A SUBSET OF THE CLASS S

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Abstract. We investigate some properties of the class of univalent functions $f(z) = z + a_2 z^2 + a_3 z^3$, analytic in the unit disc and satisfying $\left| \frac{z}{f(z)} \right|^n \leq a \leq 2$.

Introduction

Let $A$ denote the class of functions analytic in $U = \{ z : |z| < 1 \}$ and have the Taylor series
\begin{equation}
 f(z) = z + a_2 z^2 + a_3 z^3 + \ldots
\end{equation}

and let $S$ denote the well-known subclass of $A$ consisting of univalent functions. A function $f(z) \in S$ is said to be starlike in $U$ if and only if it satisfies
\begin{equation}
 \Re \frac{zf'(z)}{f(z)} > 0, \quad z \in U.
\end{equation}

In [1] the following theorem was established.

Theorem 1. Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let
\begin{equation}
 \left| \left( \frac{z}{f(z)} \right)^n \right| < 1, \quad z \in U.
\end{equation}

Then, $f \in S$.

For $0 < \alpha \leq 2$, let $S(\alpha)$ denote the class of functions $f(z) \in A$ that satisfy
\begin{equation}
 \left| \left( \frac{z}{f(z)} \right)^n \right| \leq \alpha, \quad z \in U, \quad f(z) \neq 0, \quad 0 < |z| < 1.
\end{equation}

In [3] Theorem 1 was extended to the class $S(\alpha)$ and some results for the class $S(\alpha)$ were obtained. Here we prove the following theorem.

Theorem 2. Let $f(z) \in S(\alpha)$ such that $f''(0) = 0$, then
\begin{itemize}
  \item[(i)] $\Re \frac{f(z)}{z} \geq \frac{2}{2+\alpha}, \quad z \in U$,
  \item[(ii)] $f$ is starlike in $|z| \leq \frac{\sqrt{\alpha}}{\sqrt{\alpha}} \quad (\sqrt{\alpha} < \alpha < 2)$. In particular, if $0 < \alpha \leq \sqrt{2}$, then $f(z)$ is starshaped in $U$.
  \item[(iii)] $\Re f'(z) > 0$ in $|z| \leq \frac{1}{\sqrt{\alpha}}$.
\end{itemize}

Items (i), (ii) and (iii) are improvements of results in [3] and in fact (i) and (iii) are sharp as shown by the function
\begin{equation}
 f(z) = \frac{z}{1 - \frac{\alpha}{2} z^2}.
\end{equation}
In (ii) the starlikeness of \( f(z) \) for \( 0 < \alpha \leq \sqrt{2} \) is in keeping with a result in [2] for a subclass of \( S(\alpha) \).

We need the notion of subordination. Let \( f(z) \) and \( g(z) \) be analytic functions in \( U \) with \( f(0) = g(0) \). Then \( f(z) \) is said to be subordinate to \( g(z) \), written \( f(z) \prec g(z) \) if there exists a function \( \omega(z) \) analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \), \( z \in U \) such that \( f(z) = g(\omega(z)) \), \( z \in U \).

**Proof of the Theorem**

Let

\[
p(z) = \frac{z}{f(z)} - \frac{z}{f(z)} \left( \frac{z}{f(z)} \right)' = z^2 \frac{f''(z)}{f^2(z)}
\]

In view of the expansion (1) it is easily checked that

\[
\left( \frac{1}{z} - \frac{1}{f(z)} \right)' \Bigg|_{z=0} = a_2
\]

and

\[
z \left( \frac{z^2 f'(z)}{f^2(z)} \right)' = 2z^2(a_3 - a_2^2) + \ldots
\]

From (6) we obtain

\[
p'(z) = -z \left( \frac{z}{f(z)} \right)'' = \left( \frac{z^2 f'(z)}{f^2(z)} \right)'.
\]

Thus

\[
\left| \left( \frac{z}{f(z)} \right)'' \right| \leq \alpha \iff z^2 \left( \frac{z}{f(z)} \right)'' \prec \alpha z
\]

and equivalently

\[
z \left( \frac{z^2 f'(z)}{f^2(z)} \right)' \prec \alpha z.
\]

On account of (8), (11) can be written in the form

\[
z \left( \frac{z^2 f'(z)}{f^2(z)} \right)' = \alpha \omega(z), \quad \omega(0) = \omega'(0) = 0, \quad |\omega(z)| \leq |z|^2.
\]

Therefore

\[
z \left( \frac{z^2 f'(z)}{f^2(z)} \right)' = 1 + \alpha \int_0^1 \frac{\omega(tz)}{t} dt.
\]

Further, because of the identity \( z^2 \left( \frac{1}{z} - \frac{1}{f(z)} \right)' = \frac{z^2 f'(z)}{f^2(z)} - 1 \), and using (7) with the hypothesis \( a_2 = 0 \), we obtain in view of (13)

\[
z \left( \frac{z}{f(z)} \right) = 1 - \alpha \int_0^1 \frac{\omega(tz)(1-t)}{t^2} dt.
\]

Indeed, if \( \omega(z) = \sum_{n=2}^\infty b_n z^n \) then by (13), \( z^2 \left( \frac{1}{z} - \frac{1}{f(z)} \right)' = \alpha \sum_{n=2}^\infty \frac{b_n z^{n-1}}{n} \), and on dividing by \( z^2 \) and integrating we obtain, because \( a_2 = 0 \),

\[
\frac{1}{z} - \frac{1}{f(z)} = \alpha \sum_{n=2}^\infty \frac{b_n z^{n-1}}{n(n-1)} = \frac{\alpha}{z} \int_0^1 \frac{\omega(tz)}{t^2} (1-t) dt
\]

which is (14).
As \(|\varphi(z)| \leq |z|^2\), (14) gives
\[(15) \quad \left| \frac{z}{f(z)} - 1 \right| \leq \frac{\alpha}{2} |z|^2.\]
Since \(0 < \alpha \leq 2\) this yields
\[(16) \quad \text{Re} \frac{z}{f(z)} \geq 1 - \frac{\alpha}{2}.\]
which is sharp in view of (5). Further, (15) is equivalent to
\[\left| \frac{f(z)}{z} - \frac{1}{1 - \frac{\alpha}{2} |z|^2} \right| \leq \frac{\alpha |z|^2}{2 - \frac{\alpha}{2} |z|^2}.\]
This yields
\[(17) \quad \text{Re} \frac{f(z)}{z} \geq \frac{1}{1 + \frac{\alpha}{2} |z|^2} \geq \frac{1}{1 + \frac{\alpha}{2}}\]
which establishes (i).

From (13) we obtain
\[z \frac{f'(z)}{f(z)} = \frac{f(z)}{z} (1 + \alpha \omega_1(z)), \quad \omega_1(z) = \int_0^1 \frac{\omega_1(t) z}{t} dt\]
which leads to
\[\left| \arg \left( \frac{z \frac{f'(z)}{f(z)}}{f(z)} \right) \right| = \left| \arg \frac{f(z)}{z} + \arg (1 + \alpha \omega_1(z)) \right| \leq \left| \arg \frac{f(z)}{z} \right| + \left| \arg (1 + \alpha \omega_1(z)) \right| \leq 2 \sin^{-1} \left( \frac{\alpha |z|^2}{2} \right)\]
because of (15) and the fact that \(|\omega_1(z)| \leq \frac{1 |z|^2}{2}\).
As \(\text{Re} \frac{z \frac{f'(z)}{f(z)}}{f(z)} > 0 \iff \left| \arg \left( \frac{z \frac{f'(z)}{f(z)}}{f(z)} \right) \right| \leq \frac{\pi}{2}\), we obtain from (18)
\[\text{Re} \frac{f(z)}{z} > 0 \iff |z|^2 \leq \frac{\sqrt{\alpha}}{\alpha}.\]
This establishes (ii).

Once again, from (13) we get
\[f'(z) = \left( \frac{f(z)}{z} \right)^2 (1 + \alpha \omega_1(z))\]
and a similar argument gives
\[|\arg f'(z)| \leq 3 \sin^{-1} \left( \frac{\alpha |z|^2}{2} \right)\]
which gives (iii). This completes the proof of the theorem.

It may be mentioned that in [2] the class of functions \(f(z)\) satisfying \(|z^2 \frac{f'(z)}{f(z)} - 1| \leq \mu\)
had been considered and in the present situation \(\mu = \frac{\alpha}{2}\).
References


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