ON n-INNER PRODUCTS, n-NORMS, AND THE CAUCHY-SCHWARZ INEQUALITY

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Abstract. Our observation on the Cauchy-Schwarz inequality in an inner product space and 2-inner product space suggests how the concepts of inner products and 2-inner products, as well as norms and 2-norms, can be generalized to those of n-inner products and n-norms for any $n \in \mathbb{N}$. In this paper, we offer a definition of n-inner products which is simpler than (but equivalent to) the one formulated by Misiak [9]. We also reprove the Cauchy-Schwarz inequality and give a necessary and sufficient condition for the equality.

1. Introduction

We are already familiar with inner products and norms. So, let us begin with the definition of 2-inner products and 2-norms.

Let $X$ be a real vector space of dimension $d \geq 2$. A 2-inner product on $X$ is a function $\langle \cdot, \cdot \rangle : X \times X \times X \to \mathbb{R}$ satisfying the following properties:

(1) $\langle x, y \rangle \geq 0$ for all $x, y \in X$; $\langle x, y \rangle = 0$ if and only if $x$ and $y$ are linearly dependent;
(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$;
(3) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$;
(4) $\langle x, y, z \rangle = \langle x, y \rangle \langle y, z \rangle$ for all $x, y, z \in X$.

The pair $(X, \langle \cdot, \cdot \rangle)$ is called a 2-inner product space (see [2] and [3]). Note that, for generalization purpose, we use a slightly different notation for 2-inner products.

Meanwhile, a 2-norm on $X$ is a function $\| \cdot, \| : X \times X \to \mathbb{R}$ satisfying the following properties:

(N1) $\| x, y \| = 0$ if and only if $x$ and $y$ are linearly dependent;
(N2) $\| x, y \| = \| y, x \|$ for all $x, y \in X$;
(N3) $\| x, \alpha y \| = |\alpha| \| x, y \|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$;
(N4) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$ for all $x, y, z \in X$.

The pair $(X, \| \cdot, \|)$ is called a 2-normed space (see [4]).

If $X$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, then we can define a norm $\| \cdot \|$ on $X$ by $\| x \| := \langle x, x \rangle^{\frac{1}{2}}$. One of the properties of the norm is that it satisfies the triangle inequality

$$\| x + y \| \leq \| x \| + \| y \|,$$

which is easy to prove by using the Cauchy-Schwarz inequality

$$\langle x, y \rangle^2 \leq \| x \|^2 \| y \|^2.$$
By rewriting it as a determinantal inequality involving a $2 \times 2$ Gram matrix

$$\begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix} \geq 0,$$

we see that the Cauchy-Schwarz inequality holds since the matrix is positive semidefinite (see [6], pp. 407-408, for Gram matrices).

At the same time, we can also define a 2-inner product $\langle \cdot, \cdot \rangle$ on $X$ by

$$\langle x | y, z \rangle := \begin{vmatrix} \langle x, x \rangle & \langle x, z \rangle \\ \langle y, x \rangle & \langle y, z \rangle \end{vmatrix}$$

from which we obtain a 2-norm $\| \cdot, \cdot \|$ on $X$ defined by $\| x, y \| := \langle x | y, y \rangle^{\frac{1}{2}}$, that is,

$$\| x, y \| = \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix}^{\frac{1}{2}}.$$

Let us examine this 2-norm. As usual, the properties (N1), (N2) and (N3) are easy to check. To verify the property (N4) or the triangle inequality, it suffices to prove the Cauchy-Schwarz inequality

$$\langle x | y, z \rangle^2 \leq \| x, y \|^2 \| x, z \|^2.$$

But, again, by rewriting it as

$$\begin{vmatrix} \langle x | y, y \rangle & \langle x | y, z \rangle \\ \langle y, y \rangle & \langle y, z \rangle \end{vmatrix} \geq 0,$$

and noting that the matrix is positive semidefinite, that is,

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} \langle x | y, y \rangle & \langle x | y, z \rangle \\ \langle y, y \rangle & \langle y, z \rangle \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \langle x | \alpha y + \beta z, \alpha y + \beta z \rangle \geq 0$$

for any $\alpha, \beta \in \mathbb{R}$, we see that the Cauchy-Schwarz inequality holds.

Alternatively, one may observe that, under the assumption $x \neq 0$, the Cauchy-Schwarz inequality

$$\begin{vmatrix} \langle x, x \rangle & \langle x, z \rangle \\ \langle y, x \rangle & \langle y, z \rangle \end{vmatrix}^2 \leq \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix} \begin{vmatrix} \langle x, x \rangle & \langle x, z \rangle \\ \langle z, x \rangle & \langle z, z \rangle \end{vmatrix}$$

is equivalent to

$$\begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle & \langle x, z \rangle \\ \langle y, x \rangle & \langle y, y \rangle & \langle y, z \rangle \\ \langle z, x \rangle & \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} \geq 0$$

(see [5]). Since the matrix is positive semidefinite, the inequality follows and we also see that the equality holds if and only if $x, y$ and $z$ are linearly dependent.

The above observation on the Cauchy-Schwarz inequality in an inner product space and 2-inner product space suggests how the concepts of inner products and 2-inner products, as well as norms and 2-norms, can be generalized to those of $n$-inner products and $n$-norms for any $n \in \mathbb{N}$. In this paper, we shall offer a definition of $n$-inner products which is slightly simpler than (but equivalent to) the one offered by Misiak [9]. We shall also reprove the Cauchy-Schwarz inequality and give a necessary and sufficient condition for the equality. For related work, see another paper of Misiak [10].
2. AN NATURAL EXAMPLE OF $n$-INNER PRODUCTS AND $n$-NORMS

We shall first show that we can actually define an $n$-inner product and accordingly an $n$-norm on any inner product space provided the dimension is sufficiently large.

Let $n \in \mathbb{N}$ and $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space of dimension $d \geq n$. Define the following function $\langle \cdot, \ldots, \cdot, \cdot \rangle$ on $X \times \cdots \times X$ ($n + 1$ factors) by

$$
\langle x_1, \ldots, x_{n-1}, y, z \rangle := \begin{vmatrix}
\langle x_1, x_1 \rangle & \cdots & \langle x_1, x_{n-1} \rangle & \langle x_1, z \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle x_{n-1}, x_1 \rangle & \cdots & \langle x_{n-1}, x_{n-1} \rangle & \langle x_{n-1}, z \rangle \\
\langle y, x_1 \rangle & \cdots & \langle y, x_{n-1} \rangle & \langle y, z \rangle
\end{vmatrix}
$$

Then one may check that this function satisfies the following five properties:

1. $\langle x_1, \ldots, x_{n-1}, x_n, x_n \rangle \geq 0$; $\langle x_1, \ldots, x_{n-1}, x_n, x_n \rangle = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
2. $\langle x_1, \ldots, x_{n-1}, x_n, x_n \rangle = \langle x_{i_1}, \ldots, x_{i_n}, x_{i_n}, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;
3. $\langle x_1, \ldots, x_{n-1}, y, z \rangle = \langle x_1, \ldots, x_{n-1}, z, y \rangle$;
4. $\langle x_1, \ldots, x_{n-1}, y, \alpha z \rangle = \alpha \langle x_1, \ldots, x_{n-1}, y, z \rangle$;
5. $\langle x_1, \ldots, x_{n-1}, y, z + z' \rangle = \langle x_1, \ldots, x_{n-1}, y, z \rangle + \langle x_1, \ldots, x_{n-1}, y, z' \rangle$.

Accordingly, we can define $\| \cdot \|$ on $X \times \cdots \times X$ ($n$ factors) by

$$
\|x_1, \ldots, x_n\| := \langle x_1, \ldots, x_{n-1}, x_n, x_n \rangle^{1/2},
$$

that is,

$$
\|x_1, \ldots, x_n\| = \begin{vmatrix}
\langle x_1, x_1 \rangle & \cdots & \langle x_1, x_{n-1} \rangle & \langle x_1, x_n \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle x_{n-1}, x_1 \rangle & \cdots & \langle x_{n-1}, x_{n-1} \rangle & \langle x_{n-1}, x_n \rangle \\
\langle x_n, x_1 \rangle & \cdots & \langle x_n, x_{n-1} \rangle & \langle x_n, x_n \rangle
\end{vmatrix}^{1/2}.
$$

For $n = 1$, we know that $\| \cdot \|$ is a norm, while for $n = 2$, $\| \cdot \|$ defines a 2-norm. Note further that for $n = 1$, $\| x_1 \|$ gives the length of $x_1$, while for $n = 2$, $\| x_1, x_2 \|$ represents the area of the parallelogram spanned by $x_1$ and $x_2$. One may also observe that, for $n = 3$ and $X = \mathbb{R}^3$, $\| x_1, x_2, x_3 \|$ is nothing but the volume of the parallelepiped spanned by $x_1, x_2$, and $x_3$, that is,

$$
\| x_1, x_2, x_3 \| = | \langle x_1, (x_2 \times x_3) \rangle |.
$$

Thus, in general, $\| x_1, \ldots, x_n \|$ can be interpreted as the volume of the $n$-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in $X$. Further, it satisfies the following four properties:

1. $\| x_1, \ldots, x_n \| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
2. $\| x_1, \ldots, x_n \|$ is invariant under permutation;
3. $\| x_1, \ldots, x_{n-1}, \alpha x_n \| = | \alpha | \| x_1, \ldots, x_n \|$;
4. $\| x_1, \ldots, x_{n-1}, y + z \| \leq \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|$.

Again, the first three properties are easy to see. To prove the last property or the triangle inequality, we need to establish the Cauchy-Schwarz inequality. Indeed, we have the following:

**Fact 2.1 (The Cauchy-Schwarz Inequality).** For all $x_1, \ldots, x_{n-1}, y, z \in X$, we have

\[ \langle x_1, \ldots, x_{n-1}, y, z \rangle^2 \leq \| x_1, \ldots, x_{n-1}, y \|^2 \| x_1, \ldots, x_{n-1}, z \|^2, \]

and the equality holds if and only if $x_1, \ldots, x_{n-1}, y, z$ are linearly dependent.
Proof. First observe that the inequality may be rewritten as
\[
\begin{pmatrix}
\langle x_1, \ldots, x_{n-1} | y, z \rangle \\
\langle x_1, \ldots, x_{n-1} | z, z \rangle
\end{pmatrix}
\begin{pmatrix}
\langle x_1, \ldots, x_{n-1} | y, y \rangle \\
\langle x_1, \ldots, x_{n-1} | z, z \rangle
\end{pmatrix}
\geq 0,
\]
which obviously holds since the matrix is positive semidefinite.

Next, suppose that we have the equality
\[
\begin{pmatrix}
\langle x_1, \ldots, x_{n-1} | y, y \rangle \\
\langle x_1, \ldots, x_{n-1} | z, z \rangle
\end{pmatrix}
\begin{pmatrix}
\langle x_1, \ldots, x_{n-1} | y, y \rangle \\
\langle x_1, \ldots, x_{n-1} | z, z \rangle
\end{pmatrix} = 0.
\]
If \( \langle x_1, \ldots, x_{n-1} | y, y \rangle = 0 \) or \( \langle x_1, \ldots, x_{n-1} | z, z \rangle = 0 \), then \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent. Otherwise, there exists a \( \beta \neq 0 \) such that
\[
\langle x_1, \ldots, x_{n-1} | y, z \rangle = \beta \langle x_1, \ldots, x_{n-1} | y, y \rangle
\]
and
\[
\langle x_1, \ldots, x_{n-1} | z, z \rangle = \beta \langle x_1, \ldots, x_{n-1} | z, y \rangle.
\]
Hence
\[
\langle x_1, \ldots, x_{n-1} | y, \beta y - z \rangle = 0 \quad \text{and} \quad \langle x_1, \ldots, x_{n-1} | z, \beta y - z \rangle = 0,
\]
and so
\[
\langle x_1, \ldots, x_{n-1} | \beta y - z, \beta y - z \rangle = 0.
\]
But this implies that \( x_1, \ldots, x_{n-1}, \beta y - z \) are linearly dependent, and so are \( x_1, \ldots, x_{n-1}, y, z \).

Conversely, suppose that \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent. If \( x_1, \ldots, x_{n-1} \) are linearly dependent, then the right-hand side of (1) equals zero and so does the left-hand side. So suppose that \( x_1, \ldots, x_{n-1} \) are linearly independent. Since the equation
\[
a_1 x_1 + \cdots + a_{n-1} x_{n-1} + \beta y + \gamma z = 0
\]
has a non-trivial solution, we must have \( \beta \) or \( \gamma \neq 0 \). Without loss of generality, assume that \( \gamma \neq 0 \) so that
\[
z = a_1 x_1 + \cdots + a_{n-1} x_{n-1} + by
\]
for some scalars \( a_1, \ldots, a_{n-1}, b \in \mathbb{R} \). From its definition, we have \( \langle x_1, \ldots, x_{n-1} | y, x_k \rangle = \langle x_1, \ldots, x_{n-1} | z, x_k \rangle = 0 \) for each \( k = 1, \ldots, n - 1 \). Hence
\[
\langle x_1, \ldots, x_{n-1} | y, z \rangle = \langle x_1, \ldots, x_{n-1} | y, by \rangle = b \langle x_1, \ldots, x_{n-1} | y, y \rangle
\]
and
\[
\langle x_1, \ldots, x_{n-1} | z, z \rangle = \langle x_1, \ldots, x_{n-1} | by, by \rangle = b^2 \langle x_1, \ldots, x_{n-1} | y, y \rangle,
\]
and therefore the equality follows. \( \square \)

Moreover, as it can be predicted from our introductory observation, we have the following:

**Fact 2.2.** The Cauchy-Schwarz inequality (1) is equivalent to
\[
\left| \begin{array}{ccccc}
\langle x_1, x_1 \rangle & \cdots & \langle x_1, y \rangle & \cdots & \langle x_1, z \rangle \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\langle y, x_1 \rangle & \cdots & \langle y, y \rangle & \cdots & \langle y, z \rangle \\
\langle z, x_1 \rangle & \cdots & \langle z, y \rangle & \cdots & \langle z, z \rangle
\end{array} \right| \geq 0.
\]

To prove Fact 2.2, we shall use some facts about symmetric matrices. For \( 2 \times 2 \) matrices \( A_2 = [a_{ij}] \), we have \( |A_2| = a_{11}^2 - a_{12} a_{21} \). Particularly, when \( a_{12} = a_{21} \), we have \( |A_2| = a_{11}^2 - a_{12}^2 \), and so, for instance, \( |A_2| \geq 0 \) is equivalent to \( a_{12}^2 \leq a_{11} a_{22} \). For larger matrices, we have the following:
Fact 2.3. Suppose that $A_N = [a_{ij}]$ is an $N \times N$ matrix $(N \geq 3)$ such that the determinants of the sub-matrices $A_k = [a_{ij}]_{i,j=k,...,k}$ $(k = 1, \ldots, N - 2)$ are all non-zero. Then we have

\[
|A_{N-2}| |A_N| = |M_{N-1,N-1}| |M_{N,N}| - |M_{N-1,N}| |M_{N,N-1}|
\]

where $M_{ij}$ denotes the $(N-1) \times (N-1)$ matrix obtained from $A_N$ by deleting the $i$-th row and $j$-th column. In particular, if $A_N$ is symmetric, then

\[
|A_{N-2}| |A_N| = |M_{N-1,N-1}| |M_{N,N}| - |M_{N-1,N}|.
\]

Proof. The proof is elementary. One can just use Gaussian elimination to reduce $A_N$ into the following form

\[
\begin{pmatrix}
* & * & \ldots & * & * \\
0 & * & \ldots & * & * \\
0 & 0 & \ldots & * & * \\
0 & 0 & \ldots & 0 & * \\
0 & 0 & \ldots & 0 & * \\
\end{pmatrix}
\]

and then compare both sides of (3).

We are now ready to prove Fact 2.2.

Proof of Fact 2.2. First note that the Cauchy-Schwarz inequality says that

\[
\left| \begin{array}{ccc}
\langle x_1, x_1 \rangle & \ldots & \langle x_1, x_{n-1} \rangle & \langle x_1, z \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle x_{n-1}, x_1 \rangle & \ldots & \langle x_{n-1}, x_{n-1} \rangle & \langle x_{n-1}, z \rangle \\
\langle y, x_1 \rangle & \ldots & \langle y, x_{n-1} \rangle & \langle y, z \rangle \\
\langle x_1, x_1 \rangle & \ldots & \langle x_1, x_{n-1} \rangle & \langle x_1, y \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle x_{n-1}, x_1 \rangle & \ldots & \langle x_{n-1}, x_{n-1} \rangle & \langle x_{n-1}, y \rangle \\
\langle y, x_1 \rangle & \ldots & \langle y, x_{n-1} \rangle & \langle y, z \rangle \\
\langle z, x_1 \rangle & \ldots & \langle z, x_{n-1} \rangle & \langle z, z \rangle \\
\end{array} \right|^2 
\leq 
\left| \begin{array}{ccc}
\langle x_1, x_1 \rangle & \ldots & \langle x_1, x_{n-1} \rangle & \langle x_1, z \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle x_{n-1}, x_1 \rangle & \ldots & \langle x_{n-1}, x_{n-1} \rangle & \langle x_{n-1}, z \rangle \\
\langle y, x_1 \rangle & \ldots & \langle y, x_{n-1} \rangle & \langle y, z \rangle \\
\langle z, x_1 \rangle & \ldots & \langle z, x_{n-1} \rangle & \langle z, z \rangle \\
\end{array} \right|.
\]

If $x_1, \ldots, x_{n-1}$ are linearly dependent, then both (1) and (2) become the equality $0 = 0$. So suppose that $x_1, \ldots, x_{n-1}$ are linearly independent. Then $\|\langle x_i, x_j \rangle\|_{i,j=1,\ldots,k}$ $> 0$ for each $k = 1, \ldots, n - 1$, and so, by Fact 2.3, the inequality is equivalent to

\[
\left| \begin{array}{ccc}
\langle x_1, x_1 \rangle & \ldots & \langle x_1, y \rangle & \langle x_1, z \rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle y, x_1 \rangle & \ldots & \langle y, y \rangle & \langle y, z \rangle \\
\langle z, x_1 \rangle & \ldots & \langle z, y \rangle & \langle z, z \rangle \\
\end{array} \right| 
\geq 0,
\]

since the $(n+1) \times (n+1)$ matrix is symmetric.

3. A Definition of $n$-Inner Products and $n$-NORMS

Inspired by our observations in the previous sections, we shall now generalize the concepts of inner products and 2-inner products as well as norms and 2-norms to those of $n$-inner products and $n$-norms for any $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $d \geq n$. A function $\langle \cdot, \ldots, \cdot, \cdot \rangle$ on $X \times \cdots \times X$ ($n + 1$ factors) satisfying the five properties (11) - (15) listed in §2 is called an $n$-inner product on $X$, and the pair $(X, \langle \cdot, \ldots, \cdot, \cdot \rangle)$ is called an $n$-inner product space.
Meanwhile, a function \( \| \cdot, \cdot \| \) on \( X \times \cdots \times X \) (\( n \) factors) satisfying the four properties (N1) – (N4) listed in §2 is called an \( n \)-norm on \( X \), and the pair \((X, \| \cdot, \cdot \|)\) is called an \( n \)-normed space.

Note that our definition of \( n \)-inner products is slightly simpler than Misik’s [9]. To see that it is equivalent to Misik’s, one only needs to verify that

\[
\langle x_1, \ldots, x_{n-1}, y, z \rangle = \langle x_i, \ldots, x_{n-1}, y, z \rangle
\]

for every permutation \((i_1, \ldots, i_{n-1})\) of \((1, \ldots, n-1)\). But this will follow easily from the property (12) and the polarization identity

\[
\langle x_1, \ldots, x_{n-1}, y, z \rangle = \frac{1}{4} \left[ \langle x_1, \ldots, x_{n-1}, y + z, y + z \rangle - \langle x_1, \ldots, x_{n-1}, y - z, y - z \rangle \right].
\]

The following theorem confirms that Fact 2.1 is true in any \( n \)-inner product space.

**Theorem 3.1 (The Cauchy-Schwarz Inequality).** Let \((X, \langle \cdot, \cdot \rangle)\) be an \( n \)-inner product space. Then we have

\[
\langle x_1, \ldots, x_{n-1}, y, z \rangle^2 \leq \langle x_1, \ldots, x_{n-1}, y, y \rangle \langle x_1, \ldots, x_{n-1}, z, z \rangle,
\]

and the equality holds if and only if \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent.

**Proof.** The proof goes like that of Fact 2.1. The only difference is when we have to prove that, if \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent, then the equality holds. We note here that, for each \( k = 1, \ldots, n-1 \), we have \( \langle x_1, \ldots, x_{n-1}, y, x_k \rangle = 0 \) and consequently

\[
\langle x_1, \ldots, x_{n-1}, y, x_k \rangle^2 \leq \langle x_1, \ldots, x_{n-1}, y, y \rangle \langle x_1, \ldots, x_{n-1}, x_k, x_k \rangle = 0,
\]

which implies that \( \langle x_1, \ldots, x_{n-1}, y, x_k \rangle = 0 \). The same is true when \( y \) is replaced by \( z \). Thus, if \( z = a_1 x_1 + \cdots + a_{n-1} x_{n-1} + by \) for some \( a_1, \ldots, a_{n-1}, b \in \mathbb{R} \), then

\[
\langle x_1, \ldots, x_{n-1}, y, z \rangle = \langle x_1, \ldots, x_{n-1}, y, by \rangle = b \langle x_1, \ldots, x_{n-1}, y, y \rangle
\]

and

\[
\langle x_1, \ldots, x_{n-1}, z, z \rangle = \langle x_1, \ldots, x_{n-1}, by, by \rangle = b^2 \langle x_1, \ldots, x_{n-1}, y, y \rangle,
\]

and hence the equality follows. \( \Box \)

**Corollary 3.2.** On an \( n \)-inner product space \((X, \langle \cdot, \cdot \rangle)\), the following function

\[
\| x_1, \ldots, x_n \| := \langle x_1, \ldots, x_{n-1}, x_n, x_n \rangle^{1/2}
\]

defines an \( n \)-norm. In particular, the triangle inequality

\[
\| x_1, \ldots, x_{n-1}, y + z \| \leq \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|
\]

holds for all \( x_1, \ldots, x_{n-1}, y, z \in X \).

**Corollary 3.3.** Let \((X, \langle \cdot, \cdot \rangle)\) be an \( n \)-inner product space. If \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent in \( X \), then

\[
\| x_1, \ldots, x_{n-1}, y + z \| = \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|
\]

or

\[
\| x_1, \ldots, x_{n-1}, y - z \| = \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|.
\]

Conversely, if one of the above two equalities holds, then \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent in \( X \).
Proof. Suppose that \( x_1, \ldots, x_{n-1}, y, z \) are linearly dependent in \( X \). As before, we may assume that \( z = a_1 x_1 + \cdots + a_{n-1} x_{n-1} + b y \) for some \( a_1, \ldots, a_{n-1}, b \in \mathbb{R} \). If \( b \geq 0 \), then we have

\[
\| x_1, \ldots, x_{n-1}, y + z \|^2 = \| x_1, \ldots, x_{n-1}, (1 + b) y \|^2
\]

\[
= (1 + b) \| x_1, \ldots, x_{n-1}, y \|^2
\]

\[
= \| x_1, \ldots, x_{n-1}, y \|^2 + \| x_1, \ldots, x_{n-1}, by \|^2
\]

\[
= \| x_1, \ldots, x_{n-1}, y \|^2 + \| x_1, \ldots, x_{n-1}, z \|^2.
\]

If \( b < 0 \), then we have

\[
\| x_1, \ldots, x_{n-1}, y - z \|^2 = \| x_1, \ldots, x_{n-1}, (1 - b) y \|^2
\]

\[
= (1 - b) \| x_1, \ldots, x_{n-1}, y \|^2
\]

\[
= \| x_1, \ldots, x_{n-1}, y \|^2 + \| x_1, \ldots, x_{n-1}, by \|^2
\]

\[
= \| x_1, \ldots, x_{n-1}, y \|^2 + \| x_1, \ldots, x_{n-1}, z \|^2.
\]

Therefore one of the two equalities must hold.

Conversely, without loss of generality, suppose that the equality

\[
\| x_1, \ldots, x_{n-1}, y + z \| = \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|
\]

holds. Squaring both sides, we get

\[
\langle x_1, \ldots, x_{n-1}, y + z, y + z \rangle = \| x_1, \ldots, x_{n-1}, y \|^2 + \| x_1, \ldots, x_{n-1}, z \|^2.
\]

By Theorem 3.1, \( x_1, \ldots, x_{n-1}, y, z \) must be linearly dependent. \( \square \)

The notion of \( n \)-normed spaces may be of independent interest. In an \( n \)-normed space \( (X, \| \cdot, \ldots, \cdot \|) \), we have, for instance, \( \| x_1, \ldots, x_n \| \geq 0 \) and \( \| x_1, \ldots, x_{n-1}, x_n \| = \| x_1, \ldots, x_{n-1}, x_n + \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} \| \) for all \( x_1, \ldots, x_n \in X \) and \( \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R} \).

As in a 2-normed space, a sequence \( x(k) \) in an \( n \)-normed space \( (X, \| \cdot, \ldots, \cdot \|) \) is said to be convergent to some \( x \in X \) if \( \lim_{k \to \infty} \| x_1, \ldots, x_{n-1}, x(k) - x \| = 0 \) for all \( x_1, \ldots, x_{n-1} \in X \).

In such a case, we write \( \lim_{k \to \infty} x(k) = x \) and call \( x \) the limit of \( x(k) \). One may then show that, when \( \lim_{k \to \infty} x(k) \) exists, it must be unique.

Many results in 2-normed spaces, such as fixed point theorems (see [1], [7] and [8]), may have analogues in \( n \)-normed spaces.

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References


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