SOME RESULTS ON ALMOST BIIDEALS

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Abstract. In this paper, we give some relations between biideals and almost biideals, and prove some properties on some special semigroups.

Definition 1 A nonempty subset $T$ of a semigroup $S$ is a subsemigroup of $S$ if it is closed under the operation of $S$; i.e. if $a, b \in T$, then $ab \in T$.

Definition 2 A nonempty subset $T$ of a semigroup $S$ is a two-sided ideal (or simply a biideal or an ideal) if $x, y \in S, t \in T$ imply $xt, ty \in T$.

Definition 3 A nonempty subset $T$ of a semigroup $S$ is an almost biideal if for any $s \in S$, there exists $x, y \in T$ such that $xsy \in T$. It is clear that biideal are almost biideal.

Example 1. Let $S$ be a cyclic group generated by $a$ of order 4, $e$ is the identity of $S$, $\{e, a\}$ is a almost biideal of $S$, but $\{e, a^2\}$ is not a subsemigroup of $S$, $\{e, a^2\}$ is a subsemigroup of $S$, but $\{e, a^2\}$ is not a almost biideal of $S$.

Definition 4 An element $a$ of a semigroup $S$ is regular if $a = axa$ for some $x \in S$. A semigroup $S$ is regular if every element of $S$ is regular. A nonempty subset $I$ of a semigroup $S$ is a regular set of $S$ if every element of $I$ is regular.

Definition 5 An element $a$ of a semigroup $S$ is quasi regular if $a^m = a^m x a^m$ for some positive integer $m$ and some $x \in S$. A semigroup $S$ is quasi regular if every element of $S$ is quasi regular. A nonempty subset $I$ of a semigroup $S$ is a quasi regular set of $S$ if every element of $I$ is quasi regular.

Definition 6 An element $A$ of a semigroup $S$ is idempotent if $a^2 = a$. The set of all idempotent elements of semigroup $S$ is denote by $E$.

Theorem 1 Let $S$ be a quasi regular semigroup and $E = \{e\}$, then we have

1) for any $y \in S$, there exists some positive integer $m$ and some $x \in S$ satisfy

\[ xy^m = y^m x = e, y^m e = ey^m = y^m, ye = ey; \]

2) $Se = eS$ is a group.
Proof. By the definition 5, for any \( y \in S \), there exists some positive integer \( m \) and some \( x \in S \) satisfies \( y^m = y^m xy^m \), further we have \( (y^m x)^2 = y^m x (xy^m)^2 = xy^m \). On the other hand, as \( E = \{ e \} \), so we have \( xy^m = y^m x = e \), moreover \( ey^m = y^m e = y^m ; ey = xy^my = \) \( xy^m e = xyy^m y^m x = xy^{2m+1} x, ye = yy^mx = y^mxy = ey^m yx = xyy^m yx = xy^{m+1} x \), therefore \( ey = ye \), and \( So = e S \).

For any \( u \in S e \), there exists \( s \in S \), satisfy \( u = se \). Therefore \( eu = ese = see = se^2 = se = u \), \( we = see = se^2 = se = u \), then \( e \) is the identity of \( S e \). On the other hand, there exists some positive integer \( n \) and some \( t \in S \) satisfies \( ts^n = s^n t = e, s^n e = es^n = s^n, se = es \).

Let \( v = s^{n-1} t e \). Thus \( uv = se s^{n-1} te = es s^{n-1} te = es^n te = ee e = e^2 e = ee = e^2 = e \). Similarly, we have \( vu = e \). So \( S e = e S \) is a group. \( \square \)

**Corollary 1.** \( S \) is a quasi regular semigroup and \( E = \{ e \} \) (Where \( e \) is the identity of \( S \)) if and only if \( S \) is a group.

**Lemma 1.** Let \( B \) be both a almost biideal and a subsemigroup of a semigroup \( S \), then for any \( x, y \in S \), we have \( x By \) is a almost biideal of a semigroup \( S \).

**Proof.** For any \( x, y \in S, s \in S \), we have \( ysx \in S \). Since \( B \) is a almost biideal of a semigroup \( S \), there exists \( u, v \in B \) such that \( uysxv \in B \). On the other hand, \( B \) is a subsemigroup of \( S \), so \( x(uy sx v)y = (x uy) y (s xv) \in x By \). Therefore \( x By \) is a almost biideal of a semigroup \( S \). Thus, the proof is completed. \( \square \)

**Remark 1.** Generally, for a semigroup \( S \), there uncertain exists a almost biideal \( M \) of \( S \) such that for any almost biideal \( B \) of \( S \), have \( x, y \in S \) satisfies \( B = x My \).

**Example 2** Let \( S \) be a cyclic group group generated by \( a \) of order 4, \( e \) is the identity of \( S \), there not exists a almost biideal \( M \) of \( S \) such that for any almost biideal \( B \) of \( S \), have \( x, y \in S \) satisfies \( B = x My \).

**Example 3.** There a almost biideal \( B \) of a quasi regular semigroup \( S \) and \( E = \{ e \} \), but \( B \) is not a biideal of \( S \).

**Proof.** Let \( S \) be a infinitely cyclic group generated by \( a, e \) is the identity of \( S \), it is clear that \( S \) is also a quasi regular semi- group and \( E = \{ e \} \). Take \( B = \{ e, a, a^2, a^3, \cdots \} \), then \( B \) is a almost biideal of \( S \). Because for any \( a \in S \), there exists integer \( m \) such that \( a = a^m \), so there exists positive integer \( n = | m | + 1 \), \( a^n a^n \in B \), therefore \( B \) is a almost biideal of \( S \).

On the other hand, \( a^n = e a^{-1} e \not\in B \). In fact, if exists integer \( n \geq 0 \) satisfies \( a^{-1} = a^n \), then \( a^{n+1} = e \). It contradicts to the assumption \( S \) is a infinitely cyclic group. Thus, \( B \) is not a biideal of \( S \). Hence there exists a proper almost biideal \( B \) of a quasi regular semigroup \( S \) and \( E = \{ e \} \). \( \square \)

But we have the conclusions:
Theorem 2. Let $B$ be a subgroup of semigroup $S$, and also is a almost biideal of semigroup $S$, then $B = S$.

Proof. As $B$ is a almost biideal of semigroup $S$, then for any $s \in S$, exists $u, v \in B$ satisfies $usu \in B$. On the other hand, $B$ is a subgroup of semigroup $S$, so $u^{-1}, v^{-1} \in B$, therefore $s = u^{-1}(usu)v^{-1}$

$= s \in B$, moreover, $S \subseteq B$. Since $B \subseteq S$, then $B = S$. $\square$

Corollary 1. For any a semigroup $S$, $B$ is a nonempty proper subset of $S$. If $B$ is a subgroup of $S$, then $B$ is not a almost biideal of $S$.

Corollary 2. For any a semigroup $S$, $B$ is a nonempty proper subset of $S$. If $B$ is a almost biideal of $S$, then $B$ is not a subgroup of $S$.

On the other hand, by the process of the proof of the theorem 2, we have the following conclusions:

Theorem 3. There exists a regular set $B$ of a semigroup $S$, and $B$ is a subsemigroup of $S$, but $B$ is not a regular semigroup.

Similarly, there exists a quasi regular set $B$ of a semigroup $S$, and $B$ is a subsemigroup of $S$, but $B$ is not a quasi regular semigroup.

Theorem 4. There exists a proper almost biideal $B$ of a group $S$.

But we have the following conclusion:

Theorem 5. Let $S$ be a monoid. If $S$ have not exists proper almost biideal, then $S$ is a group.

Proof. Assume contrary. Take $e$ be the identity of $S$, $S$ is not a group.

Order $B = S - \{e\}$, then exists $x \in S$, $x$ is not inverse, i.e. for any $y \in S, xy \neq e$. Therefore for any $s \in S$, let $y = sx$, we have $xy = xsx \neq e$, i.e. $x \in B, xsx \in B$. So $B$ is a proper almost biideal of $S$, it contradicts to the assumption $S$ have not exists proper almost biideal and we have the conclusion. $\square$

Lemma 2[3]. Let $S$ be a semigroup, then $S$ is a group if and only if for any $a \in S$ satisfies $aSa = S$.

Example 4. There exists a regular semigroup $S$ and $u, v \in S$ such that $uSv \neq S$.

Take $S = \{0, 1\}$ with the multiplication operation, then $S$ is a regular semigroup, and $0S0 = \{0\} \neq S$. $\square$ We have the following conclusion:

Theorem 6. Let $B$ be both a minimal almost biideal and a subsemigroup of $S$, then $B$ is a subgroup of $S$, moreover $B = S$.

Proof. For any $x \in B$, by the lemma 1, we have $xBx$ is a almost biideal of $S$. Since $xBx \subseteq B, B$ is a minimal almost biideal of $S$, so $xBx = B$. On the other hand, $B$ is
also a subsemigroup of $S$, by the lemma 2, then $B$ is a group and $B$ is a subgroup of $S$. Therefore, from the theorem 2, we have $B = S$. \( \square \)

Finally, I mention the following unsolved problem:

*Let $B$ be a minimal almost biideal of a semigroup $S$, for any $x, y \in S$, is $xy$ a minimal almost biideal of $S$?*

**References**


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