A GEOMETRIC MEAN IN THE FURUTA INEQUALITY

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ABSTRACT. Uchiyama discussed the Furuta inequality from the viewpoint of the Jensen inequality. Recently Furuta and Kamei improved it as follows: Suppose that \( A, B, C > 0 \) and \( r, s \geq 0 \). If \( A^t \ll B^s \nabla_{\mu} C^t \) for all \( t \geq 0 \), then

\[
 f(t) = A^{-t} \left( \frac{1}{t+1} \right) (B^s \nabla_{\mu} C^t)
\]

is an increasing function of \( t \geq s \). On the other hand, if \( A^t \ll B^s \nabla_{\mu} C^t \) for all \( t \geq 0 \), then

\[
 h(t) = A^{-t} \left( \frac{1}{t+1} \right) (B^s \nabla_{\mu} C^t)
\]

is a decreasing function of \( t \geq s \).

In this note, we pay our attention to the assumptions in above and point out that the operator function \( F(s) = ((1 + \mu) A^s + \mu B^s)^\frac{1}{s} \) (\( s \in \mathbb{R} \)) for given \( A, B > 0 \) and \( \mu \in [0, 1] \) is monotone increasing under the chaotic order \( X \gg Y \) defined by \( \log X \geq \log Y \) and consequently \( \lim_{h \to 0} F(h) = e^{(1+\mu) \log B + \mu \log C} \). This means that we can see another geometric mean \( B \Diamond_{\mu} C = e^{(1-\mu) \log B + \mu \log C} \) in the Furuta inequality. Moreover we consider Uchiyama’s result in a general setting.

1. Introduction

First of all, we cite the Löwner-Heinz inequality (LH) which is one of the most fundamental operator inequalities: If \( A \) and \( B \) are positive operators acting on a Hilbert space \( H \) and satisfy \( A \geq B \), then \( A^p \geq B^p \) for all \( p \in [0, 1] \). In 1987, Furuta [8] established the following historical extension of (LH), see [13], [9], [2] and [15].

The Furuta inequality

If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

(i) \( (B^{\frac{r}{p}} A^{p} B^{\frac{r}{p}})^{\frac{1}{s}} \geq (B^{\frac{r}{p}} B^{p} B^{\frac{r}{p}})^{\frac{1}{s}} \)

and

(ii) \( (A^{\frac{r}{q}} A^{p} A^{\frac{r}{q}})^{\frac{1}{s}} \geq (A^{\frac{r}{q}} B^{p} A^{\frac{r}{q}})^{\frac{1}{s}} \)

hold for \( p \geq 0 \) and \( q \geq 1 \) with \((1+r)q \geq p + r\).

Motivated by Ando’s inequality [1], we introduced the chaotic order among positive invertible operators [7]: For \( A, B > 0 \), we denote by \( A \gg B \) if \( \log A \geq \log B \). Finally we

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obtained the following chaotic version (FC) of the Furuta inequality, [3] and [10], see also [4], [5] and [6]:

For $A, B > 0$, $A \cong B$, i.e., $\log A \geq \log B$, if and only if

$$
(\hat{A} \hat{B} \hat{A}^p \hat{B}^q)^{\frac{1}{p+q}} \geq (A \hat{B}^q) \hat{A}^p \hat{B}^q
$$

for all $p, q \geq 0$. This is expressed in terms of the monotonicity of an operator function.

**Theorem A.** For $A, B > 0$, $A \cong B$ if and only if for each $s \geq 0$, $G(t, r) = A^{-r} \hat{A} B^r$ is an increasing function of both $t \geq s$ and $r \geq 0$, where $\hat{A}$ is the $\alpha$-geometric mean.

Recently, Uchiyama [16] gave a new viewpoint to the Furuta inequality. He explained that it is from the Jensen inequality for operator concave functions.

**Theorem B.** If $A \leq B \mu C$ for $A, B, C > 0$, then

$$
B^t \nabla \mu C^t \leq A^{-r} \hat{A} \nabla \mu C^t \tag{B^t \nabla \mu C^t}
$$

for $r \geq 0$ and $t \geq 0$, where $\mu$ and $\nabla \mu$ are $\mu$-harmonic and arithmetic means respectively.

Afterwards, we were given an opportunity to see a paper [11] by Furuta and Kamei, in which Theorem B is improved from the viewpoint of Theorem B.

**Theorem C.** Suppose that $A, B, C > 0$ and $r, s \geq 0$. If $A^t \leq B^t \nabla \mu C^t$ for all $t \geq 0$, then

$$
f(t) = A^{-r} \hat{A} \nabla \mu C^t \tag{f(t)}
$$

is an increasing function of $t \geq s$. On the other hand, if $A^t \leq B^t \nabla \mu C^t$ for all $t \geq 0$, then

$$
h(t) = A^{-r} \hat{A} \nabla \mu C^t \tag{h(t)}
$$

is a decreasing function of $t \geq s$.

In this note, we pay our attention to the assumptions of Theorems B and C. Namely we discuss the monotonicity of the operator function

$$
F(s) = ((1 - \mu)A^s + \mu B^s)^\frac{1}{s} \tag{F(s)}
$$

for given $A, B > 0$ and $\mu \in [0, 1]$. It is not monotone increasing under the usual operator order, but we can prove that it is monotone increasing under the chaotic order and moreover $\lim_{s \to 0} F(\epsilon) = e^{(1 - \mu) \log A + \mu \log B}$. We call it the chaotically $\mu$-geometric mean $A \hat{\mu} B$ of $A$ and $B$. So we can reformulate Theorem C and generalize Theorem B. This means that we find, in the Furuta inequality, another geometric mean different from the geometric mean $\hat{A}$ in the sense of Kubo-Ando. Of course, they coincide if $A$ and $B$ commute.

2. THE CHAOTICALLY GEOMETRIC MEAN

In this section, we discuss the monotonicity of the operator function $F(s)$. First of all, we do it under the usual operator order.

**Lemma 1.** Let $B, C > 0$ and $\mu \in [0, 1]$ be given. Then the operator function $F(s) = ((1 - \mu)B^s + \mu C^s)^\frac{1}{s}$ is monotone increasing on $[1, \infty)$, i.e., $F(s) \leq F(t)$ if $1 \leq s \leq t$. In addition, $F(s) \leq F(t)$ if $1 \leq t \leq 2s$, and $F(s)$ is not monotone increasing on $(0, 1]$ in general.
Proof. The first assertion follows from the operator concavity of the function $x^r$ ($r \in [0, 1]$): If $1 \leq s \leq t$, then
\[(1 - \mu)B^r + \mu C^r \geq (1 - \mu)B^s + \mu C^s\]
and so $F(t) \geq F(s)$ by (LH). On the other hand, the second one follows from the operator convexity of $x^r$ for $1 \leq r \leq 2$: If $1 \leq t \leq 2s$, then
\[(1 - \mu)B^r + \mu C^r \leq (1 - \mu)B^t + \mu C^t\]
and so $F(s) \leq F(t)$ by (LH).

Finally we give a simple counterexample to the third one as follows:

\[B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.
\]

Then
\[F(1) = \frac{1}{2}(B + C) = \begin{pmatrix} 14 & 14 \\ 14 & 20 \end{pmatrix}\]

and
\[F\left(\frac{1}{3}\right) = \left[\frac{1}{2}(B^\frac{1}{3} + C^\frac{1}{3})\right]^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix},\]

so that
\[F(1) - F\left(\frac{1}{3}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} \geq 0.\]

Next we discuss it under the chaotic order.

Lemma 2. The operator function $F(s)$ is monotone increasing under the chaotic order, i.e., $F(s) \ll F(t)$ if $s < t$. In particular,
\[s \lim_{h \to 0} F(h) = e^{(1 - \mu) \log B + \mu \log C}.
\]

Proof. It suffices to show that for $s < t$ with $s, t \neq 0$
\[\frac{1}{s} \log((1 - \mu)B^s + \mu C^s) \leq \frac{1}{t} \log((1 - \mu)B^t + \mu C^t).
\]

To prove this, the operator concavity of $x^r$ for $r \in [0, 1]$ is available. We first assume $0 < s < t$. Then
\[\log((1 - \mu)B^r + \mu C^r)^\frac{1}{r} \geq \log((1 - \mu)B^s + \mu C^s),\]
and so $\log F(t) \geq \log F(s)$. Next, if $s < t < 0$, then $\frac{1}{r} \in (0, 1)$ and hence
\[\log((1 - \mu)B^s + \mu C^s)^\frac{1}{r} \geq \log((1 - \mu)B^t + \mu C^t).
\]
Noting $t < 0$, we have $\log F(s) \leq \log F(t)$. 
Now we prove the second assertion. By the operator concavity of $\log x$ and the Krein inequality $x - 1 \geq \log x$, it implies that for any $t > 0$
\[
(1 - \mu) \log B + \mu \log C \\
= \frac{1}{t}((1 - \mu) \log B' + \mu \log C') \\
\leq \frac{1}{t} \log((1 - \mu) B' + \mu C') \\
\leq \frac{1}{t}((1 - \mu) B' + \mu C' - 1) \\
= (1 - \mu) \frac{B' - 1}{t} + \mu \frac{C' - 1}{t} \\
\to (1 - \mu) \log B + \mu \log C \quad (t \to +0).
\]
Therefore it follows that
\[
\lim_{t \to +0} \log((1 - \mu) B' + \mu C')^\tau = (1 - \mu) \log B + \mu \log C,
\]
so that
\[
\lim_{t \to +0} ((1 - \mu) B' + \mu C')^\tau = e^{(1 - \mu) \log B + \mu \log C}.
\]
On the other hand, it follows from the identity obtained above that for $s > 0$
\[
F_{B,C}(-s) = F_{B^{-1},C^{-1}}(s)^{-1} \\
\to e^{(1 - \mu) \log B^{-1} + \mu \log C^{-1}} \\
= e^{(1 - \mu) \log B + \mu \log C}.
\]
Hence we have the second assertion, which says that $s\lim_{h \to 0} F(h)$ can be regarded as $F(0)$. Therefore, if $s < 0 < t$, then
\[
F(s) \ll F(0) \ll F(t).
\]
Consequently we have the monotonicity of $F(s)$.

For the sake of convenience, we define another geometric mean:

**Definition 3.** For $B, C > 0$ and $\mu \in [0,1]$, $B \diamond_{\mu} C = e^{(1 - \mu) \log B + \mu \log C}$ is said to be the chaotically $\mu$-geometric mean of $B$ and $C$.

**Theorem 4.** For $B, C > 0$ and $\mu \in [0,1]$, both $(B' \nabla_{\mu} C')^\tau$ and $(B' \nabla_{\mu} C')^\tau$ converge to the chaotically $\mu$-geometric mean $B \diamond_{\mu} C$ as $t \searrow 0$. Consequently
\[
\lim_{t \searrow 0} (B' \nabla_{\mu} C')^\tau = B \diamond_{\mu} C.
\]

**Proof.** The first assertion follows from Lemma 2. To prove the second one, it suffices to show that $\log(B' \nabla_{\mu} C')^\tau$ converges to $(1 - \mu) \log B + \mu \log C$. By the well-known arithmetic-geometric mean inequality, we have
\[
B' \nabla_{\mu} C' \leq B' \nabla_{\mu} C' \leq B' \nabla_{\mu} C',
\]
so that
\[
\log(B' \nabla_{\mu} C') \leq \log(B' \nabla_{\mu} C') \leq \log(B' \nabla_{\mu} C').
\]
By multiplying $\frac{1}{t}$ on each term, it follows from Lemma 2 that the middle term $\frac{1}{t} \log(B' \nabla_{\mu} C')$ must converge to $(1 - \mu) \log B + \mu \log C$. 
Remark. The second assertion of Theorem 4 appeared in [12, Lemma 3.3].

3. UCHIYAMA’S GENERALIZATION ON THE FURUTA INEQUALITY

As stated in Theorem B, Uchiyama gave an interesting viewpoint to the Furuta inequality. Recently it is considered under the chaotic order by Furuta-Kamei, which we cite as Theorem C. We now reformulate it by using the chaotically $\mu$-geometric mean.

Theorem 5. For $A, B, C > 0$ and $\mu \in [0,1]$, the following statements are mutually equivalent:

1. $A \preceq B \triangleleft_{\mu} C$.
2. $B^t \triangledown_{\mu} C^t \preceq A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$ for $r \geq 0$ and $t \geq s \geq 0$.
3. For each $r, s \geq 0$, $f(t) = A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$ is an increasing function of $t \geq s$.

Proof. First of all, we note that (1) is equivalent to the condition $A^t \preceq B^{-1} \triangledown_{\mu} C^t$ for all $t \geq 0$ by Lemma 2 and Theorem 4. That is, (1) implies (3) has been proved in Theorem C. If (3) holds, then (2) is obtained by putting $t = s$. Finally, if (2) holds for $s = 0$, then for each $t > 0$, $1 \leq A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$ for all $r \geq 0$. It is equivalent to (1) by (FC) stated in §1.

The following theorem is a complement to Theorem 5, which is corresponding to the second assertion of Theorem C.

Theorem 6. For $A, B, C > 0$ and $\mu \in [0,1]$, the following statements are mutually equivalent:

1. $A \succ B \triangleleft_{\mu} C$.
2. $B^t \triangledown_{\mu} C^t \preceq A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$ for $r \geq 0$ and $t \geq s \geq 0$.
3. For each $r, s \geq 0$, $h(t) = A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$ is a decreasing function of $t \geq s$.

Proof. Clearly (1) is equivalent to the condition $A^{-1} \preceq B^{-1} \triangledown_{\mu} C^{-1}$. So it follows from Theorem 5 that (1) means $f_{A^{-1}, B^{-1}, C^{-1}}(t)$ is monotone increasing. Moreover, since $h_{A, B, C}(t)^{-1} = f_{A^{-1}, B^{-1}, C^{-1}}(t)$, (1) holds if and only if $h(t)$ is monotone decreasing, i.e., (3) holds. The proof of the others is similar to that of Theorem 5.

We note that Theorems 4 - 6 require an improvement of Theorem B. As a matter of fact, we can reply as follows:

Theorem 7. Suppose that $A, B, C > 0$ satisfy $A \preceq (B^t \triangledown_{\mu} C^t)^{1/t_0}$ for some $t_0 > 0$. If $t_0 \geq 0$, then

$$B^t \triangledown_{\mu} C^t \preceq A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$$

for all $r \geq 0$ and $t \geq s \geq 0$ with $t \geq t_0$. On the other hand, if $t_0 < 0$, then

$$(B^t \triangledown_{\mu} C^t)^{\frac{1}{t_0}} \preceq A^{-r} \triangleleft_{\frac{1}{1+r}} (B^t \triangledown_{\mu} C^t)$$

for all $r \geq 0$ and $-t_0 \geq t \geq s \geq 0$.

Proof. We need the following fact [14, Theorem 2 (3)] obtained by (FC): If $A \preceq B$, then $B^t \preceq A^{-r} \triangleleft_{\frac{1}{1+r}} B^t$ for all $r \geq 0$ and $t \geq s \geq 0$. We first suppose that $A \preceq F(t)$ for some $t_0 > 0$. Since $A \preceq F(t)$ for $t \geq t_0$ by Lemma 2, we have

$$F(t)^{\frac{1}{t_0}} \leq A^{-r} \triangleleft_{\frac{1}{1+r}} F(t)^{\frac{1}{t_0}}$$
for all $r \geq 0$ and $t \geq s \geq 0$ with $t \geq t_0$. On the other hand, since $F(t)^t = (B^t \nabla_\mu C^t)^t \geq B^t \nabla_\mu C^t$ for $t \geq s \geq 0$, it follows that

$$B^t \nabla_\mu C^t \leq F(t)^t \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1}, \quad F(t)^t = A^{-T}\frac{\nabla_\mu C^t}{t + 1} B^t \nabla_\mu C^t.$$

Next we suppose that $A \ll F(t_0)$ for some $t_0 < 0$. Since

$$A \ll F(t_0) \ll F(-t) = (B^t \nabla_\mu C^t)^t$$

for $-t_0 \geq t \geq s \geq 0$, we have the desired inequality

$$(B^t \nabla_\mu C^t)^t \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1} (B^t \nabla_\mu C^t)$$

by applying the inequality cited in above again.

For the sake of convenience, we cite a mean theoretic proof of the inequality [14, Theorem 2 (3)] used above: For this, (FC) under $A \ll B$ is expressed as $1 \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1} B^t = B^t \frac{\nabla_\mu C^t}{t + 1} A^{-T}$ for $t, r \geq 0$. Thus, if $A \ll B$ and $r \geq 0$, then for $t \geq s \geq 0$,

$$A^{-T}\frac{\nabla_\mu C^t}{t + 1} B^t = B^t \frac{\nabla_\mu C^t}{t + 1} A^{-T} = B^t \frac{\nabla_\mu C^t}{t + 1} (B^t \frac{\nabla_\mu C^t}{t + 1} A^{-T}) \geq B^t \frac{\nabla_\mu C^t}{t + 1} 1 = 1 \frac{\nabla_\mu C^t}{t + 1} B^t = B^t.$$

We now remark that Theorem 7 can be rephrased as a similar form to Theorem C.

**Corollary 8.** Suppose that $A, B, C > 0, \mu \in [0,1]$ and $t_0 > 0$. Then the following statements are mutually equivalent:

1. $A \ll (B^t \nabla_\mu C^t)^{1/t_0}$.
2. $B^t \nabla_\mu C^t \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1} (B^t \nabla_\mu C^t)$ for all $r \geq 0$ and $t \geq s \geq 0$ with $t \geq t_0$.
3. For each $r, s \geq 0$, $f(t) = A^{-T}\frac{\nabla_\mu C^t}{t + 1} (B^t \nabla_\mu C^t)$ is an increasing function of $t$, precisely, $f(t) \geq f(t_1)$ for $t \geq t_1 \geq s$ with $t \geq t_0$.

**Proof.** (1) $\rightarrow$ (3): It is similar to that of Theorem C. Since $A \ll F(t_0) \ll F(t)$ for $t \geq t_0$ by Lemma 2, Theorem A implies that

$$A^{-T}\frac{\nabla_\mu C^t}{t + 1} F(t)^t \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1} F(t)^t = f(t)$$

for $t \geq t_1 \geq s \geq 0$. Moreover, since the operator convexity of $x^\alpha$ ($\alpha \in [0,1]$) ensures that

$$F(t)^t = (B^t \nabla_\mu C^t)^{1/t} \geq B^t \nabla_\mu C^t = F(t_1)^t,$$

we have

$$f(t_1) = A^{-T}\frac{\nabla_\mu C^t}{t + 1} F(t_1)^t \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1} F(t)^t \leq f(t).$$

(3) $\rightarrow$ (2): If we take $t_1 = s$ in (3), then $f(s) \leq f(t)$ for $t \geq t_0$. Since $f(s) = B^s \nabla_\mu C^s$, we have (2). (2) $\rightarrow$ (1): We take $s = 0$ and $t = t_0$ in (2).

**Corollary 9.** Suppose that $A, B, C > 0, \mu \in [0,1]$ and $t_0 < 0$. Then the following statements are mutually equivalent:

1. $A \ll (B^t \nabla_\mu C^t)^{1/t_0}$.
2. $(B^t \nabla_\mu C^t)^t \leq A^{-T}\frac{\nabla_\mu C^t}{t + 1} (B^t \nabla_\mu C^t)$ for all $r \geq 0$ and $-t_0 \geq t \geq s \geq 0$.
3. For each $t \in [s, -t_0]$ and $s \geq 0$, $k(r) = A^{-T}\frac{\nabla_\mu C^t}{t + 1} (B^t \nabla_\mu C^t)$ is an increasing function of $r \geq 0$.

**Proof.** (1) $\rightarrow$ (3): Lemma 2 implies that $A \ll F(-t)$ for $t \leq -t_0$. Since $F(-t)^t = B^t \nabla_\mu C^t$, it follows from Theorem A that $k(r)$ is an increasing function of $r \geq 0$. Moreover (3) implies that $k(0) \leq k(r)$ for $r \geq 0$, that is, (2) holds, and (2) $\rightarrow$ (1) follows from putting $s = 0$ in (2).
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