduced the concept of a $p$-$\mathcal{I}$-ideal in an IS-algebra, and gave necessary and sufficient conditions for an $\mathcal{I}$-ideal to be a $p$-$\mathcal{I}$-ideal, and also stated a characterization of PS-algebras by $p$-$\mathcal{I}$-ideals. Y. B. Jun and E. H. Roh [6] considered the fuzzification of a $p$-$\mathcal{I}$-ideal in an IS-algebra, and investigated some of their properties. E. H. Roh et al. [10] discussed the notion of $a$-$\mathcal{I}$-ideals in IS-algebras. In this paper, we consider the fuzzification of $a$-$\mathcal{I}$-ideals in IS-algebras. We give relations between fuzzy $p$-$\mathcal{I}$-ideals and fuzzy $a$-$\mathcal{I}$-ideals. We state characterizations of fuzzy $a$-$\mathcal{I}$-ideals. We finally establish the extension property for fuzzy $a$-$\mathcal{I}$-ideals.

2. Preliminaries

By a BCI-algebra we mean an algebra $(X, \ast, 0)$ of type $(2,0)$ satisfying the following conditions:

- $(x \ast y) \ast (x \ast z) = 0$,
- $(x \ast (x \ast y)) \ast y = 0$,
- $x \ast x = 0$,
- $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$

for all $x, y, z \in X$. A BCI-algebra $X$ satisfying $0 \leq x$ for all $x \in X$ is called a BCK-algebra. In any BCK/BCI-algebra $X$ one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x \ast y = 0$.

A BCI-algebra $X$ has the following properties:

(P1) $x \ast 0 = x$,  
(P2) $(x \ast y) \ast z = (x \ast z) \ast y$.

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(P3) \((x \ast z) \ast (y \ast z) \leq x \ast y\)
for all \(x, y, z \in X\). A nonempty subset \(I\) of a BCK/BCI-algebra \(X\) is called an ideal of \(X\) if it satisfies
(i) \(0 \in I\),
(ii) \(x \ast y \in I\) and \(y \in I\) imply \(x \in I\) for all \(x, y \in X\).

**Definition 2.1.** (Jun et al. [8]) An IS-algebra is a non-empty set \(X\) with two binary operations “\(*\)” and “\(\cdot\)”, and constant 0 satisfying the axioms

- \(I(X) := (X, *, 0)\) is a BCI-algebra.
- \(S(X) := (X, \cdot)\) is a semigroup.
- The operation “\(\cdot\)” is distributive (on both sides) over the operation “\(*\)”, that is,
  \[x \cdot (y \ast z) = (x \cdot y) \ast (x \cdot z) \quad \text{and} \quad (x \ast y) \cdot z = (x \cdot z) \ast (y \cdot z), \quad \forall x, y, z \in X.\]

**Example 2.2.** Let \(X = \{0, a, b, c\}\) be a set with the following Cayley tables:

<table>
<thead>
<tr>
<th>(\ast)</th>
<th>0</th>
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Then \(X\) is an IS-algebra (see [8]).

**Definition 2.3.** [1, Definition 2.3] A nonempty subset \(A\) of an IS-algebra \(X\) is said to be left (resp. right) stable if \(x \cdot a \in A\) (resp. \(a \cdot x \in A\)) whenever \(x \in S(X)\) and \(a \in A\).

In what follows, the terminology “stable” means “left stable”, and let \(X\) denote an IS-algebra unless otherwise specified.

**Definition 2.4.** [8, Definition 3] A nonempty subset \(A\) of \(X\) is called an \(I\)-ideal of \(X\) if it satisfies
(i) \(A\) is a stable subset of \(S(X)\),
(ii) for any \(x, y \in I(X), x \ast y \in A\) and \(y \in A\) imply that \(x \in A\).

**Definition 2.5.** [9, Definition 3.1] A nonempty subset \(A\) of \(X\) is called a \(p \& I\)-ideal of \(X\) if it satisfies
(i) \(A\) is a stable subset of \(S(X)\),
(ii) for any \(x, y, z \in I(X), (x \ast z) \ast (y \ast z) \in A\) and \(y \in A\) imply that \(x \in A\).

We place a bar over a symbol to denote a fuzzy set so \(\bar{p}, \bar{A}, \bar{X}, \cdots\) all represent fuzzy set in a set.

**Definition 2.6.** [3, Definition 4] A fuzzy set \(\bar{A}\) in \(X\) is called a fuzzy \(I\)-ideal of \(X\) if it satisfies
(i) \(\bar{A}\) is a fuzzy ideal of a \(BCI\)-algebra \(X\),
(ii) \(\bar{A}(x) \geq \bar{A}(y) \quad \forall x, y \in X.\)

**Definition 2.7.** [6, Definition 3.2] A fuzzy set \(\bar{A}\) in \(X\) is called a fuzzy \(p \& I\)-ideal of \(X\) if it satisfies
(i) \(\bar{A}\) is a fuzzy stable set in \(S(X)\),
(ii) \(\bar{A}(x) \geq \min\{\bar{A}((x \ast z) \ast (y \ast z)), \bar{A}(y)\} \quad \forall x, y, z \in X.\)
Note that every fuzzy $p$-$\mathcal{I}$-ideal is a fuzzy $\mathcal{I}$-ideal, but the converse is not true (see [6, Theorem 3.4 and Example 3.5]).

3. Fuzzy $a$-$\mathcal{I}$-ideals

**Definition 3.1.** [10, Definition 3.1] A non-empty subset $A$ of an $\textbf{IS}$-algebra $X$ is called an $a$-$\mathcal{I}$-ideal of $X$ if it satisfies

(i) $A$ is a stable subset of $S(X)$,
(ii) for any $x, y, z \in I(X)$, $(x \ast z) \ast (0 \ast y) \in A$ and $z \in A$ imply that $y \ast x \in A$.

**Definition 3.2.** A fuzzy set $\tilde{A}$ in $X$ is called a fuzzy $a$-$\mathcal{I}$-ideal of $X$ if it satisfies

(i) $\tilde{A}$ is a fuzzy stable set in $S(X)$,
(ii) $\tilde{A}(y \ast x) \geq \min \left\{ \tilde{A}((x \ast z) \ast (0 \ast y)), \tilde{A}(z) \right\}$ $\forall x, y, z \in I(X)$.

**Example 3.3.** Let $X = \{0, a, b, c\}$ be a set with the following Cayley tables:

<table>
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Then $X$ is an $\textbf{IS}$-algebra (see [8]). We can easily check that a fuzzy set $\tilde{A}$ in $X$ given by $\tilde{A}(0) = \tilde{A}(a) = 0.6$ and $\tilde{A}(b) = \tilde{A}(c) = 0.2$ is a fuzzy $a$-$\mathcal{I}$-ideal of $X$.

**Proposition 3.4.** If $\tilde{A}$ is a fuzzy $a$-$\mathcal{I}$-ideal of $X$, then $\tilde{A}(0 \ast x) \geq \tilde{A}(x) \geq \tilde{A}(0 \ast (0 \ast x))$ for all $x \in X$.

**Proof.** Since $\tilde{A}$ is fuzzy stable, it follows that $\tilde{A}(0) = \tilde{A}(0x) \geq \tilde{A}(x)$ for all $x \in X$. Taking $z = y = 0$ in Definition 3.2(ii), we have

$$\tilde{A}(0 \ast x) \geq \min \left\{ \tilde{A}((x \ast z) \ast (0 \ast 0)), \tilde{A}(0) \right\} \tilde{A}(x)$$

for all $x \in X$. Now putting $y = x$ and $x = z = 0$ in Definition 3.2(ii), we have

$$\tilde{A}(x) \geq \min \left\{ \tilde{A}((0 \ast 0) \ast (0 \ast x)), \tilde{A}(0) \right\} \tilde{A}(0 \ast (0 \ast x))$$

for all $x \in X$. \hfill \Box

**Theorem 3.5.** Every fuzzy $a$-$\mathcal{I}$-ideal is a fuzzy $\mathcal{I}$-ideal.

**Proof.** Let $\tilde{A}$ be a fuzzy $a$-$\mathcal{I}$-ideal of $X$. Taking $y = 0$ in Definition 3.2(ii) and using Proposition 3.4, we get

$$\tilde{A}(x) \geq \tilde{A}(0 \ast (0 \ast x)) \geq \tilde{A}(0 \ast x) \geq \min \left\{ \tilde{A}((x \ast z) \ast (0 \ast 0)), \tilde{A}(z) \right\} = \min \left\{ \tilde{A}(x \ast z), \tilde{A}(z) \right\}.$$ 

for all $x, z \in X$. Hence $\tilde{A}$ is a fuzzy $\mathcal{I}$-ideal of $X$. \hfill \Box

The following example shows that the converse of Theorem 3.5 may not be true.
Example 3.6. Let $X$ be an IS-algebra in Example 3.3 and let $	ilde{B}$ be a fuzzy set in $X$ defined by $\tilde{B}(0) = \tilde{B}(b) = 0.8$ and $\tilde{B}(a) = \tilde{B}(c) = 0.5$. Then $\tilde{B}$ is a fuzzy $\mathcal{I}$-ideal of $X$, but it is not a fuzzy $a\&\mathcal{I}$-ideal of $X$ since
\[
\tilde{B}(a * b) = 0.5 < 0.8 = \min \left\{ \tilde{B} \left( (b * b) * (0 * a) \right), \tilde{B}(b) \right\}.
\]

We provide conditions for a fuzzy $\mathcal{I}$-ideal to be a fuzzy $a\&\mathcal{I}$-ideal.

**Theorem 3.7.** Let $\tilde{A}$ be a fuzzy $\mathcal{I}$-ideal of $X$. Then the following are equivalent.

(i) $\tilde{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of $X$.

(ii) $\tilde{A} \left( y * (x * z) \right) \geq \tilde{A} \left( (x * z) * (0 * y) \right)$, $\forall x, y, z \in I(X)$,

(iii) $\tilde{A} \left( y * x \right) \geq \tilde{A} \left( x * (0 * y) \right)$, $\forall x, y \in I(X)$.

**Proof.** Assume that $\tilde{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of $X$. For every $x, y, z \in I(X)$, we have
\[
\tilde{A} \left( y * (x * z) \right) \geq \min \left\{ \tilde{A} \left( ((x * z) * (x * z)) * (0 * y) \right), \tilde{A} \left( (x * z) * (0 * y) \right) \right\}
= \min \left\{ \tilde{A} \left( ((x * z) * (0 * y)) * (x * z) \right), \tilde{A} \left( (x * z) * (0 * y) \right) \right\}
= \min \left\{ \tilde{A}(0), \tilde{A} \left( (x * z) * (0 * y) \right) \right\}
= \tilde{A} \left( (x * z) * (0 * y) \right).
\]

(iii) is by taking $z = 0$ in (ii) and using (P1). Suppose that (iii) holds. Note that
\[
(x * (0 * y)) * (x * z) \leq x * (x * z) \leq z
\]
for all $x, y, z \in I(X)$. Since $\tilde{A}$ is order reversing, it follows that
\[
\tilde{A} \left( x * (0 * y) \right) * (x * z) \geq (x * z) * (0 * y)
\]
Hence
\[
\tilde{A} \left( y * x \right) \geq \tilde{A} \left( x * (0 * y) \right)
\geq \min \left\{ \tilde{A} \left( x * (0 * y) \right) * (x * z), \tilde{A} \left( (x * z) * (0 * y) \right) \right\}
= \min \left\{ \tilde{A} \left( x * (0 * y) \right), \tilde{A} \left( (x * z) * (0 * y) \right) \right\},
\]
and so $\tilde{A}$ is a fuzzy $a\&\mathcal{I}$-ideal of $X$. \qed

**Lemma 3.8.** [6, Theorem 3.9] Let $\tilde{A}$ be a fuzzy $\mathcal{I}$-ideal of $X$. Then $\tilde{A}$ is a fuzzy $p\&\mathcal{I}$-ideal of $X$ if and only if it satisfies
\[
\tilde{A}(x) \geq \tilde{A}(0 * (0 * x)), \forall x \in I(X).
\]

Combining Proposition 3.4 and Lemma 3.8, we have the following theorem.

**Theorem 3.9.** Every fuzzy $a\&\mathcal{I}$-ideal is a fuzzy $p\&\mathcal{I}$-ideal.
The converse of Theorem 3.9 is false, as is shown in the following example.

**Example 3.10.** Let $X$ be an IS-algebra in Example 2.2 and let $\tilde{A}$ be a fuzzy set in $X$ defined by $\tilde{A}(0) = \tilde{A}(a) = 0.7$ and $\tilde{A}(b) = \tilde{A}(c) = 0.5$. Then $\tilde{A}$ is a fuzzy $p$-ideal of $X$, but it is not a fuzzy $a$-ideal of $X$ because

$$\tilde{A}(b * c) = 0.5 < 0.7 = \min \left\{ \tilde{A}(c * a * (0 * b)), \tilde{A}(a) \right\}.$$ 

**Theorem 3.11.** In an associative IS-algebra $X$, that is, the identity $(x * y) * z = x * (y * z)$ holds in $X$, every fuzzy $I$-ideal is a fuzzy $a$-ideal.

**Proof.** Let $\tilde{A}$ be a fuzzy $I$-ideal of $X$. Note that

$$(y * x) * (x * (0 * y)) = (y * x) * (x * 0) * y = (y * x) * (x * y)$$

$$= (y * (x * y)) * x = (y * x) * y * x$$

$$= (y * y) * x = 0 * x = 0$$

for all $x, y \in X$. Hence

$$\tilde{A}(y * x) \geq \min \left\{ \tilde{A}(y * (x * (0 * y))), \tilde{A}(x * (0 * y)) \right\}$$

$$= \min \left\{ \tilde{A}(0), \tilde{A}(x * (0 * y)) \right\}$$

$$= \tilde{A}(x * (0 * y)).$$

It follows from Theorem 3.7 that $\tilde{A}$ is a fuzzy $a$-ideal of $X$. 

**Theorem 3.12.** (Extension property for fuzzy $a$-ideals) Let $\tilde{A}$ and $\tilde{B}$ be fuzzy $I$-ideal of $X$ such that $\tilde{A}(0) = \tilde{B}(0)$ and $\tilde{A} \subseteq \tilde{B}$, that is, $\tilde{A}(x) \leq \tilde{B}(x)$ for all $x \in X$. If $\tilde{A}$ is a fuzzy $a$-ideal of $X$, then so is $\tilde{B}$.

**Proof.** Let $x, y \in X$. Using Theorem 3.7(ii) and (P2), we have

$$\tilde{B}(y * (x * (0 * y))) \geq \tilde{A}(y * (x * (0 * y)))$$

$$\geq \tilde{A}(x * (0 * y)) * (0 * y)$$

$$= \tilde{A}(x * (0 * y)) * (x * (0 * y))$$

$$= \tilde{A}(0) = \tilde{B}(0).$$

Note that

$$\left( (y * x) * (x * (0 * y)) \right) * (y * (x * (0 * y)))$$

$$\geq (y * (x * (0 * y))) * (x * (0 * y))$$

$$\geq x * (x * (0 * y)) * (0 * y)$$

$$= (x * (0 * y)) * (x * (0 * y)) = 0.$$
and so $\overline{B}\left((y \ast x) \ast (x \ast (0 \ast y))\right) \ast (y \ast (x \ast (0 \ast y))) \geq \overline{B}(0)$. It follows that

$$\overline{B}\left((y \ast x) \ast (x \ast (0 \ast y))\right)$$

$$\geq \min\left\{\overline{B}\left((y \ast x) \ast (x \ast (0 \ast y))\right) \ast (y \ast (x \ast (0 \ast y)))\right\}, \overline{B}\left(y \ast (x \ast (0 \ast y))\right)\}$$

$$\geq \min\left\{\overline{B}\left((y \ast x) \ast (x \ast (0 \ast y))\right) \ast (y \ast (x \ast (0 \ast y)))\right\}, \overline{B}(0)\right\}$$

$$= \overline{B}\left((y \ast x) \ast (x \ast (0 \ast y))\right) \ast (y \ast (x \ast (0 \ast y)))$$

$$\geq \overline{B}(0),$$

so that

$$\overline{B}(y \ast x) \geq \min\left\{\overline{B}\left((y \ast x) \ast (x \ast (0 \ast y))\right), \overline{B}\left(x \ast (0 \ast y)\right)\right\}$$

$$\geq \min\left\{\overline{B}(0), \overline{B}\left(x \ast (0 \ast y)\right)\right\}$$

$$= \overline{B}\left(x \ast (0 \ast y)\right).$$

Using Theorem 3.7, we know that $\overline{B}$ is a fuzzy $\alpha$-$\mathcal{I}$-ideal of $X$.

**References**


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